Local limit of random discrete surfaces with (or without!) a statistical physics model.

Marie Albenque (CNRS, LIX, École Polytechnique)

based on joint works with Laurent Ménard (Univ. Paris Nanterre – NYU Shanghai) and Gilles Schaeffer (CNRS, LIX, École Polytechnique)
I - Definition of planar maps
Maps – Definition(s)

A map is a collection of polygons glued along their sides (with some technical conditions).
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Here, the resulting surface is the sphere: this is a planar map.

If all the polygons have \( p \) sides, the resulting map is called a \( p \)-angulation.

\( 3 \)-angulation = triangulation, \( 4 \)-angulation = quadrangulation
Maps – Definition(s)

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Here, the resulting surface is the sphere: this is a **planar map**.

If all the polygons have $p$ sides, the resulting map is called a **$p$-angulation**.

$$	ext{3-angulation} = \text{triangulation}, \quad \text{4-angulation} = \text{quadrangulation}$$

Side remark: we could also obtain a surface different from the sphere (and even not connected!)

$$\text{equivalent to}$$
Maps – Definition(s)

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To avoid dealing with symmetries: maps are **rooted** (a corner is marked).
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Planar map = planar graph + cyclic order of edges around each vertex. To avoid dealing with symmetries: maps are rooted (a corner is marked).

A map $M$ defines a discrete metric space:

- points: set of vertices of $M = V(M)$.
- distance: graph distance $= d_{gr}$. 
Maps – Motivations

Maps appear in various fields of mathematics, computer science and statistical physics (connections with representation theory, KP-hierarchies, topological recurrence, ...).

They can be studied from many angles.
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They can be studied from many angles.

Today, I focus on the study of limits of random planar maps and, more precisely on local limits of random planar triangulations.

Model: $\mathcal{T}_n = \{\text{Triangulations of size } n\}$

$= n + 2$ vertices, $2n$ faces, $3n$ edges

$T_n = \text{Uniform}$ random element of $\mathcal{T}_n$
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Today, I focus on the study of limits of random planar maps and, more precisely on local limits of random planar triangulations.

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\( T_n = \text{Uniform} \) random element of \( \mathcal{T}_n \)

Spoiler: In the second half of the talk, we will change the probability distribution.
Model: \( \mathcal{T}_n = \{ \text{Triangulations of size } n \} \)
\[ = n + 2 \text{ vertices, } 2n \text{ faces, } 3n \text{ edges} \]
\[ \mathcal{T}_n = \text{Uniform random element of } \mathcal{T}_n \]

Two possible points of view:

Global point of view (scaling limit):

Local point of view (Benjamini-Schramm topology):

Simulation by T.Budd
Simulation by I.Kortchemski
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(scaling limit):

Local point of view  
(Benjamini-Schramm topology):

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Scaling limit of random maps

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When the size of the map goes to infinity, so does the typical distance between two vertices.

**Idea:** ”scale” the map = length of edges decreases with the size of the map.

**Goal:** obtain a limiting (non-trivial) compact object
Scaling limit of random maps

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**Motivations + Results:**

- Discretization of a continuous surface.
- Construction of a 2-dim. analogue of the Brownian motion: The Brownian Map, homeomorphic to the sphere, Hausdorff dimension = 4 [Miermont 13], [Le Gall 13].
- Link with Liouville Quantum Gravity, (will maybe be discussed at the end of the talk) [Duplantier, Sheffield 11], [Duplantier, Miller, Sheffield 14], [Miller, Sheffield 16, 16, 17]
- **Universality:** the scaling is “always” \( n^{-1/4} \) + the limiting object does not depend on the precise combinatorics of the model (\( p \)-angulations, simple triangulations,...)
Local limits of random maps

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When the size of the map goes to infinity, so does the typical distance between two vertices.

Idea: Do NOT scale the distances

Look at neighborhoods of the root

Goal: obtain some (probability distribution on) infinite random maps.
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When the size of the map goes to infinity, so does the typical distance between two vertices.

Idea: Do NOT scale the distances
Look at neighborhoods of the root
Goal: obtain some (probability distribution on) infinite random maps.

Motivations + Results:

• Nice model of random discrete geometry.
• Construction of the Uniform Infinite Planar Triangulation (= UIPT). [Angel, Schramm]
• Connection with some models on Euclidean lattices via the KPZ formula (for Knizhnik, Polyakov and Zamolodchiko), [Duplantier, Sheffield 11]
• Universality: the number of vertices at distance \( R \) from the root is “always” of order \( R^4 \).

Simulation by I.Kortchemski
II - Local limits
Definitions and first examples
Local topology (∼ Benjamini–Schramm convergence)

\( \mathcal{G} := \text{family of (locally finite) rooted graphs} \)

For \( g \in \mathcal{G} \) and \( R \in \mathbb{N}^* \),

\[ B_R(g) = \text{ball of radius } R \text{ around the root vertex of } g \]
Local topology (Benjamini–Schramm convergence)

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\[
B_R(g) = \text{ball of radius } R \text{ around the root vertex of } g
\]

**Definition:**
The local topology on \( G \) is induced by the distance:

\[
d_{loc}(g, g') := \frac{1}{1 + \max\{R \geq 0 : B_R(g) = B_R(g')\}}
\]
Local topology ($\sim$ Benjamini–Schramm convergence)

$\mathcal{G} :=$ family of (locally finite) rooted graphs

For $g \in \mathcal{G}$ and $R \in \mathbb{N}^*$,

$$B_R(g) = \text{ball of radius } R \text{ around the root vertex of } g$$

**Definition:**

The **local topology** on $\mathcal{G}$ is induced by the distance:

$$d_{loc}(g, g') := \frac{1}{1 + \max\{R \geq 0 : B_R(g) = B_R(g')\}}$$

$g_n \to g$ for the local topology $\iff$

For all fixed $R$, there exists $n_0$ s.t.:

$$B_R(g_n) = B_R(g) \text{ for } n \geq n_0$$
Local topology (\sim Benjamini–Schramm convergence)

\( \mathcal{G} := \) family of (locally finite) rooted graphs

For \( g \in \mathcal{G} \) and \( R \in \mathbb{N}^* \),

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**Definition:**
The **local topology** on \( \mathcal{G} \) is induced by the distance:

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\]

\( g_n \to g \) for the local topology \iff \( g_n \to g \) for the local topology,

For all fixed \( R \), there exists \( n_0 \) s.t.:

\[ B_R(g_n) = B_R(g) \text{ for } n \geq n_0 \]

And for **random** graphs?

\[ (\mu_n) = \text{sequence of probability distributions on } \mathcal{G} \text{ (e.g. uniform distribution on } \mathcal{T}_n) \]

if \( \mu_n \xrightarrow{n \to \infty} \mu \) in distribution for the local topology,

we say that \( \mu \) is the **local weak limit** of \( (\mu_n) \).
Local convergence: simple examples

\[ \text{Root} = 0 \]
Local convergence: simple examples

\[0 \rightarrow (\mathbb{Z}^+, 0)\]

Root = 0
Local convergence: simple examples

\[ \begin{align*}
\text{Root} &= 0 \\
0 &\quad 1 \quad 2 \quad n \quad \rightarrow (\mathbb{Z}_+, 0)
\end{align*} \]

Uniformly chosen root
Local convergence: simple examples

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\[
\begin{array}{cccc}
0 & 1 & 2 & n \\
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Root does not matter
Local convergence: simple examples

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Root \Rightarrow 0

Root does not matter

0 1 2 \rightarrow (\mathbb{Z}, 0)

Uniformly chosen root

0 1 2 \rightarrow (\mathbb{Z}, 0)

Root does not matter

n \rightarrow (\mathbb{Z}^2, 0)

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Local convergence: simple examples

\[ 0 \rightarrow (\mathbb{Z}_+, 0) \]

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\[ n \rightarrow (\mathbb{Z}^2, 0) \]

Uniformly chosen root
III - Local limits of random trees and maps
Local convergence: more complicated examples

$\mu_n =$ uniform measure on plane trees with $n$ vertices:

\[ \begin{array}{c|c|c|c|c|c}
\mu_1 & \mu_2 & \mu_3 & \mu_4 & \mu_5 \\
\hline
1/2 & 1/2 & 1/5 & 1/5 & 1/5 \\
\end{array} \]
Local convergence: more complicated examples

\( \mu_n = \) uniform measure on plane trees with \( n \) vertices:

\( \mu_1 \)

\( \mu_2 \)

\( \mu_3 \)

\( \mu_4 \)

\( n = 500 \)

\( n = 1000 \)
Local convergence: more complicated examples

\[ \mu_n = \text{uniform measure on plane trees with } n \text{ vertices:} \]

\[ \begin{align*}
\mu_1 & = 1/2 \\
\mu_2 & = 1/2 \\
\mu_3 & = 1/5 \\
\mu_4 & = 1/5 \\
\end{align*} \]

The limit is a \textbf{probability distribution} on infinite trees with one infinite branch [Kesten].
Local limit of large uniformly random triangulations

**Theorem** [Angel – Schramm, ’03]

Let \( P_n \) = uniform distribution on triangulations of size \( n \).

\[
P_n \xrightarrow{(d)} \text{UIPT}, \quad \text{for the local topology}
\]

\( \text{UIPT} \) = Uniform Infinite Planar Triangulation

= measure supported on infinite planar triangulations.
Local limit of large uniformly random triangulations

A very short idea of the proof:

Need to evaluate the probability that a given neighborhood $\Delta$ of the root appears:

$$P_n \left( \begin{array}{c} ??? \\ ??? \end{array} \right)$$
Local limit of large uniformly random triangulations

A very short idea of the proof:

Need to evaluate the probability that a given neighborhood $\Delta$ of the root appears:

$$P_n \left( \begin{array} {ccc} \text{???)} & \text{???)} \\ \ell(\Delta) = 11 \end{array} \right) = \frac{|T^{(k)}_{3n-e(\Delta)+\ell(\Delta)}|}{|T_n|}$$

$T^{(k)}_n = \{\text{triangulations with } n \text{ edges and perimeter } k\}$

\[
\begin{align*}
\ell(\Delta) & = \text{perimeter of } \Delta \\
e(\Delta) & = \#\{\text{edges of } \Delta\}
\end{align*}
\]
Local limit of large uniformly random triangulations

A very short idea of the proof:

Need to evaluate the probability that a given neighborhood $\Delta$ of the root appears:

$$\mathbb{P}_n \left( \begin{array}{c} ??? \\ \ell(\Delta) = 11 \end{array} \right) = \frac{|\mathcal{T}_n^{(k)}|}{|\mathcal{T}_n|} \to \mathbb{P}_\infty \left( \begin{array}{c} ??? \\ \ell(\Delta) = \text{perimeter of } \Delta \end{array} \right)$$

$\mathcal{T}_n^{(k)} = \{\text{triangulations with } n \text{ edges and perimeter } k\}$
Local limit of large uniformly random triangulations

**Theorem** [Angel – Schramm, ’03]

Let $\mathbb{P}_n = \text{uniform distribution on triangulations of size } n$.

$$\mathbb{P}_n \xrightarrow{(d)} \text{UIPT}, \quad \text{for the local topology}$$

UIPT = Uniform Infinite Planar Triangulation
= measure supported on infinite planar triangulations.

**Some properties of the UIPT:**

- The UIPT has almost surely one end [Angel – Schramm, 03]
- Volume (nb. of vertices) and perimeters of balls known to some extent.
  $$\mathbb{E} \left[ |B_R(T_\infty)| \right] \sim \frac{2}{7} R^4 \quad [\text{Angel 04, Curien – Le Gall 12}]$$
- Simple random Walk is recurrent [Gurel-Gurevich – Nachmias 13]
Local limit of large uniformly random triangulations

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- Simple random Walk is recurrent [Gurel-Gurevich – Nachmias 13]

Universality: we expect the same behavior for other uniform models of maps.

In particular, we expect the volume growth to be 4.

(proved for quadrangulations [Krikun 05], simple triangulations [Angel 04])
Intermezzo: why should we care about local limits?

Suppose that a sequence of random graphs $G_n$ admits a local weak limit $G_\infty$,

$$f(G_n) \overset{\text{proba}}{\longrightarrow} f(G_\infty)$$

for any $f$ which is continuous for $d_{loc}$.

e.g.: $f = |B_R(.)|$
Intermezzo: why should we care about local limits?

Suppose that a sequence of random graphs $G_n$ admits a local weak limit $G_\infty$,

Then, $f(G_n) \xrightarrow{\text{proba}} f(G_\infty)$ for any $f$ which is continuous for $d_{loc}$.

E.g.: $f = |B_R(.)|

Main idea: The limiting object is often “nicer”.

Hence, it is easier to compute $f(G_\infty)$, from which we can deduce the behavior of $f(G_n)$. 
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For graphs, it has been formalized as the objective method [Aldous-Steele 94].
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e.g.: $f = |BR(.)|

Main idea: The limiting object is often “nicer”.

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For graphs, it has been formalized as the objective method [Aldous-Steele 94].

Two example for maps:

• one-endedness in the UIPT:

Allows to give an explicit description of what can happen when the map gets disconnected.

This is crucial to study a “peeling” exploration spiraling around the root, which gives the volume of the balls [Angel 03].

• spatial Markov property

Conditionally on their perimeter, the interior and exterior of a ball are independent
IV - Local limits of Ising-weighted triangulations
Escaping universality: adding matter

First, Ising model on a finite deterministic planar triangulation $T$:

**Spin configuration** on $T$:

$$\sigma : V(T) \rightarrow \{-1, +1\} = \{\oplus, \ominus\}.$$

**Ising model** on $T$: take a random spin configuration with probability:

$$P(\sigma) \propto e^{\beta J \sum_{v \sim v'} \mathbf{1}_{\{\sigma(v) = \sigma(v')\}}}$$

$\beta > 0$: inverse temperature.
$J = \pm 1$: coupling constant.
$h = 0$: no magnetic field.
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**Ising model** on $T$: take a random spin configuration with probability:

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**Combinatorial formulation**: \( P(\sigma) \propto \nu^{m(\sigma)} \)

with \( m(\sigma) = \) number of monochromatic edges \( (\nu = e^{\beta J}) \).
Escaping universality: adding matter

First, **Ising model** on a finite deterministic planar triangulation $T$:

**Spin configuration** on $T$:

$$\sigma : V(T) \to \{-1, +1\} = \{\bigcirc, \bigoplus\}.$$  

**Ising model** on $T$: take a random spin configuration with probability:

$$P(\sigma) \propto e^{\beta J \sum_{v \sim v'} 1_{\{\sigma(v) = \sigma(v')\}}} \quad \beta > 0: \text{inverse temperature.}$$  
$$J = \pm 1: \text{coupling constant.}$$  
$$h = 0: \text{no magnetic field.}$$

**Combinatorial formulation:**  

$$P(\sigma) \propto \nu^{m(\sigma)} \quad \text{with } m(\sigma) = \text{number of monochromatic edges (} \nu = e^{\beta J} \text{)}.$$ 

Next step: Sample a triangulation of size $n$ **together** with a spin configuration, with probability $\propto \nu^{m(T, \sigma)}$.

$$\mathbb{P}^{\nu}_n \left( \{(T, \sigma)\} \right) = \frac{\nu^{m(T, \sigma)} \delta_{|e(T)|=3n}}{Z_n}.$$  
$$Z_n = \text{normalizing constant.}$$
Escaping universality: adding matter

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Ising model on $T$: take a random spin configuration with probability:

$$P(\sigma) \propto e^{\beta J \sum_{v \sim v'} 1_{\{\sigma(v) = \sigma(v')\}}}$$  

with $m(\sigma) =$ number of monochromatic edges ($\nu = e^{\beta J}$).

Combinatorial formulation: $P(\sigma) \propto \nu^{m(\sigma)}$

Next step: Sample a triangulation of size $n$ together with a spin configuration, with probability $\propto \nu^{m(T,\sigma)}$.

$$\mathbb{P}_n^\nu \left( \{(T, \sigma)\} \right) = \frac{\nu^{m(T,\sigma)} \delta_{|e(T)|=3n}}{\mathcal{Z}_n}.$$  

$\mathcal{Z}_n =$ normalizing constant.

Remark: This is a probability distribution on triangulations with spins. But, forgetting the spins gives a probability a distribution on triangulations without spins different from the uniform distribution.
Escaping universality: new asymptotic behavior

Counting exponent for undecorated maps:

number of (undecorated) maps of size \( n \sim \kappa \rho^{-n} n^{-5/2} \)

(e.g.: triangulations, quadrangulations, general maps, simple maps,...)

where \( \kappa \) and \( \rho \) depend on the combinatorics of the model.
Escaping universality: new asymptotic behavior

Counting exponent for undecorated maps:

number of (undecorated) maps of size $n \sim \kappa \rho^{-n} n^{-5/2}$
(e.g.: triangulations, quadrangulations, general maps, simple maps,...)
where $\kappa$ and $\rho$ depend on the combinatorics of the model.

Generating series of Ising-weighted triangulations:

$$Q(\nu, t) = \sum_{T \in \mathcal{T}_f} \sum_{\sigma : V(T) \to \{-1, +1\}} \nu^{m(T, \sigma)} t^{e(T)}.$$  

Theorem [Bernardi – Bousquet-Mélou 11]
For every $\nu > 0$, $Q(\nu, t)$ is algebraic and satisfies

$$[t^{3n}] Q(\nu, t) \sim_{n \to \infty} \begin{cases} \kappa \rho_{\nu c}^{-n} n^{-7/3} & \text{if } \nu = \nu_c = 1 + \frac{1}{\sqrt{7}}, \\ \kappa \rho_\nu^{-n} n^{-5/2} & \text{if } \nu \neq \nu_c. \end{cases}$$

See also [Boulatov – Kazakov 1987], [Bousquet-Melou – Schaeffer 03] and [Bouttier – Di Francesco – Guitter 04].

This suggests a different behavior of the underlying maps for $\nu = \nu_c$. 
Local convergence of triangulations with spins

**Theorem** [A. – Ménard – Schaeffer, 21]

Let $\mathbb{P}_n^\nu = \nu$–Ising weighted probability distribution for triangulations of size $n$:

$$\mathbb{P}_n^\nu \xrightarrow{(d)} \nu\text{-IIPT}, \quad \text{for the local topology}$$

$\nu\text{-IIPT} = \nu$-Ising Infinite Planar Triangulation

= measure supported on infinite planar triangulations.
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**A very very short idea of the proof:** Computations + Maple = ❤

Need to evaluate the probability that a given neighborhood of the root appears:
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Need to evaluate the probability that a given neighborhood of the root appears:

$$\mathbb{P}_{n}^{\nu} \left( \begin{array}{c} ??? \\ ??? \\ ??? \end{array} \right) = \frac{\nu \cdot m(\Delta) - m(\omega) \cdot [t^{3n} - e(\Delta) + |\omega|] \cdot Z_{\omega}(\nu, t)}{[t^{3n}] Q(\nu, t)}$$

Generating series of triangulations with boundary conditions given by $\omega$.

Here $\omega = + - + - - - + - + + + -$
Theorem [A. – Ménard – Schaeffer, 21]

Let $\mathbb{P}_n^\nu = \nu$–Ising weighted probability distribution for triangulations of size $n$:

$$\mathbb{P}_n^\nu \xrightarrow{(d)} \nu$-IIPT, \quad \text{for the local topology}$$

$\nu$-IIPT = $\nu$-Ising Infinite Planar Triangulation

$\quad = \text{measure supported on infinite planar triangulations.}$

Moreover:

- The $\nu$-IIPT has almost surely one end
- Simple random Walk is recurrent on $\nu_c$-IIPT.
Local convergence of triangulations with spins

**Theorem** [A. – Ménard – Schaeffer, 21]

Let $\mathbb{P}_n^\nu = \nu$–Ising weighted probability distribution for triangulations of size $n$:

$$\mathbb{P}_n^\nu \xrightarrow{(d)} \nu$-$\text{IPT}$, \quad \text{for the local topology}$$

$\nu$-$\text{IPT} = \nu$-Ising Infinite Planar Triangulation

$= \text{measure supported on infinite planar triangulations}$. 

Moreover:

- The $\nu$-$\text{IPT}$ has almost surely one end
- Simple random Walk is recurrent on $\nu_c$-$\text{IPT}$.

But:

- Volume (nb. of vertices) and perimeters of balls is **unknown**.

**Non-universality**: we expect a **different** behavior for $\nu = \nu_c$

In particular, we expect the volume growth to be different from 4.
V - Connections and motivations from statistical physics
Motivations from statistical physics

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In general relativity, the underlying space is not Euclidean anymore but is a Riemannian space, whose curvature describes the gravity.

One of the main challenge of modern physics is to make two theories consistent:

- quantum mechanics (which governs microscopic scales)
- general relativity (which governs macroscopic scales)
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One of the main challenge of modern physics is to make two theories consistent:

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One attempt to reconcile these two theories, is the Liouville Quantum gravity which replaces the deterministic Riemannian space by a random metric space.
Link with Liouville Quantum Gravity

$\gamma \in (0, 2)$, $\gamma$-Liouville Quantum Gravity = measure on a surface [Duplantier, Sheffield 11].

Simulation of the Brownian sphere by T.Budd

Simulation of $\sqrt{\frac{8}{3}}$-LQG by T.Budd
Link with Liouville Quantum Gravity

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\[ \text{Construction in the continuum.} \]

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A priori, there is no canonical way to embed a planar map in the sphere.

But, for simple triangulations: the circle packing theorem gives a canonical embedding.

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Simulation of $\sqrt{\frac{8}{3}}$-LQG by T.Budd
The critical Ising model is *believed* to converge to $\sqrt{3}$-LQG.

Similar statements for other models of decorated maps
(with a spanning subtree ($\gamma = \sqrt{2}$), with a bipolar orientation ($\gamma = \sqrt{4/3}$),...).

For $\gamma \in (0, 2)$, there exists $d_\gamma =$ “fractal dimension of $\gamma$-LQG”

$d_\gamma =$ ball volume growth exponent for corresponding maps ??

YES, in some cases [Gwynne, Holden, Sun ’17], [Ding, Gwynne ’18]

The connection is not proven for Ising, but $d_{\sqrt{3}}$ is a good candidate for the volume growth exponent.
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General bounds for $d_{\gamma}$ [Ding, Gwynne ’18], which give $4.18 \leq d_{\sqrt{3}} \leq 4.25$.

In particular $d_{\sqrt{3}} \neq 4$ and growth volume would then be different than the uniform models.
**Perspectives and related works**

- Compute the volume growth of the $\nu$-IIPT
  or, at least, prove that it is different from 4 for $\nu = \nu_c$

- Study the connected components of the $+\ell$ spins [A-Ménard, 22+]
  gives some insights about the Ising model on $\mathbb{Z}^2$ via the KPZ formula

- Investigate the different statistical physics models and their link with $\gamma$-LQG
Thank you!