Combinatorial proof of the rationality scheme for maps in higher genus

Marie Albenque (CNRS, LIX, École Polytechnique)
joint work with Mathias Lepoutre

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A **map** is a collection of polygons glued along their sides (with some technical conditions).
Maps – Definition(s)

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We will also encounter maps on other closed orientable surfaces: torus of genus $g$, disk, ...

[Diagram of maps on various surfaces]
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Euler’s formula: for every map $m$ (on a closed surface without boundary),

$$|V(m)| + |F(m)| = 2 + |E(m)| - 2g(m)$$

vertices faces edges genus
Maps – Definition(s)

A map of genus $g$ is a proper embedding of a connected graph in the torus with $g$ holes (such that all its faces are homeomorphic to disks and considered up to orientation-preserving homeomorphisms).
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\text{map} = \text{graph} + \text{cyclic order of edges around each vertex}.
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map = graph + cyclic order of edges around each vertex.

To avoid dealing with symmetries: maps are rooted (a corner is marked).
Enumeration of planar maps

In the 60’s, Tutte obtained closed enumerative formulas for many families of planar maps.

\[
\#\{\text{rooted planar maps with } n \text{ edges}\} = \frac{2 \cdot 3^n}{n + 2} \text{Catalan}(n) \quad \text{[Tutte 63]}
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\[
= \#\{\text{binary plane trees with } n \text{ inner vertices}\}
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Combinatorial proof ? Bijection ?

Yes ! [Cori & Vauquelin 81], [Schaeffer 97, 98]
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As a corollary: combinatorial proof of Tutte’s formula.
A blossoming tree is a plane tree where vertices can carry opening stems or closing stems:

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\# \text{ closing stems} = \# \text{ opening stems}
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**Diagram:**

- Left: A blossoming tree with opening stems indicated by arrows pointing outward.
- Right: A blossoming tree with closing stems indicated by arrows pointing inward, with a dotted red path indicating a transformation.
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Can we reverse the construction?

i.e. can we determine a canonical spanning tree?
and give a characterization of the possible trees?
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Yes...

Many works in: [Schaeffer, Bousquet-Mélou, Bouttier, Di Francesco, Guitter, Poulalhon, Fusy, Bernardi, A.]
Schaeffer’s blossoming bijection

Blossoming bijection

[Schaeffer 97]
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Blossoming bijection

Turning ccw

If the encountered edge is not a bridge, delete it!

Otherwise, continue!
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Schaeffer’s blossoming bijection

Theorem: [Schaeffer 97]
This is a bijection between 4-valent maps with \( n \) vertices and a family of blossoming 4-valent plane trees with \( n \) vertices.
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This is a bijection between 4-valent maps with \( n \) vertices and a family of blossoming 4-valent plane trees with \( n \) vertices

Question:
Can we generalize it to 4-valent maps in higher genus?
**Theorem:** [Tutte 63], bijective proof in [Schaeffer 97]

\[ M(z) = \sum_{m} z^{\left| E(m) \right|}, \text{ where } m \in \{ \text{planar maps} \}. \]

Then:
\[ M = \frac{1 - 4T}{(1 - 3T)^2} \text{ where } T = \text{unique formal power series defined by } T = z + 3T^2 \]
Rationality scheme in higher genus

**Theorem:** [Tutte 63], bijective proof in [Schaeffer 97]

\[ M(z) = \sum_m z^{|E(m)|}, \quad \text{where} \quad m \in \{\text{planar maps}\}. \]

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**Theorem:** [Bender, Canfield 91], **first bijective proof** in [Lepouvre 19]

For any \( g \geq 1 \), let \( M_g(z) = \sum_m z^{E(m)}, \quad \text{where} \quad m \in \{\text{maps of genus} \ g\}. \)

Then \( M_g \) is a rational function of \( T \).
Rationality scheme in higher genus

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**Remark:**
Result not available with the “mobile-type” bijection of [Chapuy – Marcus – Schaeffer]
Blossoming bijections in higher genus

**Theorem:** [Tutte 63], bijective proof in [Schaeffer 97]

\[ M(\mathbf{z}_\bullet, \mathbf{z}_\circ) = \sum_m \mathbf{z}_\bullet^{\left| V(m) \right|} \mathbf{z}_\circ^{\left| F(m) \right|}, \quad \text{where } m \in \{ \text{planar maps} \}. \]

Then

\[ M = T_\circ T_\bullet (1 - 2T_\circ - 2T_\bullet) \]

where

\[
\begin{align*}
T_\bullet &= \mathbf{z}_\bullet + T_\bullet^2 + 2T_\circ T_\bullet \\
T_\circ &= \mathbf{z}_\circ + T_\circ^2 + 2T_\bullet T_\circ
\end{align*}
\]
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Euler’s formula: \[ |V(m)| + |F(m)| = 2 + |E(m)| - 2g(m) \]
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**Theorem:** [Tutte 63], bijective proof in [Schaeffer 97]

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M(z_\bullet, z_\circ) = \sum_m z_\bullet |V(m)| z_\circ |F(m)|, \text{ where } m \in \{\text{planar maps}\}.
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**Euler’s formula:** \(|V(m)| + |F(m)| = 2 + |E(m)| - 2g(m)|

Already for planar maps, this result is not accessible with mobile-type bijections.
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**Euler’s formula:** \[ |V(m)| + |F(m)| = 2 + |E(m)| - 2g(m) \]

**Theorem:** [Bender, Canfield, Richmond 95], bijective proof in [A., Lepoutre 20+]

For any \( g \geq 1 \), let

\[ M_g(z_\bullet, z_\circ) = \sum_{m} z_\bullet |V(m)| z_\circ |F(m)|, \text{ where } m \in \{\text{maps of genus } g\}. \]

Then \( M_g \) is a rational function of \( T_\bullet \) and \( T_\circ \).
Reformulation of Schaeffer’s blossoming bijection

Aparte: dual of a tree-decorated map (\(=\) map endowed with a spanning tree).
Reformulation of Schaeffer’s blossoming bijection

Aparté: dual of a tree-decorated map (≡ map endowed with a spanning tree).
Reformulation of Schaeffer’s blossoming bijection

Aparté: dual of a tree-decorated map (= map endowed with a spanning tree).
Reformulation of Schaeffer’s blossoming bijection

Aparte: dual of a **tree-decorated map** (= map endowed with a spanning tree).
Reformulation of Schaeffer’s blossoming bijection

Aparte: dual of a tree-decorated map (= map endowed with a spanning tree).

Prop (folklore): This is a bijection for the set of tree-decorated maps.

Abuse of language: “dual of a tree” = corresponding spanning tree of the dual map
Reformulation of Schaeffer’s blossoming bijection

If the encountered edge is not a bridge, delete it!
Otherwise, continue!

Label the faces by their distance to the root face in the dual graph

Turning ccw

Blossoming bijection
[Schaeffer 97]
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Label the faces by their distance to the root face in the dual graph
Consider the “leftmost” breadth-first tree
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Blossoming bijection [Schaeffer 97]

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Label the faces by their distance to the root face in the dual graph
Consider the “leftmost” breadth-first tree

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Bridge
Turning ccw
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Reformulation of Schaeffer’s blossoming bijection

Consider the “leftmost” breadth-first tree

Claim: The dual of the leftmost breadth-first tree is the blossoming tree given by the first description of the bijection.

Label the faces by their distance to the root face in the dual graph

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Turning ccw
Caracterization of the blossoming trees
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Caracterization of the blossoming trees

Good labeling of the corners:
Caracterization of the blossoming trees

Theorem: The blossoming trees are 4-valent trees, that can be endowed with a non-negative good labeling of their corners.
Caracterization and enumeration of the blossoming trees

Good labeling of the corners:
Caracterization and enumeration of the blossoming trees

**Good labeling** of the corners:

Locally around a vertex of a 4-valent tree with a non-negative good labeling:

2 incoming edges and 2 outgoing edges:

- or

\[ i + 1 \quad i \]

\[ i + 1 \quad i \]

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Caracterization and enumeration of the blossoming trees

Good labeling of the corners:

Locally around a vertex of a 4-valent tree with a non-negative good labeling:

2 incoming edges and 2 outgoing edges:

\[ T(z) = z + 3T(z)^2 \]

we retrieve the enumerative result of [Schaeffer]
In higher genus

Theorem [Lepoutre ’19]:

4-valent bicolorable maps of genus $g$

bijection

4-valent blossoming unicellular maps of genus $g$, that can be endowed with a good non-negative labeling
In higher genus

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4-valent blossoming unicellular maps of genus $g$, that can be endowed with a good non-negative labeling

- Bicolorability comes from the radial construction

Bicolorable 4-valent map with $n$ vertices
Theorem [Lepoutre ’19]:

4-valent bicolorable maps of genus $g$

bijection

4-valent blossoming unicellular maps of genus $g$, that can be endowed with a good non-negative labeling

- Bicolorability comes from the radial construction
- **Planar** 4-valent maps are bicolorable, not true in general in higher genus.

- Radial construction [Tutte 63]

Map with $n$ edges

Bicolorable 4-valent map with $n$ vertices

Planar 4-valent maps are bicolorable, not true in general in higher genus.
Theorem [Lepoutre '19]:

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4-valent blossoming **unicellular maps of genus $g$**, that can be endowed with a good non-negative labeling

Dual of a tree-decorated map in **higher genus**.
Theorem [Lepoutre ’19]:

4-valent bicolorable maps of genus $g$

bijection

4-valent blossoming unicellular maps of genus $g$, that can be endowed with a good non-negative labeling

Dual of a tree-decorated map in higher genus.
Theorem [Lepoutre ’19]:

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Dual of a tree-decorated map in higher genus.

Prop (folklore): The dual of a tree-decorated map of genus \( g \) is a map
with a spanning unicellular map of genus \( g \).
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obtained by orienting backwards the edges in
the contour of the unique face.

On top of the local constraints around each vertex, the
fact that the labeling is good gives some compatibility
constraints for the edges of the non-contractible cycles.
In higher genus

Theorem [Lepoutre ’19]:

4-valent bicolorable maps of genus $g$

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4-valent blossoming unicellular maps of genus $g$,
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How to enumerate these objects?

How to prove the rationality schemes with this bijection?
In higher genus: scheme reduction

Unicellular map = non-contractible cycles, the core + tree-like parts
In higher genus: scheme reduction

Unicellular map = non-contractible cycles, the core + tree-like parts

1st step:
erase the trees

i.e.
• replace trees by
• tree containing the root by
In higher genus: scheme reduction

Unicellular map = non-contractible cycles, the \textbf{core} + tree-like parts

1st step:
erase the trees

\begin{itemize}
\item replace trees by \textup{0}\textup{1}\textup{2}
\item tree containing the root by \textup{0}\textup{1}\textup{2}
\end{itemize}

still a non-negative good labeling!
In higher genus: scheme reduction

Unicellular map = non-contractible cycles, the core + tree-like parts

1st step:
erase the trees
i.e.
• replace trees by
• tree containing the root by

still a non-negative good labeling!

2nd step:
reroot at a “scheme stem”

numbers of such stems depends on the shape of the scheme.
In higher genus: scheme reduction

Unicellular map = non-contractible cycles, the \textbf{core} + tree-like parts

1st step:
erase the trees
i.e.
• replace trees by $
\uparrow$
• tree containing the root by $\downarrow$
still a non-negative good labeling !

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Unicellular map = non-contractible cycles, the \textbf{core} + tree-like parts

\begin{itemize}
  \item 1st step: erase the trees
  \item i.e.
  \item \begin{itemize}
    \item replace trees by \;
    \item tree containing the root by \;
  \end{itemize}
  \item still a non-negative good labeling !
\end{itemize}

\begin{align*}
  M_s(z) &= \text{gen. series of maps that admit } s \text{ as scheme.} \\
  &\text{after applying the radial construction + Lepoutre's bijection + erasing the trees !}
\end{align*}

\begin{align*}
  \text{then:} \\
  M_s(z) &= \kappa_s \cdot R_s(T(z))
\end{align*}

\begin{itemize}
  \item 2nd step: reroot at a “scheme stem”
  \item numbers of such stems depends on the \textbf{shape of the scheme.}
  \item still a non-negative good labeling !
\end{itemize}
Theorem: [Bender, Canfield 91], first bijective proof in [Lepoutre 19]

For any $g \geq 1$, let $M_g(z) = \sum_{m} z^{|E(m)|}$, where $m \in \{\text{maps of genus } g\}$.

Then $M_g$ is a rational function of $T$, where:

$$T = \text{unique formal power series defined by } T = z + 3T^2$$
Theorem: [Bender, Canfield 91], first bijective proof in [Lepoutre 19]

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We have: $M_g(z) = \sum_{s \in S_g} M_s(z)$ where

Since, for any fixed $g$, $|S_g| < \infty$. In view of $M_s(z) = \kappa_s \cdot R_s(T(z))$,,
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"Enough" to prove that: Theorem: [Lepoutre 19] (simpler proof in [A. Lepoutre 21+])

For any $s \in S_g$, $R_s$ is a rational function.
Theorem: [Bender, Canfield 91], first bijective proof in [Lepoutre 19]

For any \( g \geq 1 \), let \( M_g(z) = \sum_{m} z^{|E(m)|} \), where \( m \in \{ \text{maps of genus } g \} \).

Then \( M_g \) is a rational function of \( T \), where:

\[
T = \text{unique formal power series defined by } T = z + 3T^2
\]

We have:

\[
M_g(z) = \sum_{s \in S_g} M_s(z) \quad \text{where}
\]

Since, for any fixed \( g \), \(|S_g| < \infty\). In view of \( M_s(z) = \kappa_s \cdot R_s(T(z)) \),

“Enough” to prove that:

Theorem: [Lepoutre 19] (simpler proof in [A. Lepoutre 21+])

For any \( s \in S_g \), \( R_s \) is a rational function.

Remark: an analogous statement does not hold for the bijection of [Chapuy – Marcus – Schaeffer]

Kind of a miracle that it does work for this bijection.

But, this seems robust: extension to bivariate enumeration and to Eulerian \( k \)-angulations

(w.i.p with Castellvi and Fusy)
Thank you for your attention!
Schemes in higher genus
In higher genus: labeled scheme

\[ w_{e_3} = (b) \]

\[ w_{e_1} = (a) \]

\[ w_{e_2} = (d) \]