Convergence of simple Triangulations

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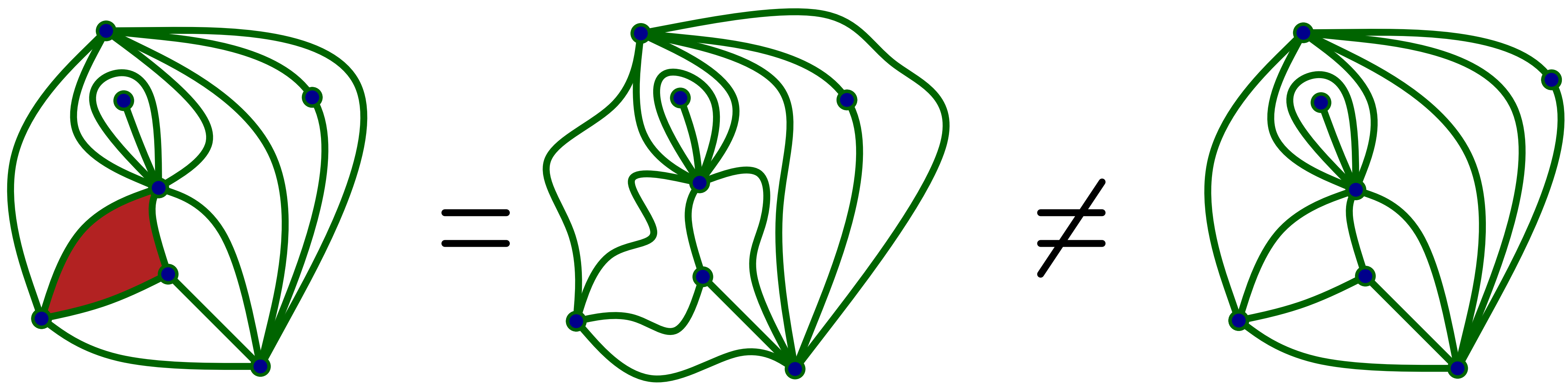
Planar Maps – Triangulations.

A **planar map** is the embedding of a connected graph in the sphere up to continuous deformations.



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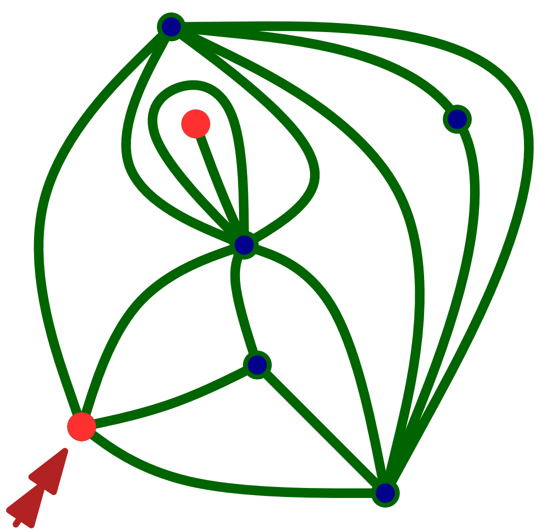

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Plane maps are **rooted**. Face that contains the root = **outer face**

Distance between two vertices = number of edges between them.
Planar map = **Metric space**

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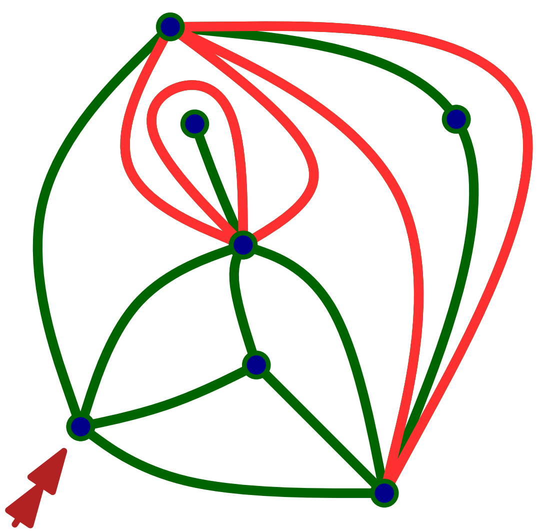
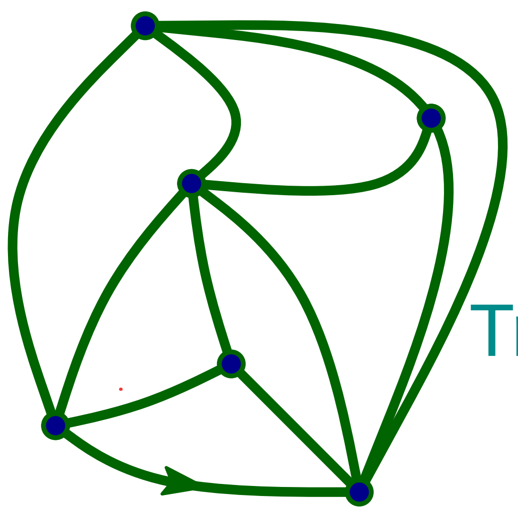


Triangulation = all faces are triangles.

Simple map = no loops nor multiple edges

Model + Motivation



Simple Triangulation

Euler Formula : $v + f = 2 + e$

Triangulation : $2e = 3f$

$\mathcal{M}_n = \{$Simple triangulations of size $n\}$

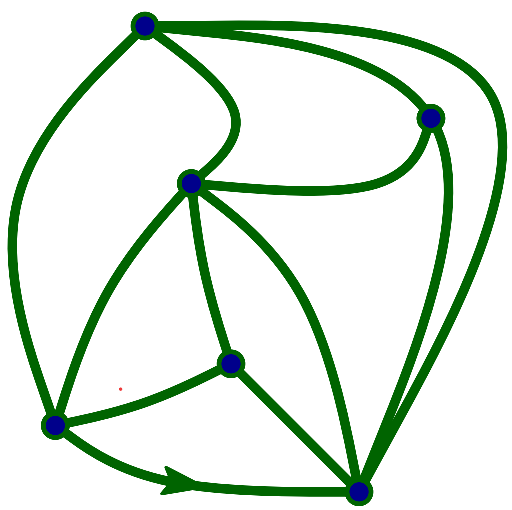
$= n + 2$ vertices, $2n$ faces, $3n$ edges

$M_n =$ Random element of $\mathcal{M}_n$

What is the behavior of M_n when n goes to infinity ?

typical distances ? convergence towards a continuous object ?

Model + Motivation



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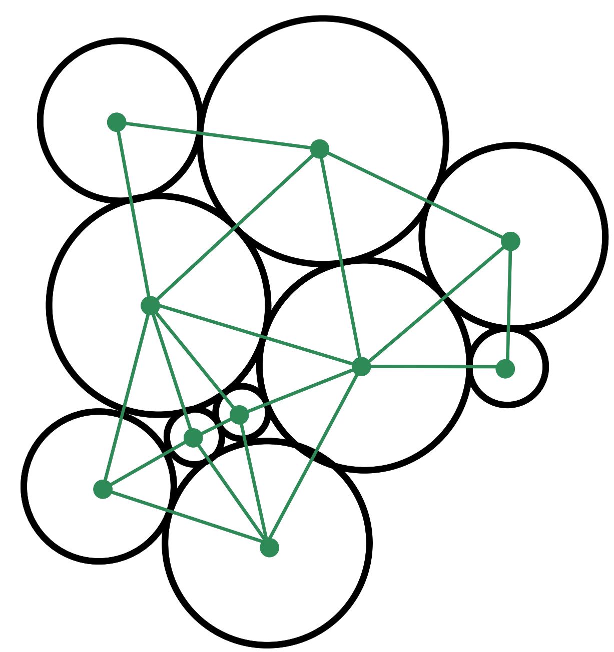
One motivation : Circle-packing theorem

Each simple triangulation M has a unique (up to Möbius transformations and reflections) circle packing whose tangency graph is M.
[Koebe-Andreev-Thurston]

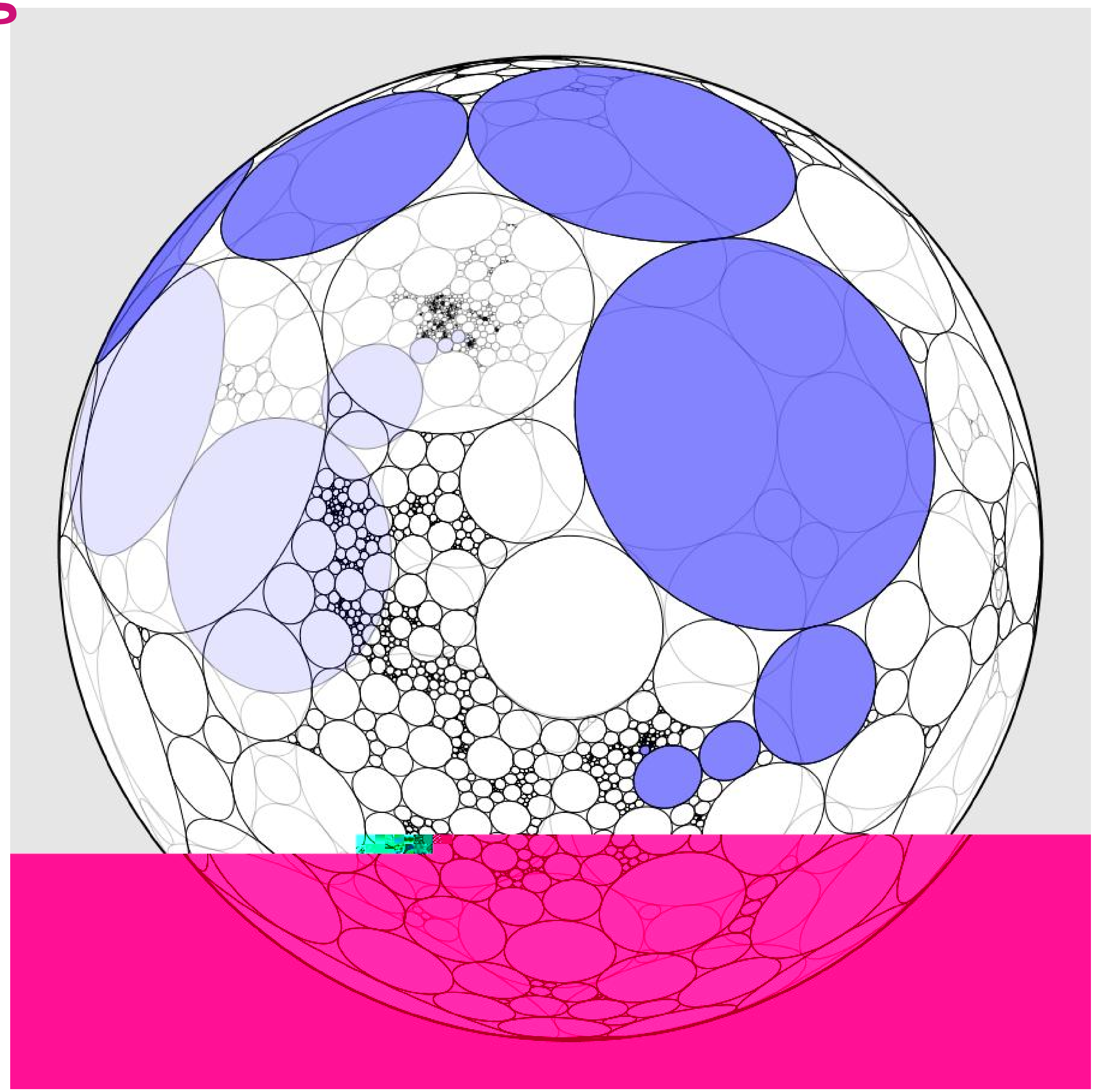


Gives a canonical embedding of simple triangulations in the sphere and possibly of their limit.

Random circle packing

Random circle packing = canonical embedding of random simple triangulation in the sphere.

Gives a way to de ne a canonical embedding of their limit ?



Team effort : code by Kenneth Stephenson, Eric Fusy and our own.

Convergence of uniform quadrangulations

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  Typical distance is $n^{1/4}$ + convergence of the pro le

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  Convergence towards the Brownian map
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Idea : The Brownian map is a **universal** limiting object.
All "reasonable models" of maps (properly rescaled) are expected to converge towards it.

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general maps
NOT simple maps

Problem : These results relie on nice bijections between maps and labeled trees [Schae er '98], [Bouttier-Di Francesco-Guitter '04].

The result

Theorem : [Addario-Berry, A.]
(M_n) = sequence of random **simple** triangulations, then:

$$\left(M_n ; \left(\frac{3}{4n}\right)^{1/4} d_{M_n} \right) \xrightarrow{(d)} (M ; D^\star);$$

for the distance of Gromov-Hausdor on the isometry classes of compact metric spaces.

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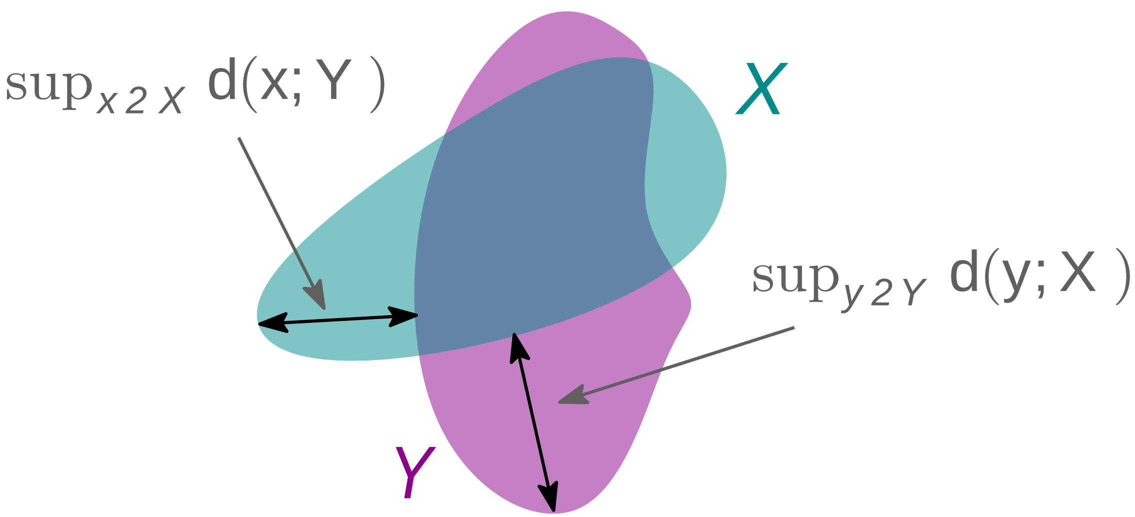
Exactly the same kind of result as Le Gall and Miermont's.

Gromov-Hausdorff distance

Hausdorff distance between X and Y two compact sets of $(E; d)$:

$$d_H(X;Y) = \max\{\sup_{x\in X} d(x;Y); \sup_{y\in Y} d(y;X)\}$$



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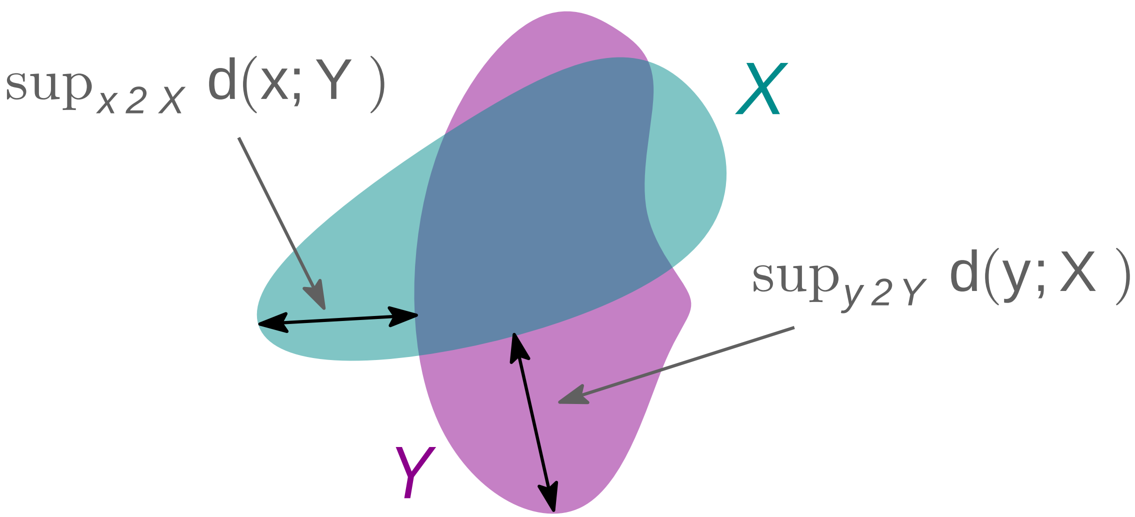

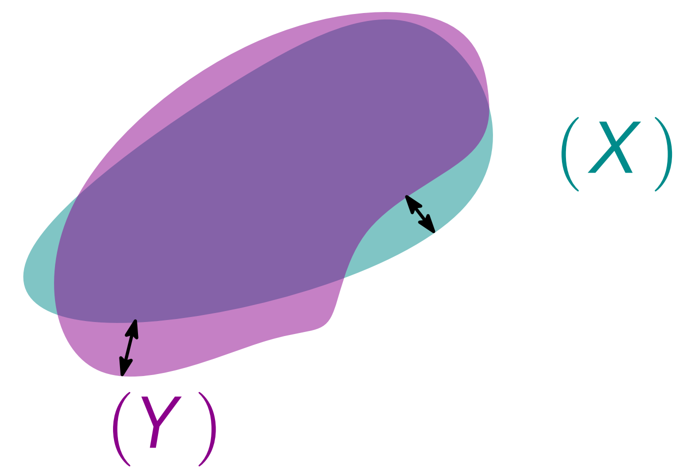

Gromov-Hausdorff distance btw two compact metric spaces E and F:

$$\mathbf{d_{GH}}(\mathbf{E};\mathbf{F}) = \inf \mathbf{d_H}(\quad(\mathbf{E});\quad(\mathbf{F}))$$

In mum taken on :
- all the metric spaces M
- all the isometric embeddings $\quad;\quad : E;F \to M$.

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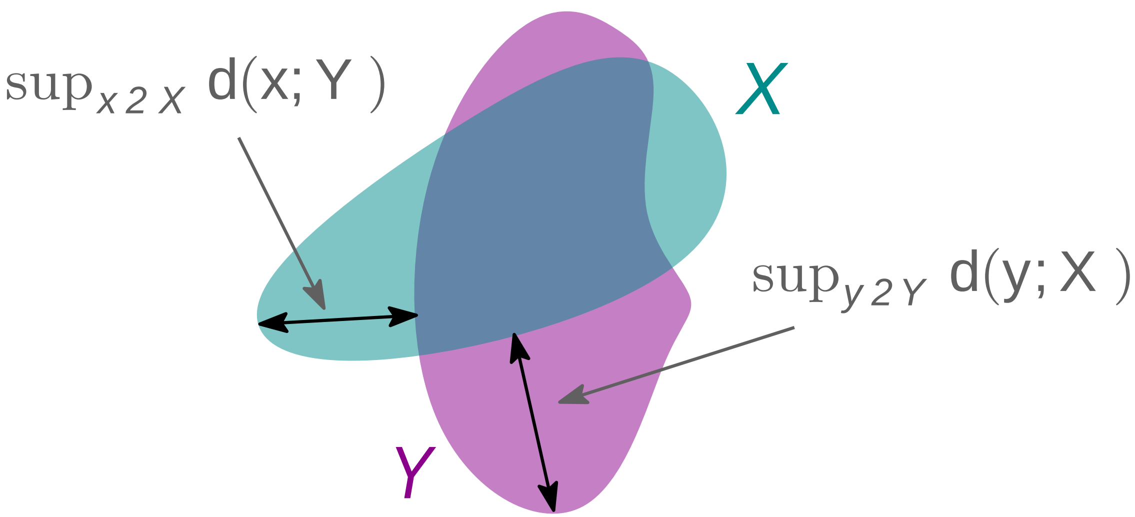

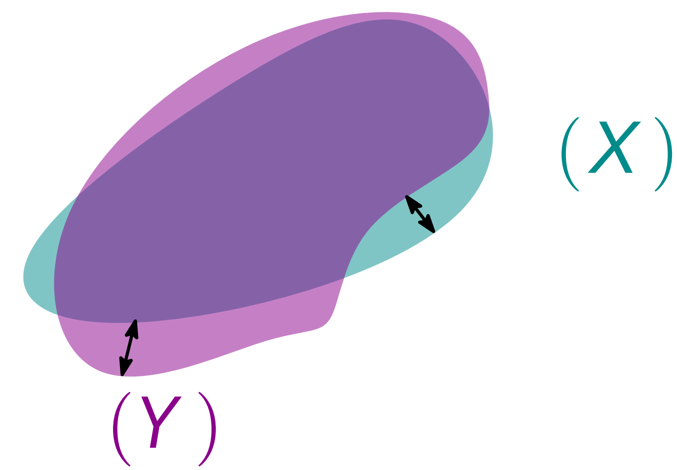

Gromov-Hausdorff distance btw two compact metric spaces E and F:

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$\{$isometry classes of compact metric spaces with GH distance$\}$
= complete and separable (= \Polish") space.

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Idea of proof :

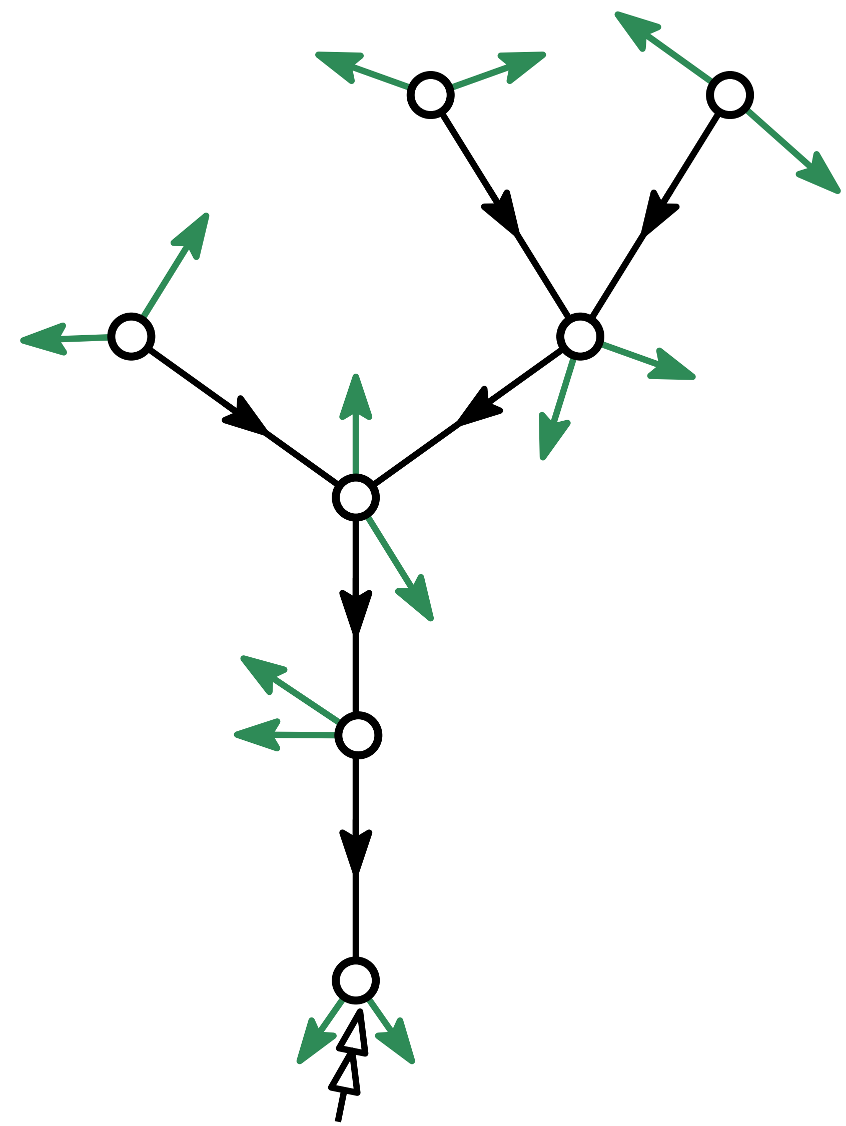
- encode the simple triangulations by some trees,
- **study the limits of trees**,
- interpret the **distance in the maps by some function of the tree**.

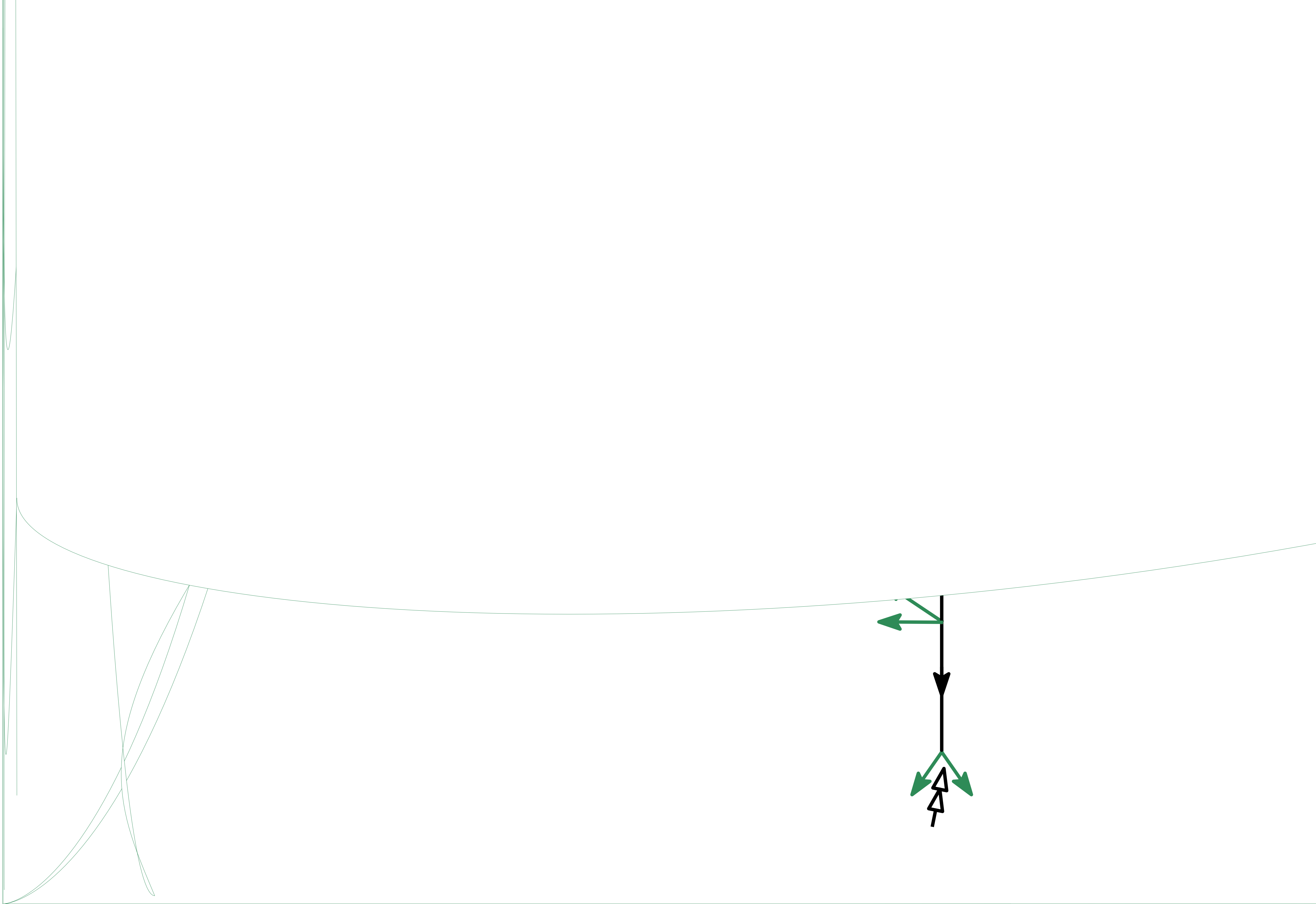
From blossoming trees to simple triangulations

plane tree:
plane map that is a tree

rooted plane tree:
one corner is distinguished

2-blossoming tree:
planted plane tree such that each vertex carries two leaves





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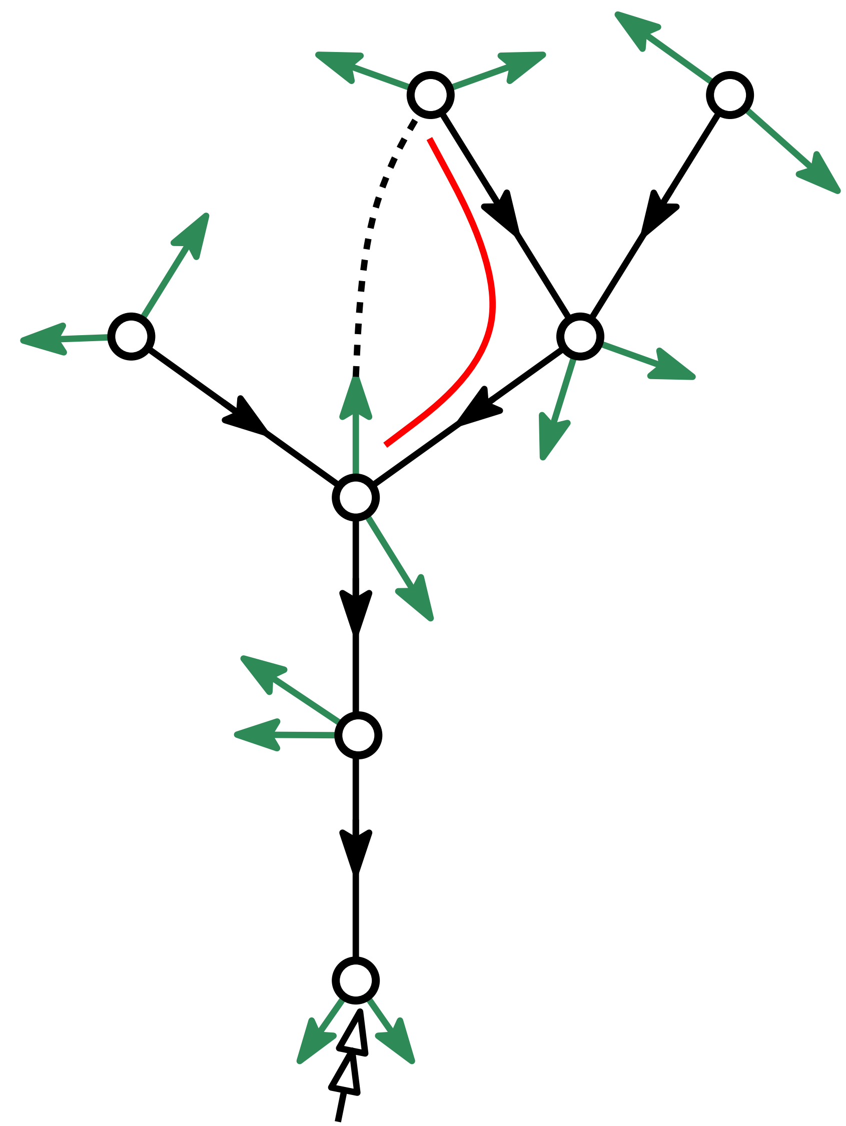
Given a planted 2-blossoming tree:

- If a leaf is followed by two internal edges,
- close it to make a triangle.



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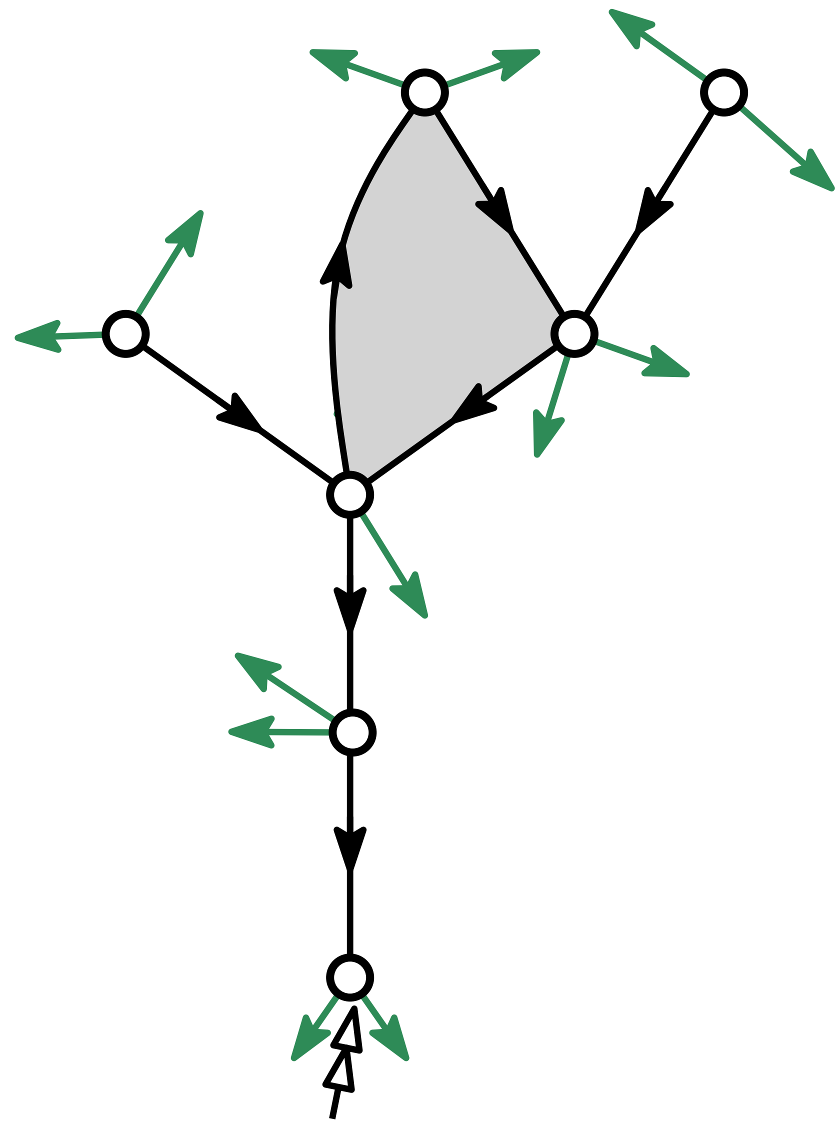

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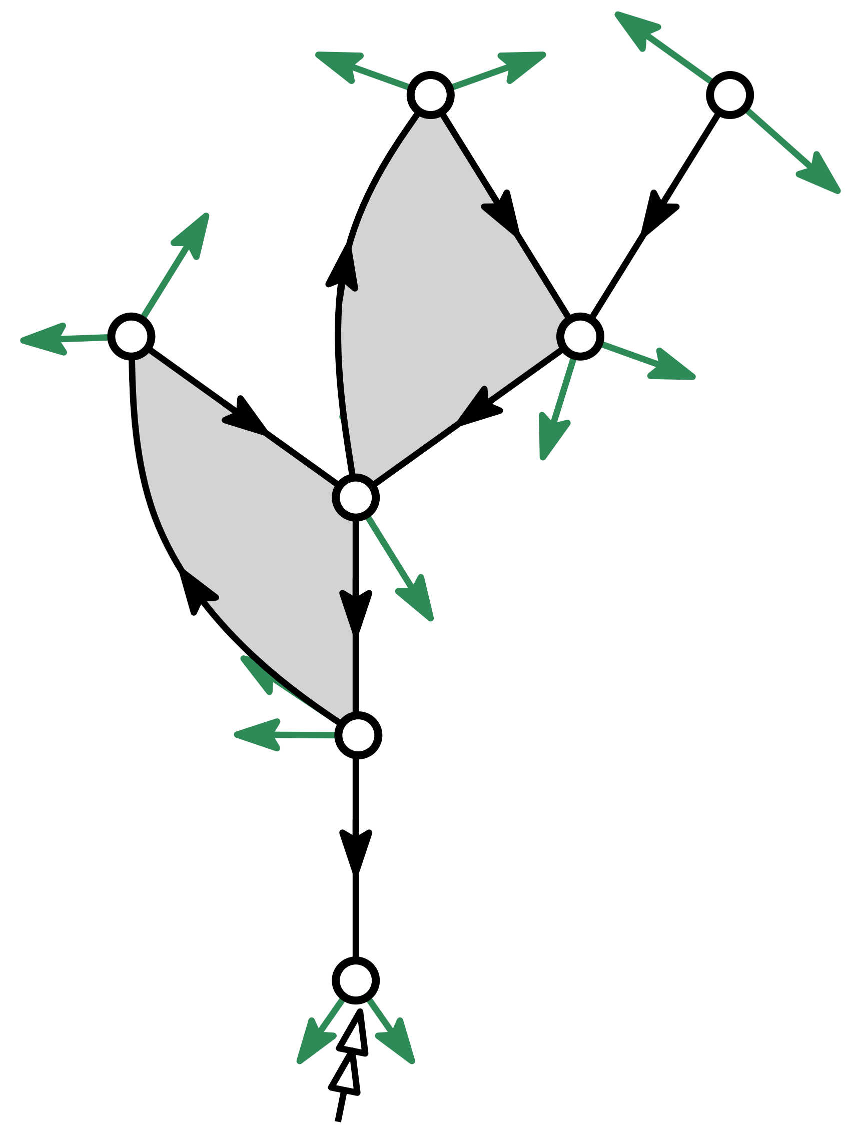

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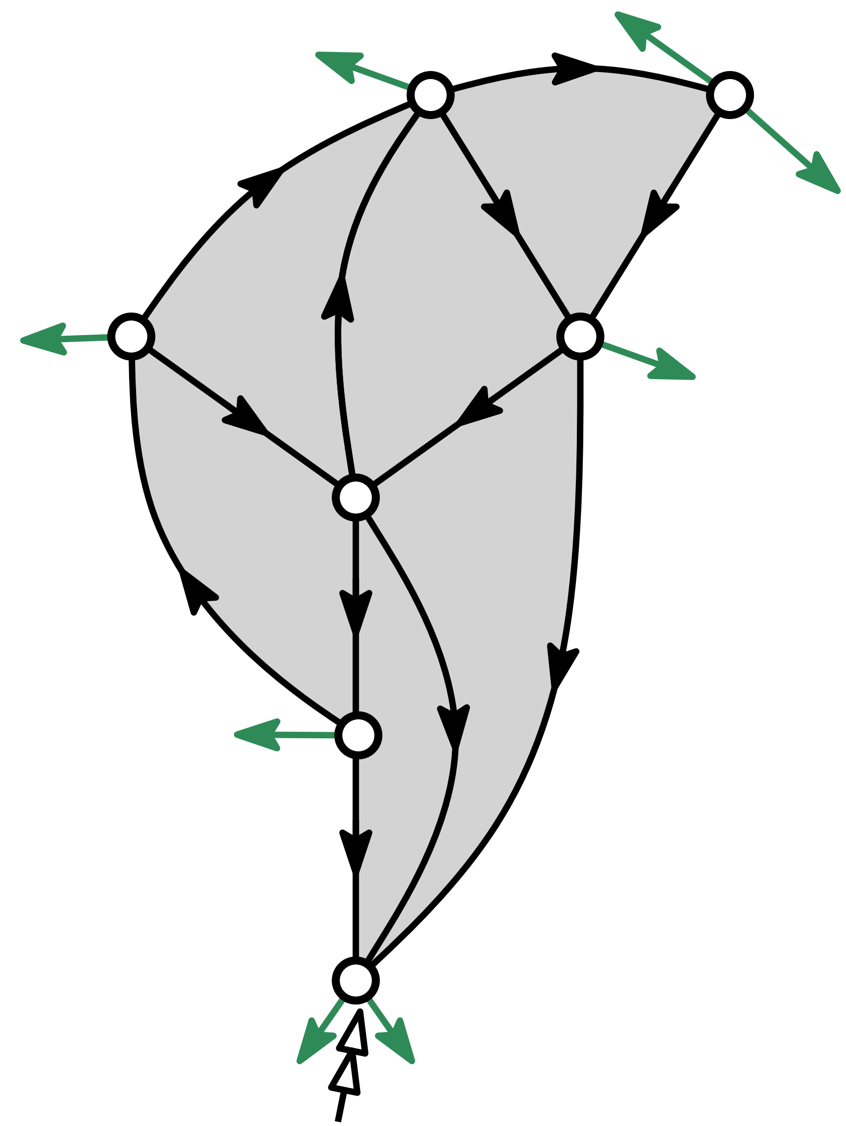
Tree **balanced** = root corner has two leaves



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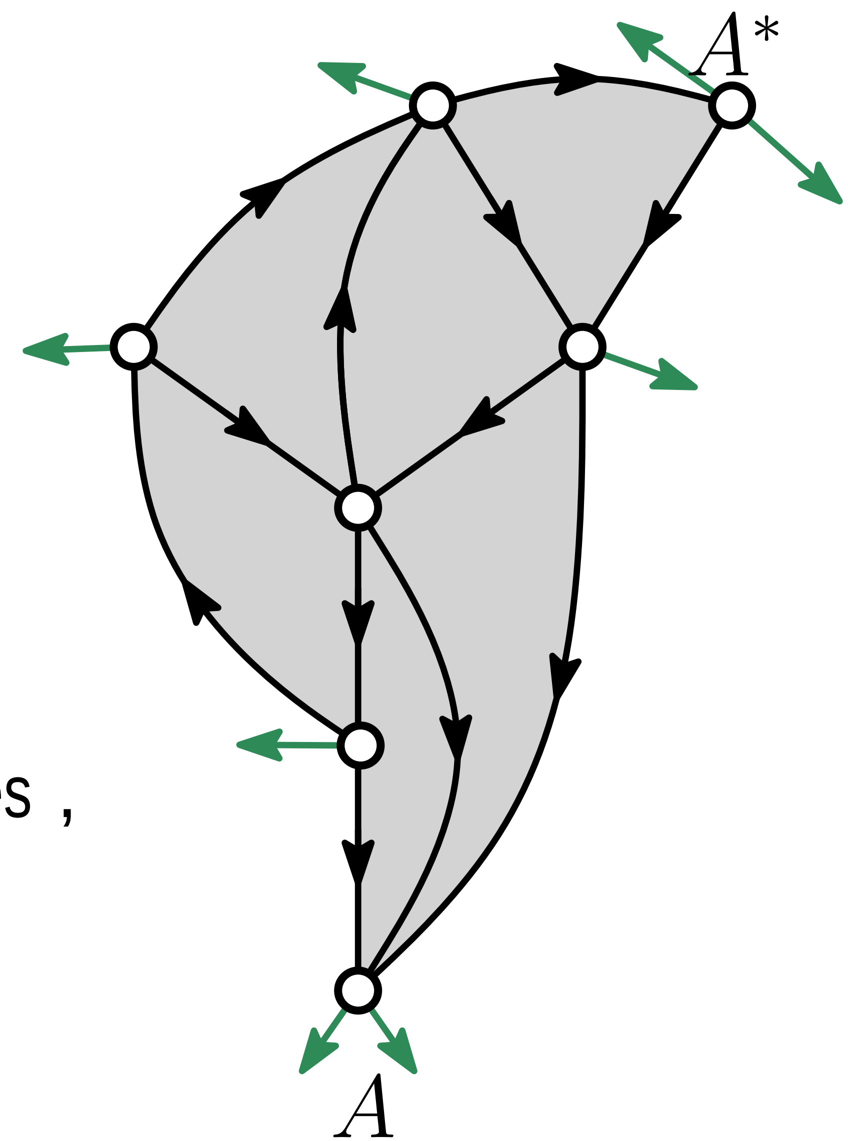

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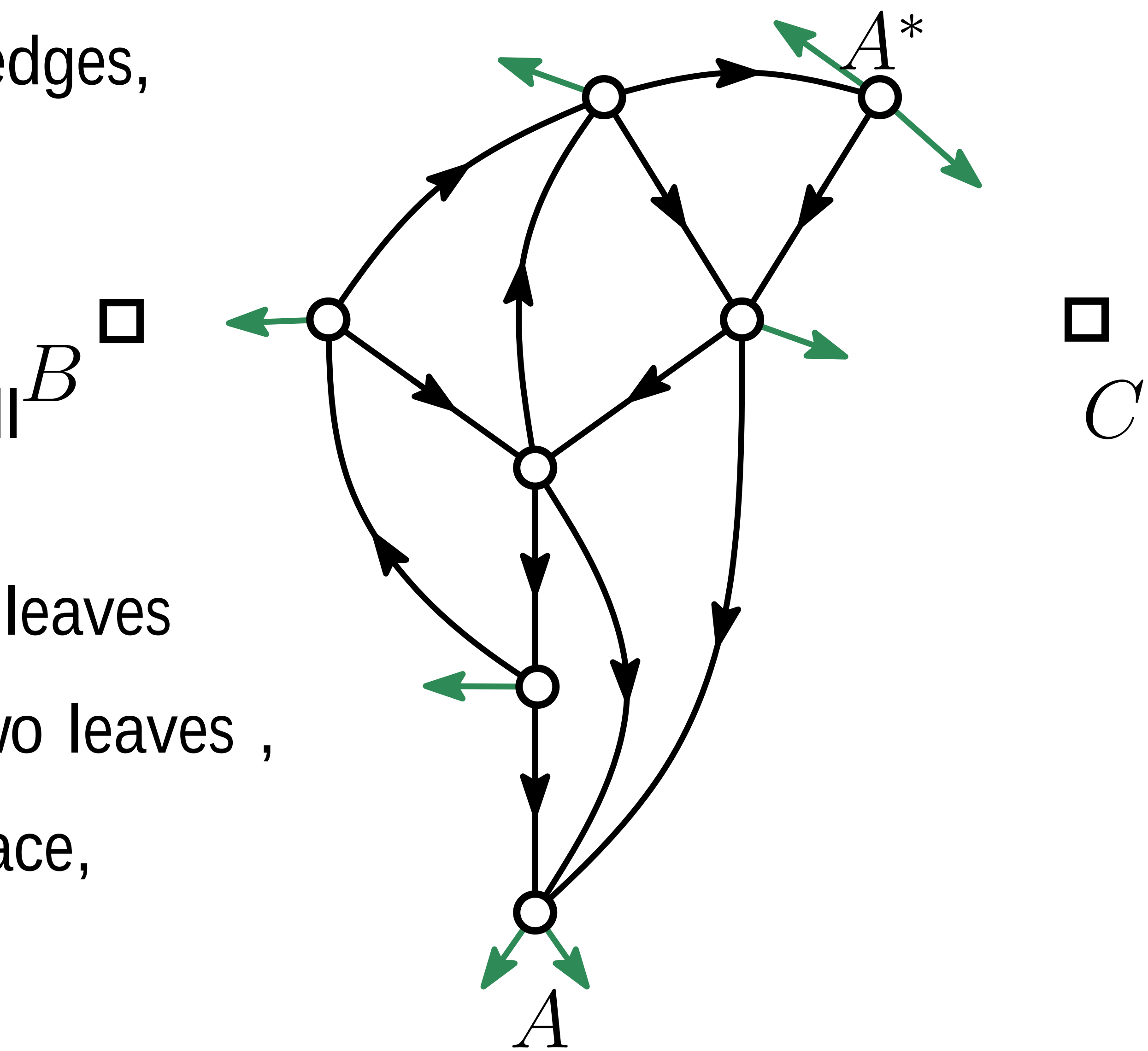

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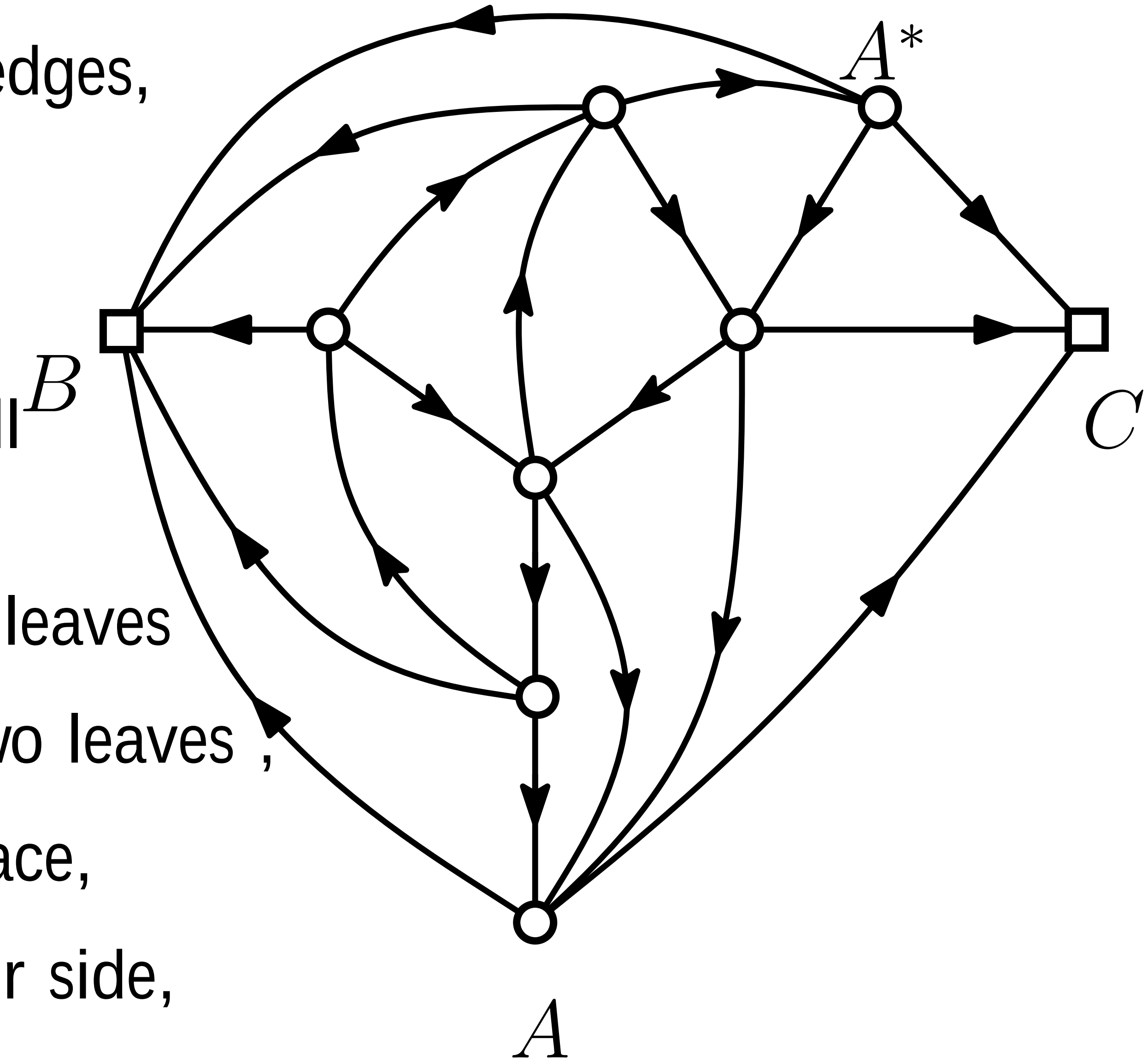

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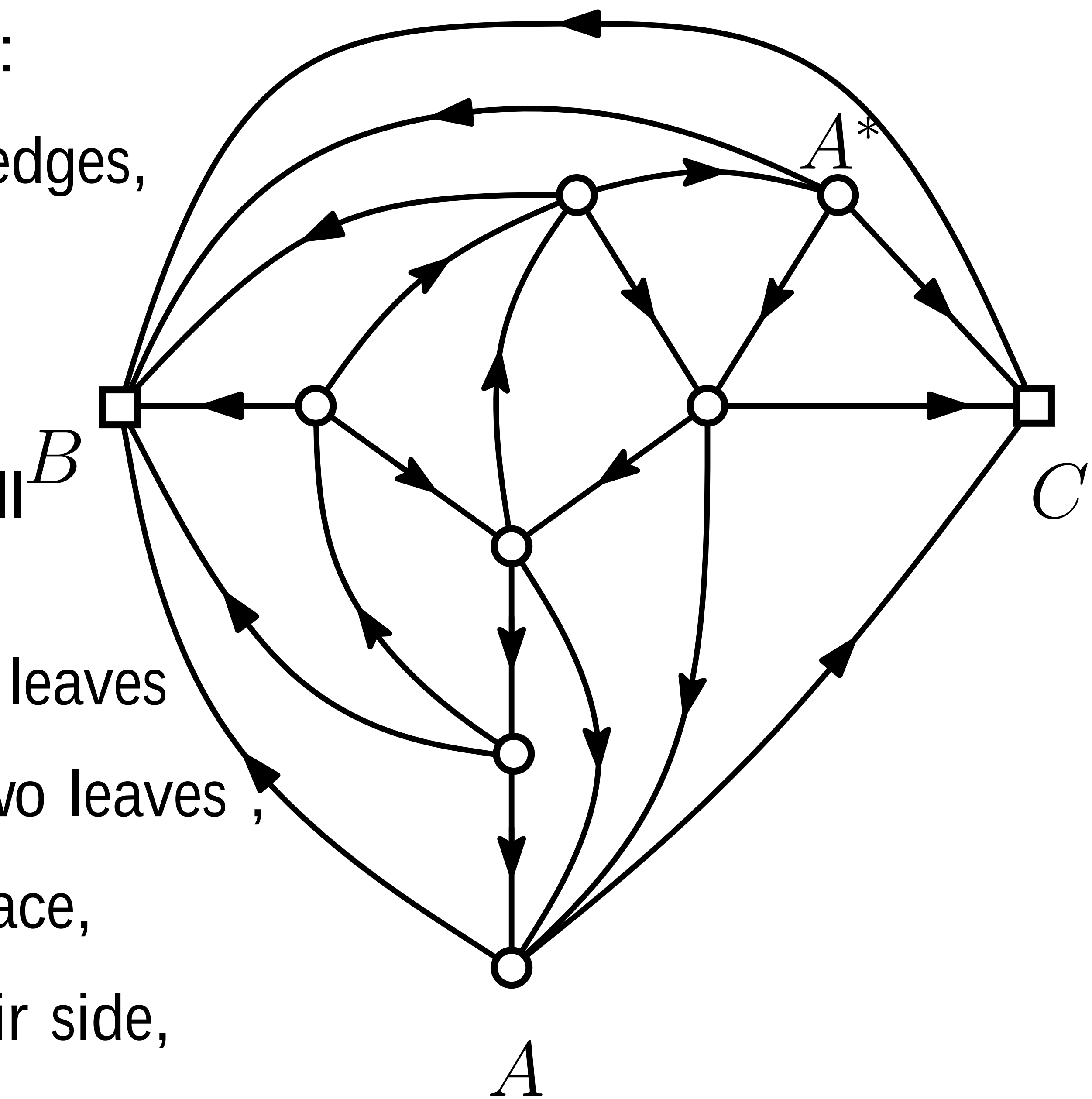
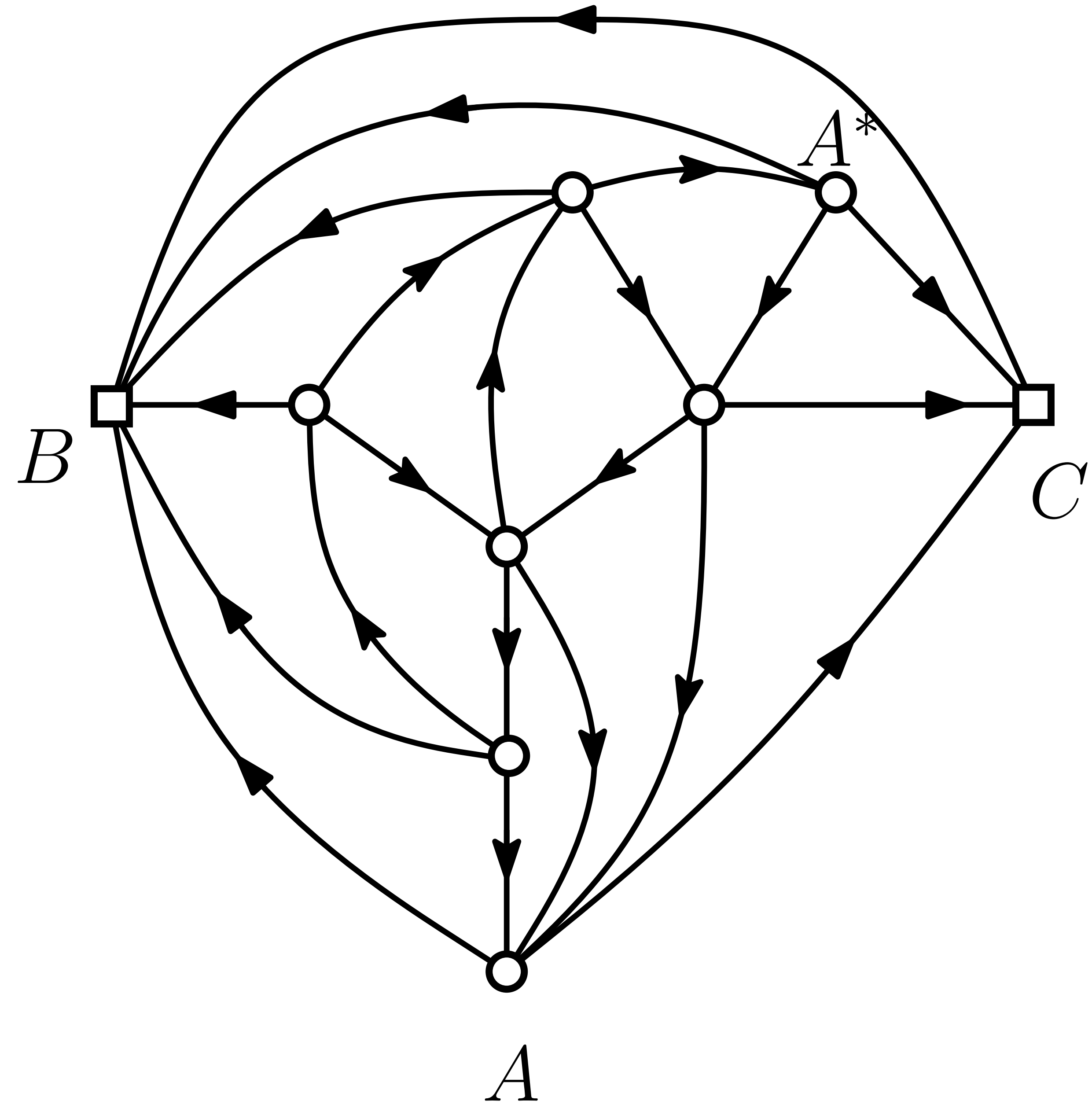
- Connect B and C.



From blossoming trees to simple triangulations

Simple triangulation endowed with its unique orientation such that :

- out$(v) = 3$ for v an inner vertex
- out$(A) = 2$, out$(B) = 1$ and out$(C) = 0$
- no counterclockwise cycle



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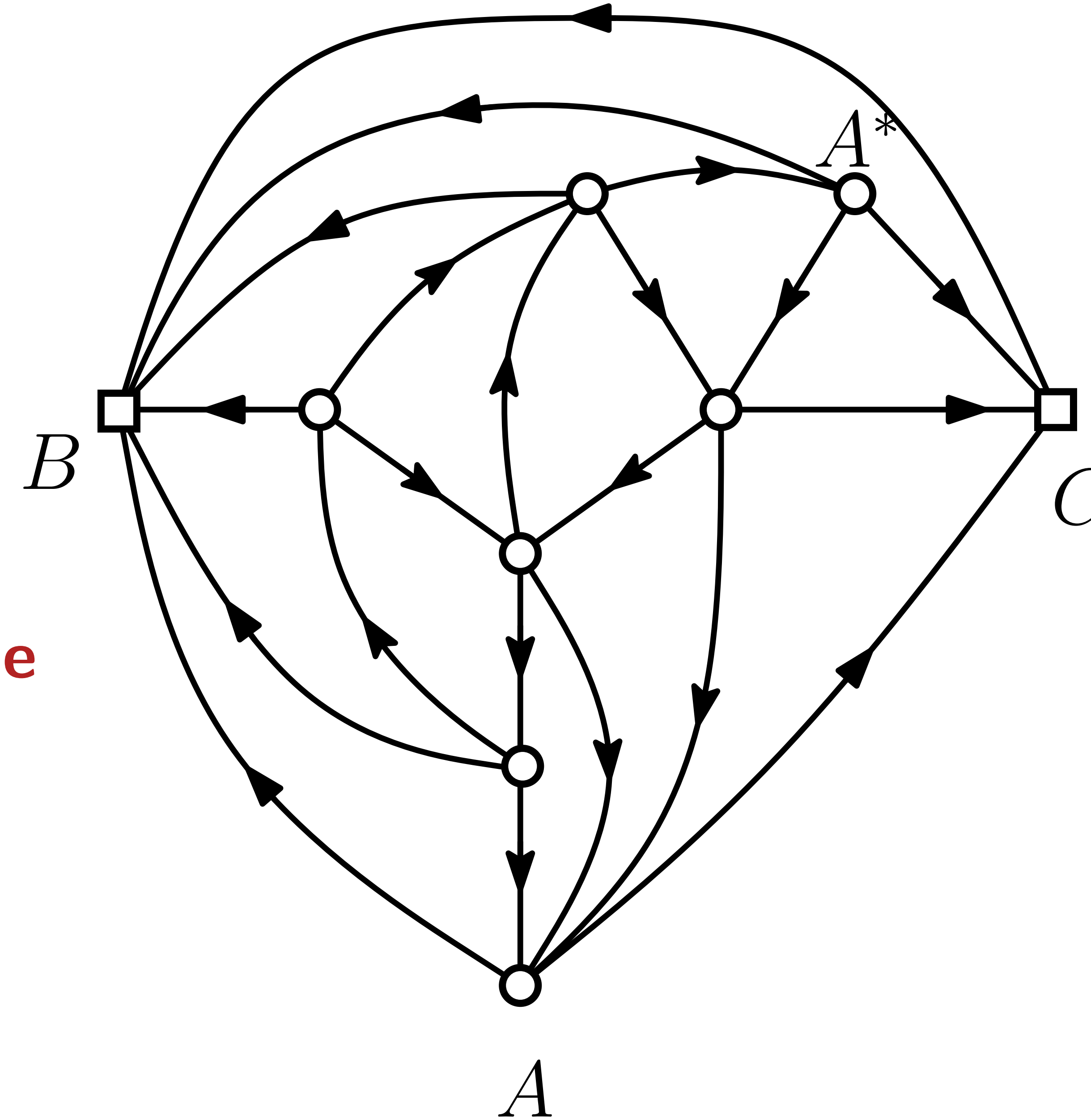
The orientations characterize simple triangulations [Schnyder]



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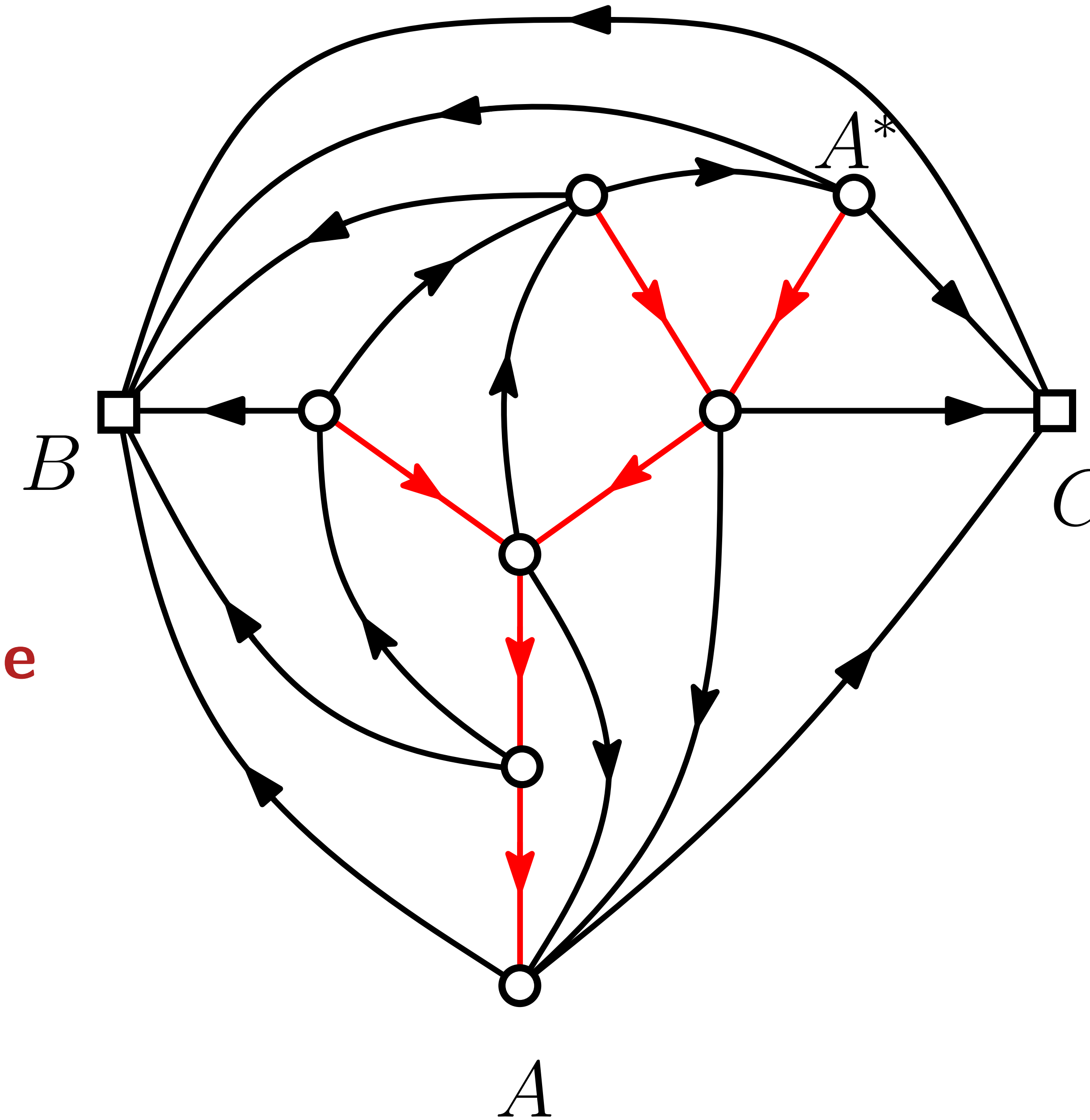
Given the orientation the blossoming tree is the leftmost spanning tree of the map (after removing B and C).



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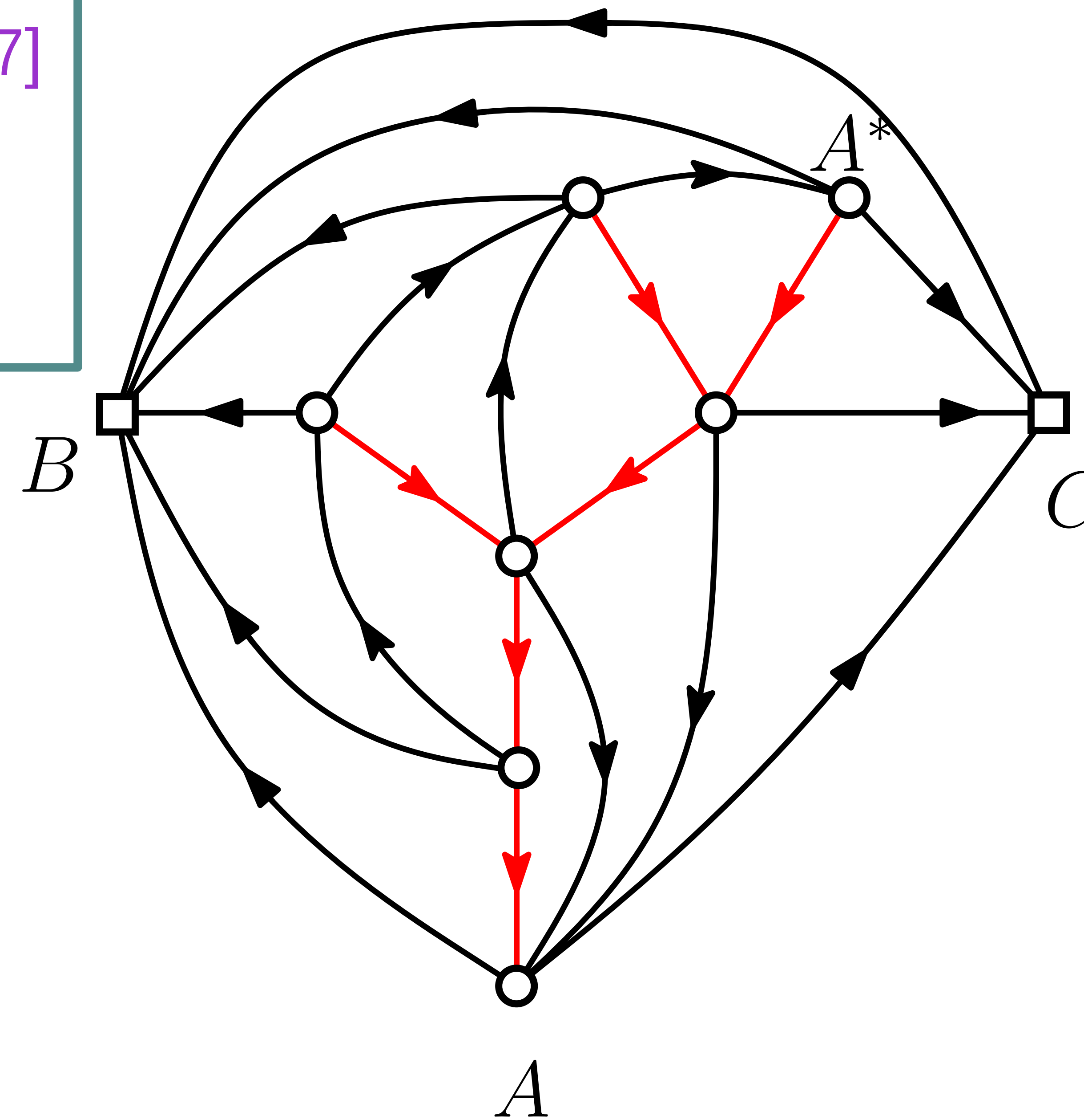
Proposition: [Poulalhon, Schae er '07]

The closure operation is a bijection between balanced 2-blossoming trees and simple triangulations.



Same bijection with corner labels

- Start with a planted 2-blossoming tree.
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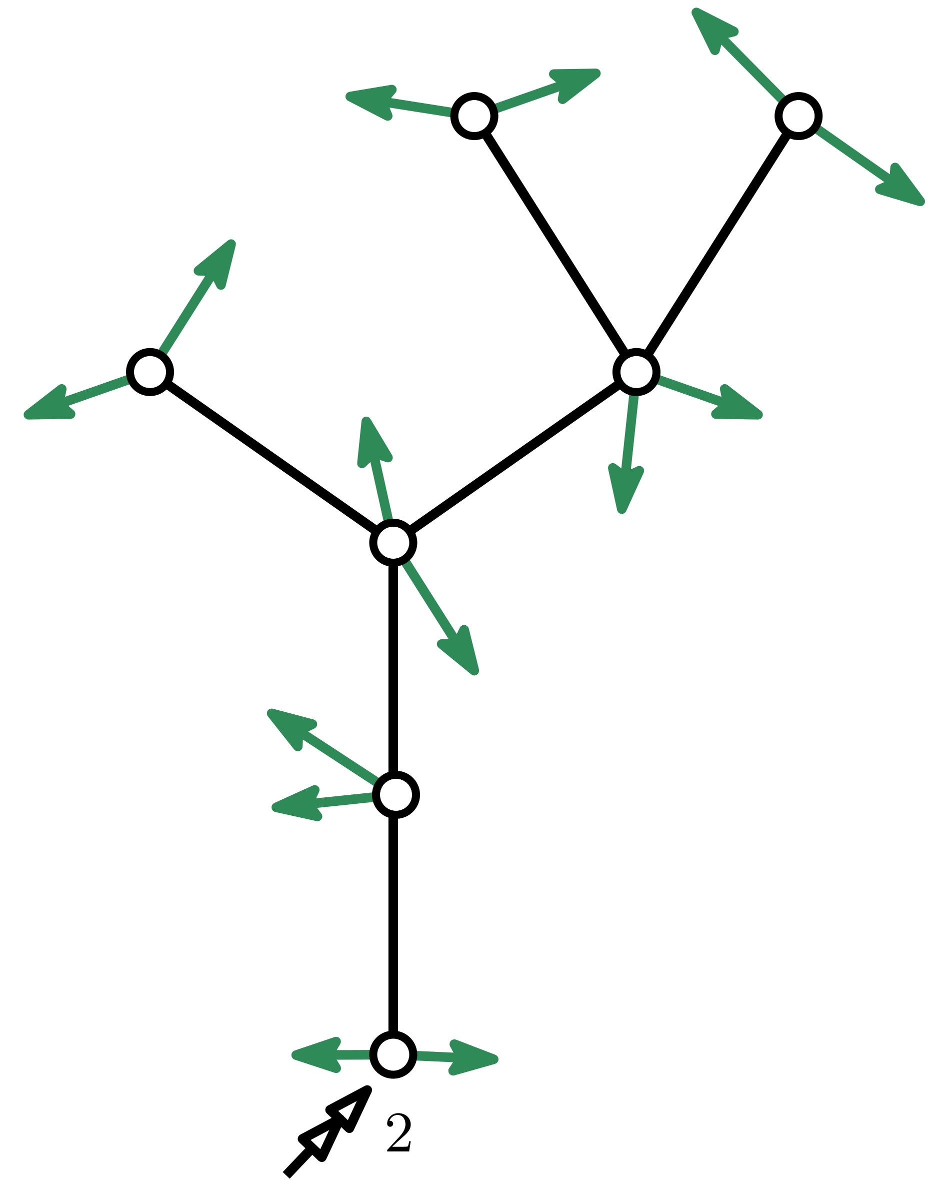

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In contour order, apply the following rules:

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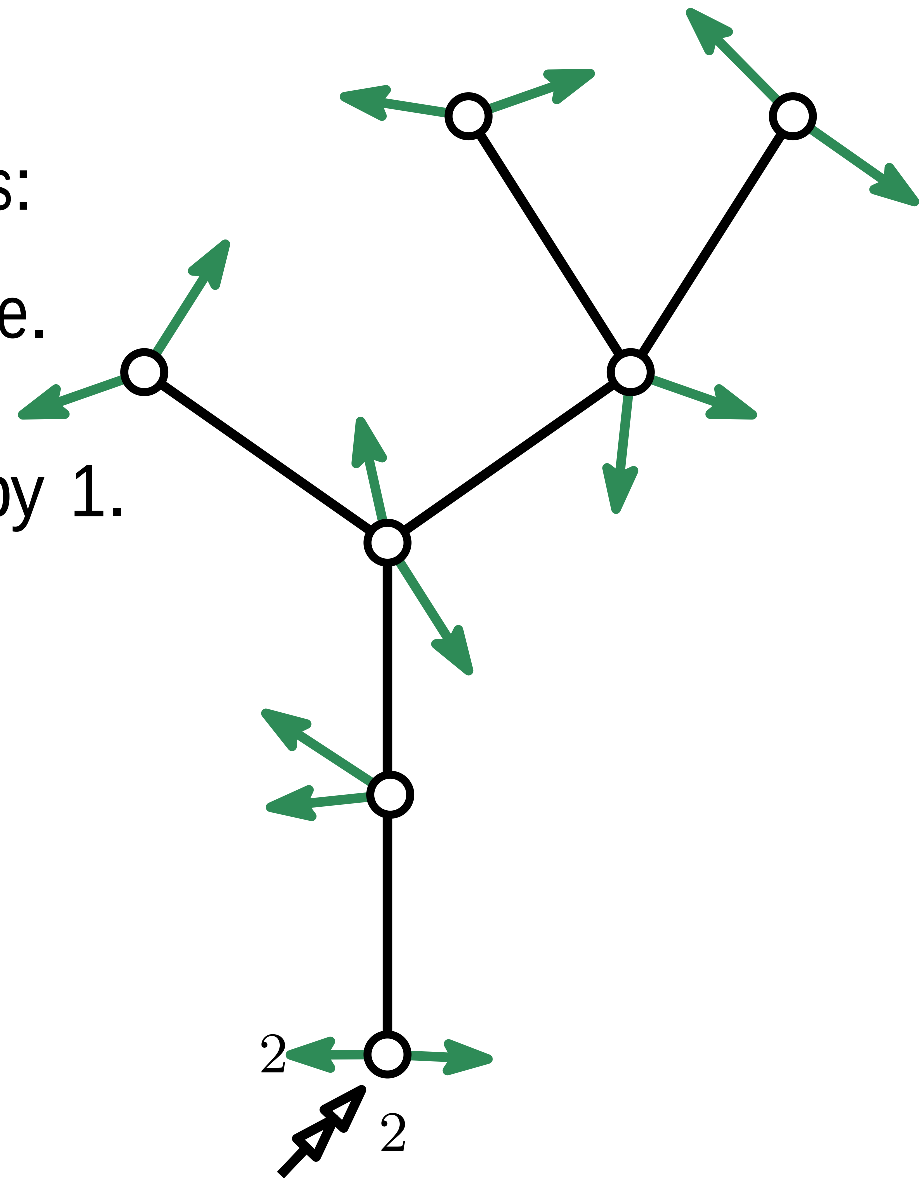

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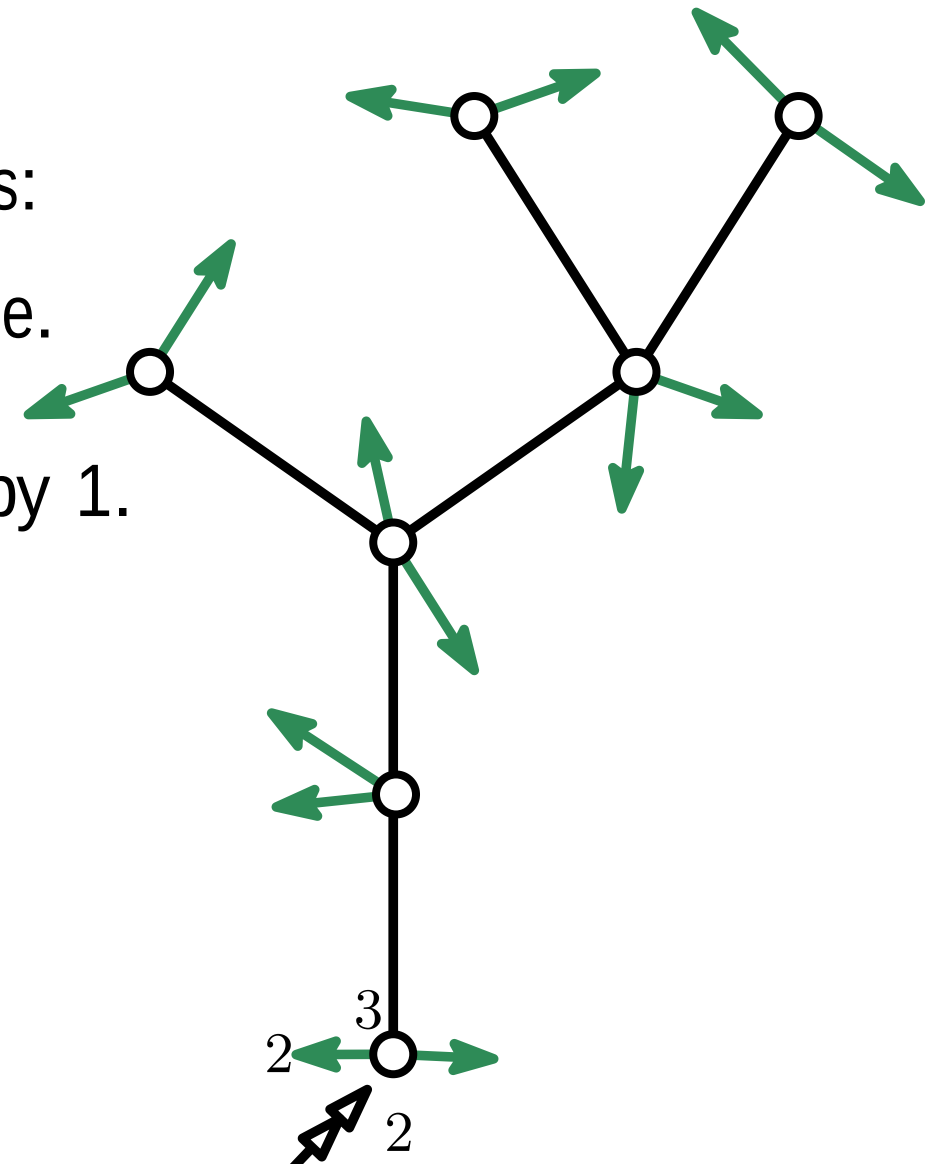

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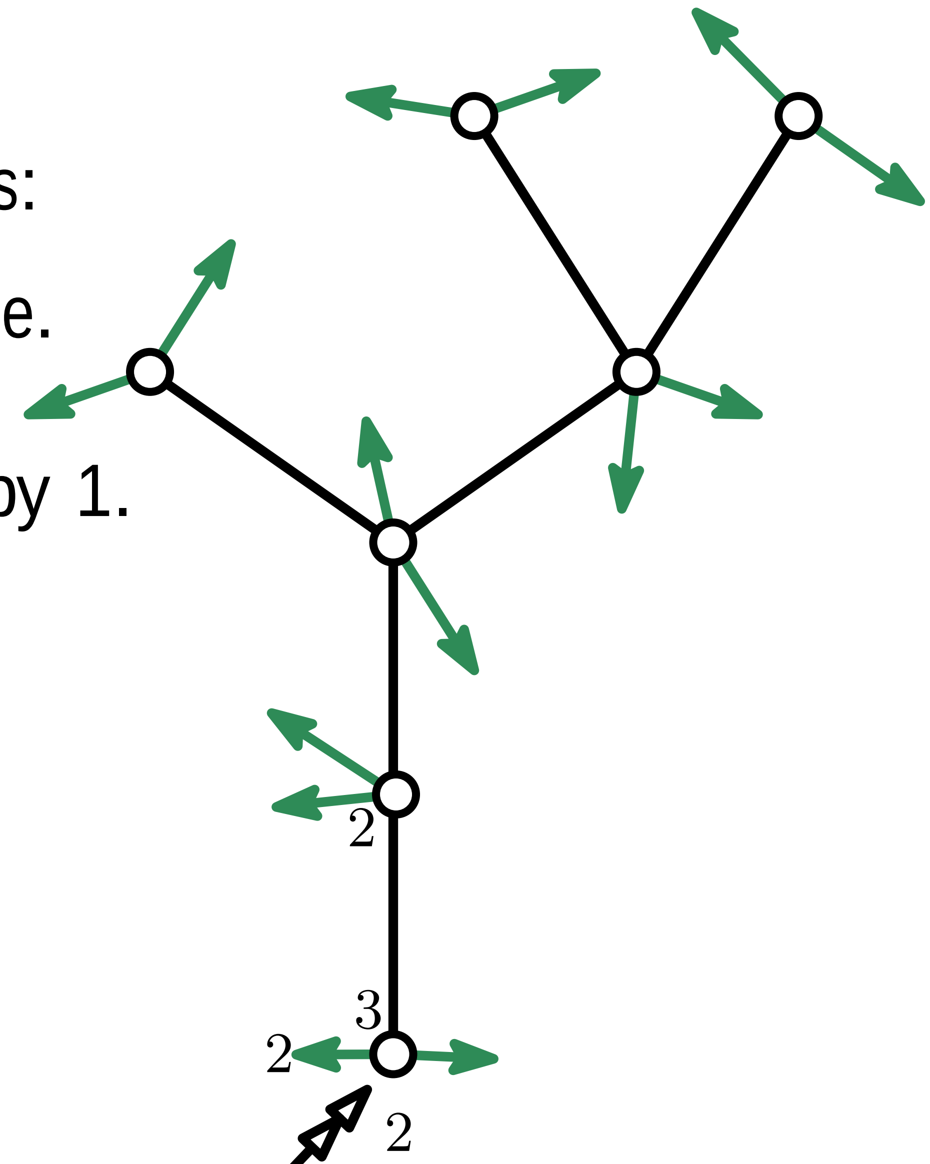

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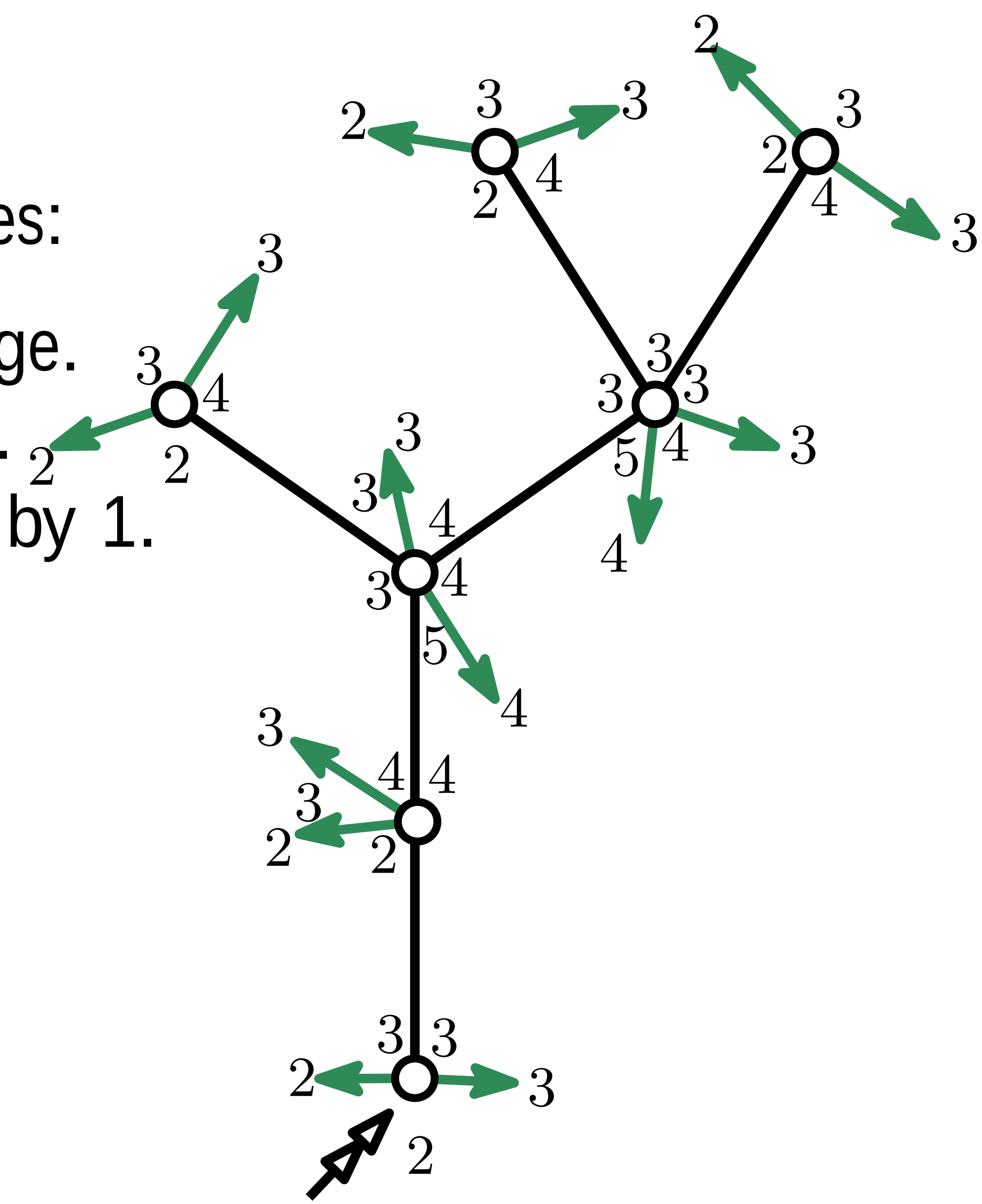

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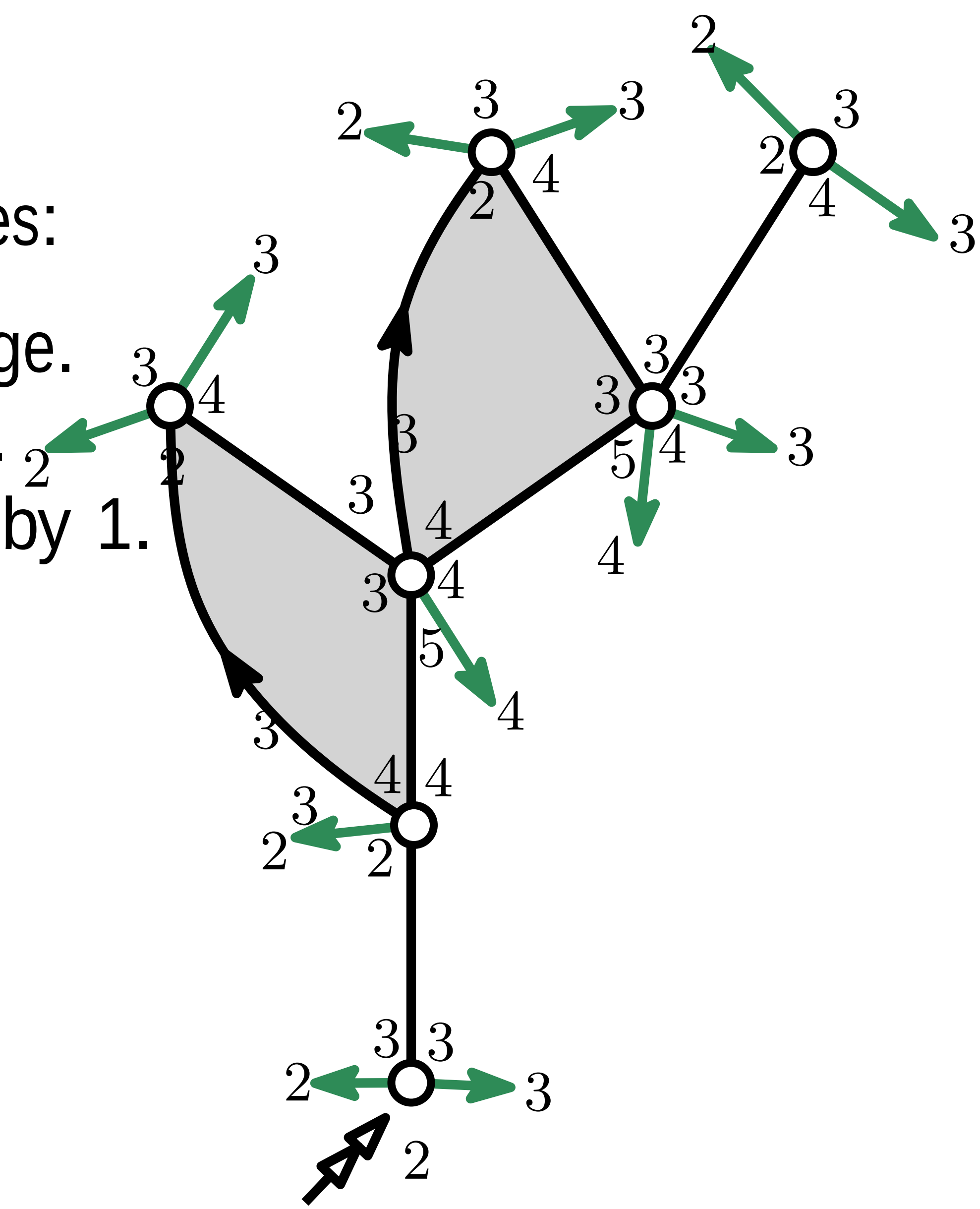
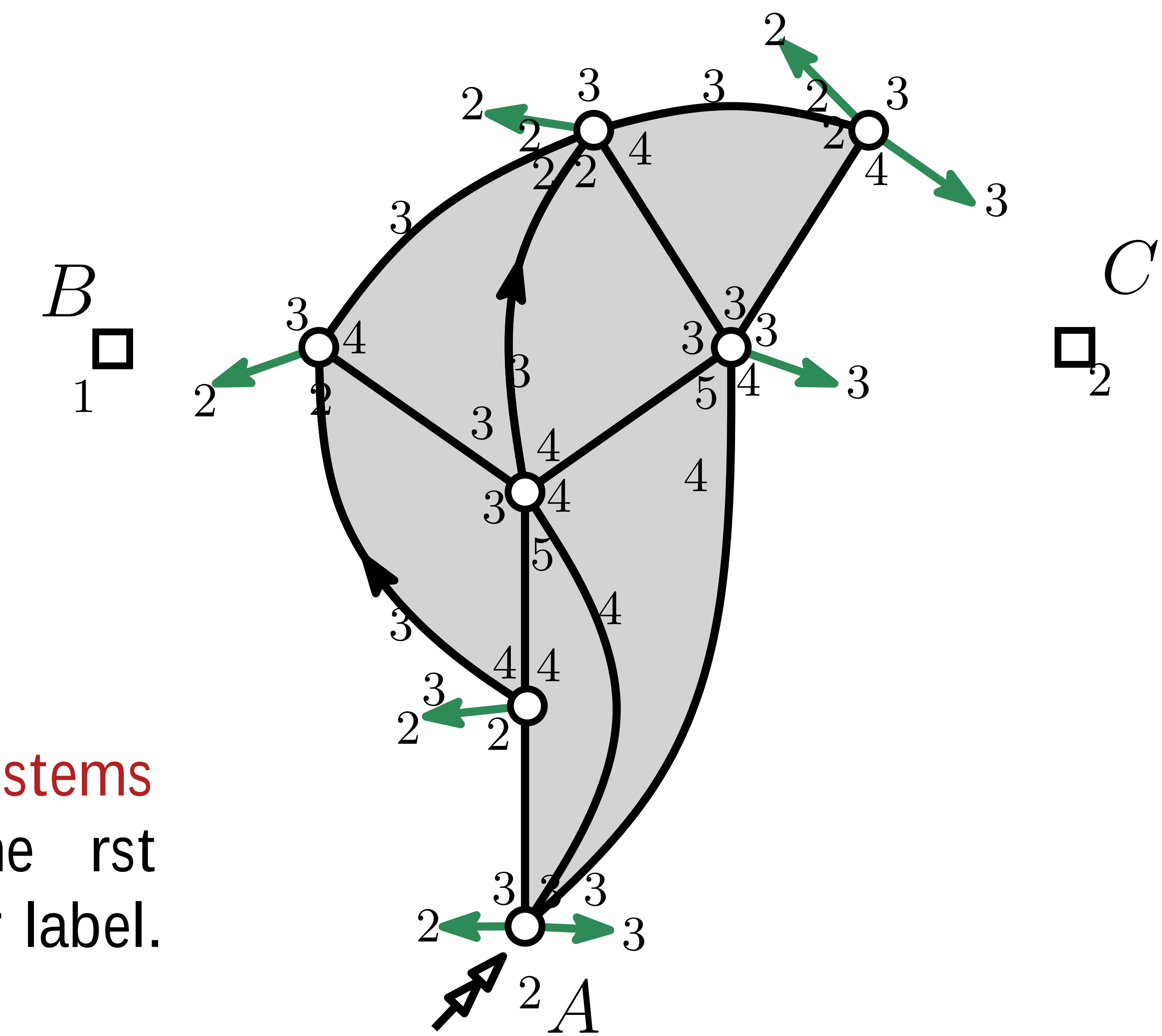


Aside: Tree is balanced $\Leftrightarrow$
all labels ≥ 2
+root corner incident to two stems

Closure: Merge each leaf with the rst
subsequent corner with a smaller label.

Same bijection with corner labels



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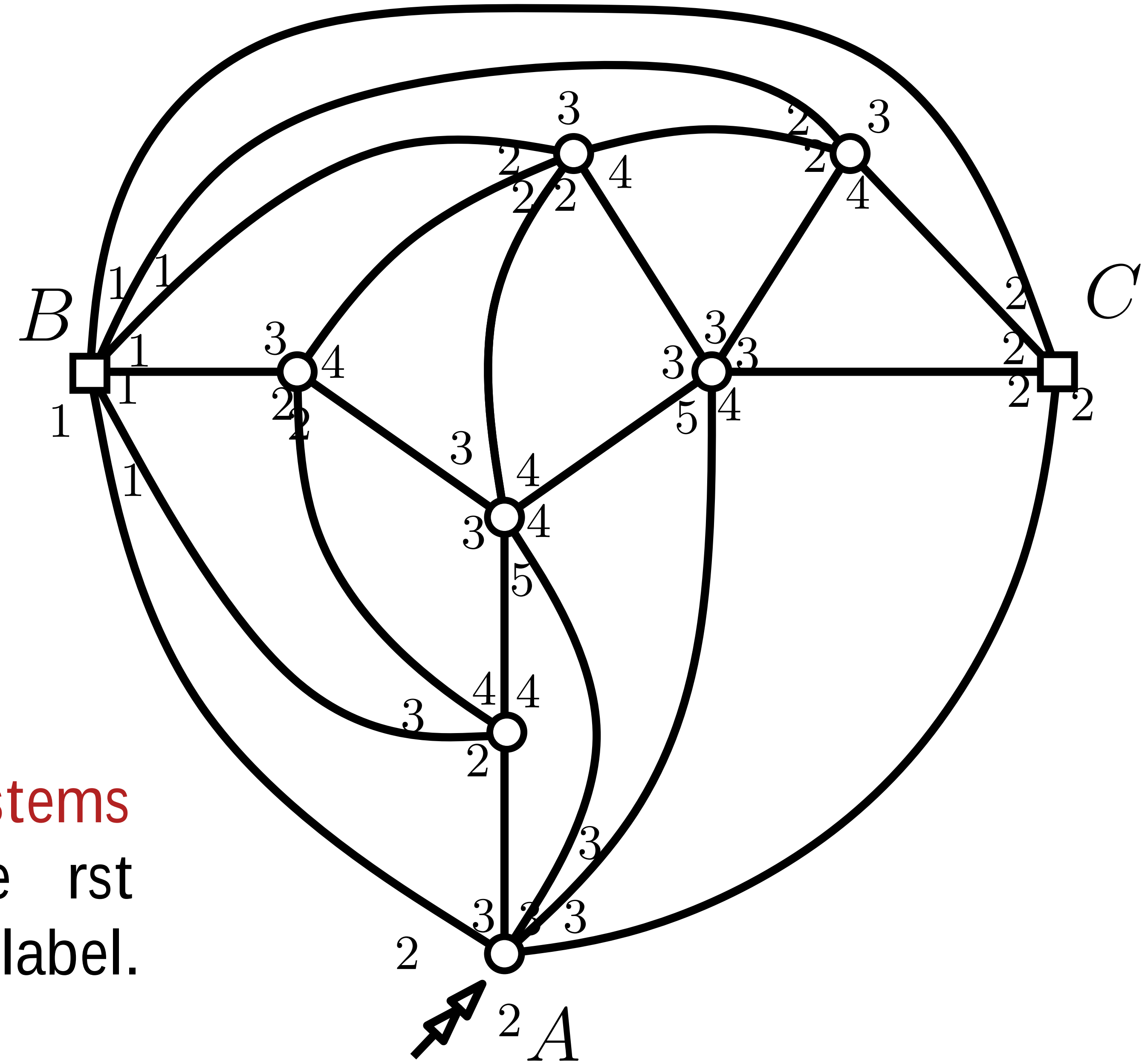

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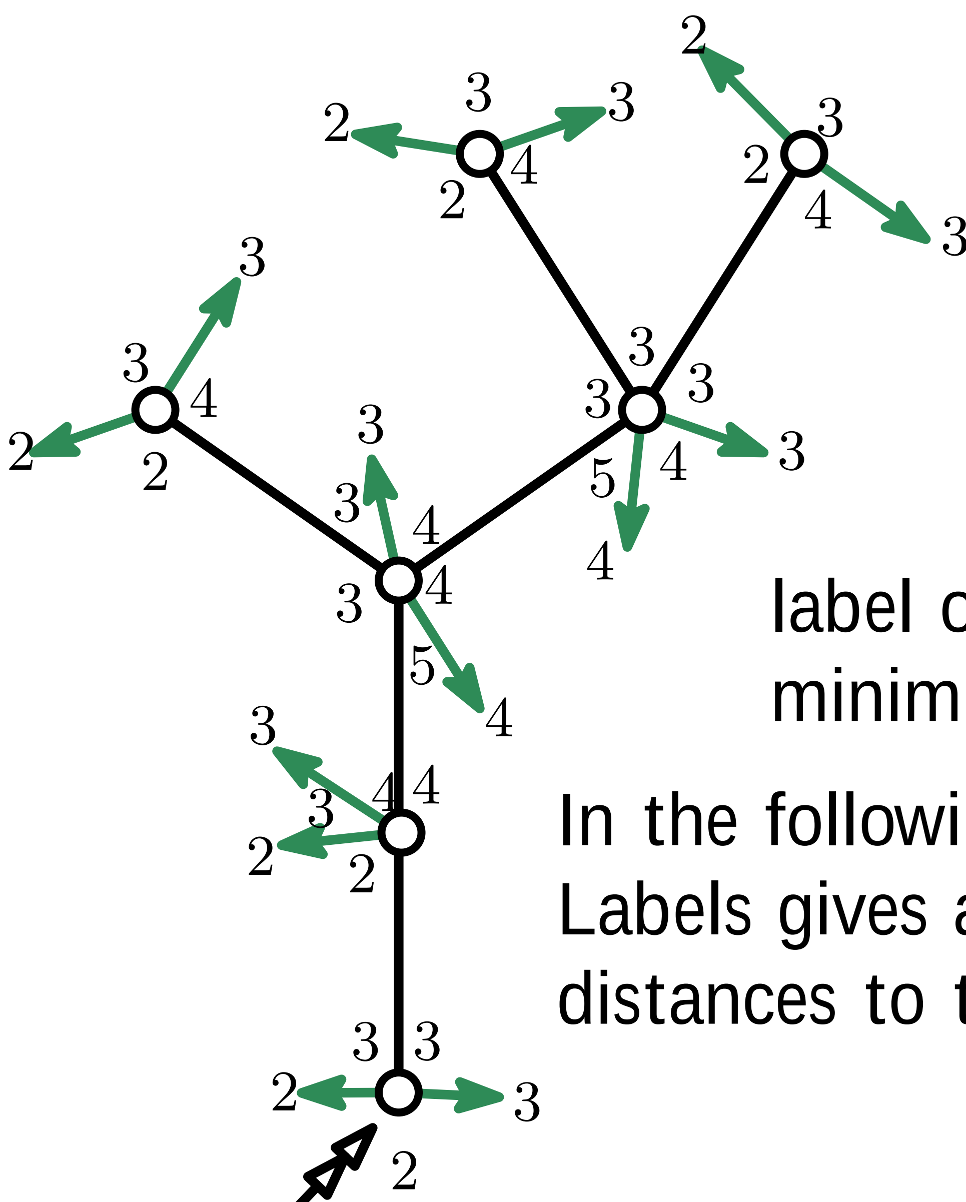
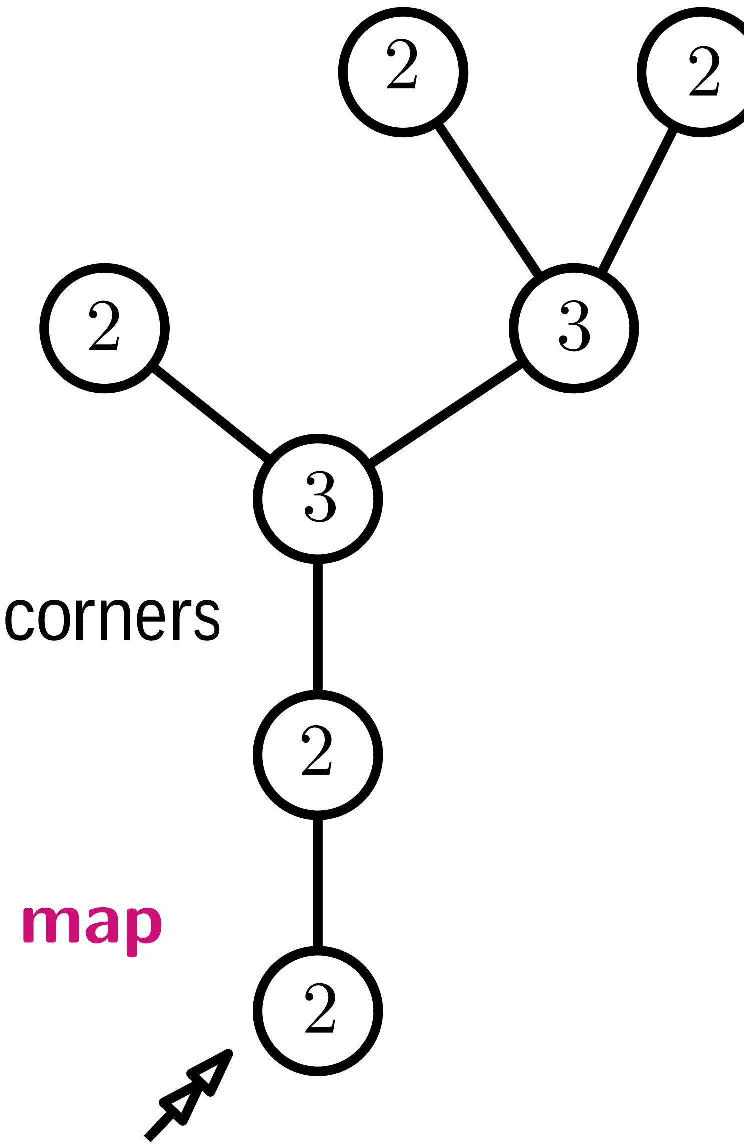
From blossoming trees to labeled trees



label of a vertex =
minimum label of its corners

In the following:
Labels gives approximate
distances to the root **in the map**



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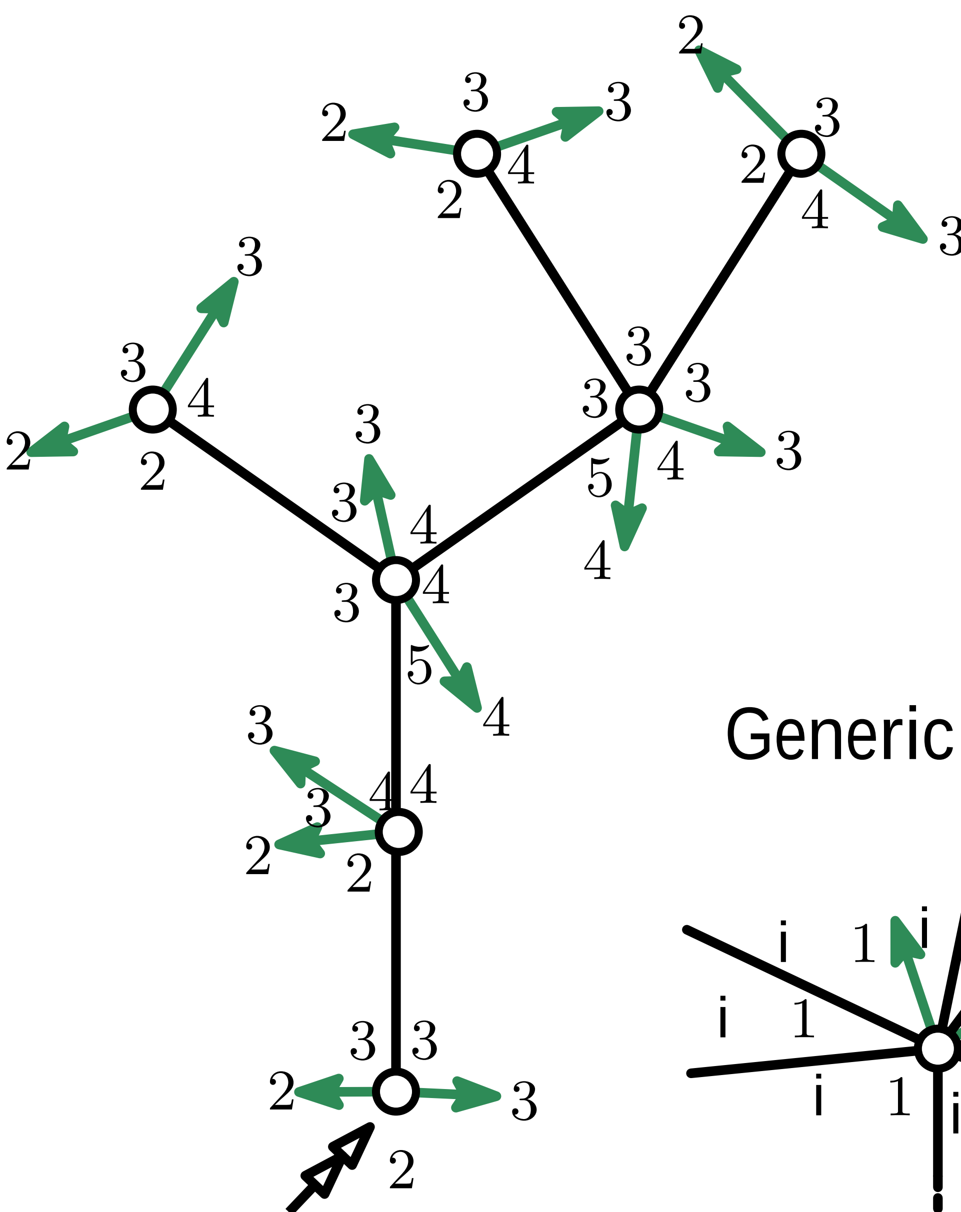
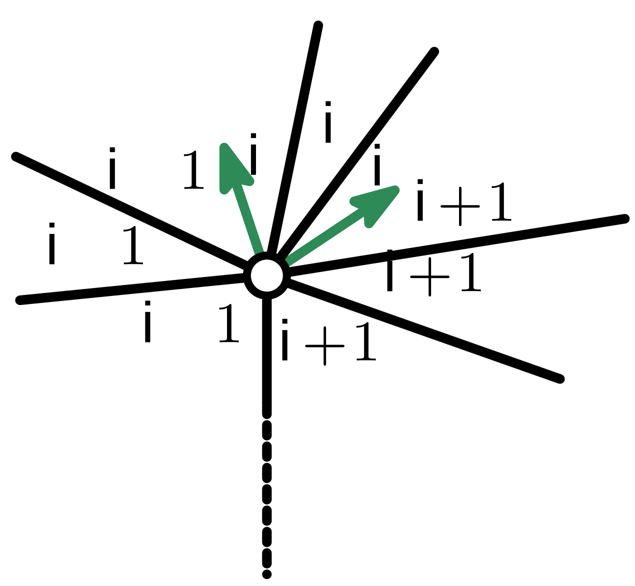
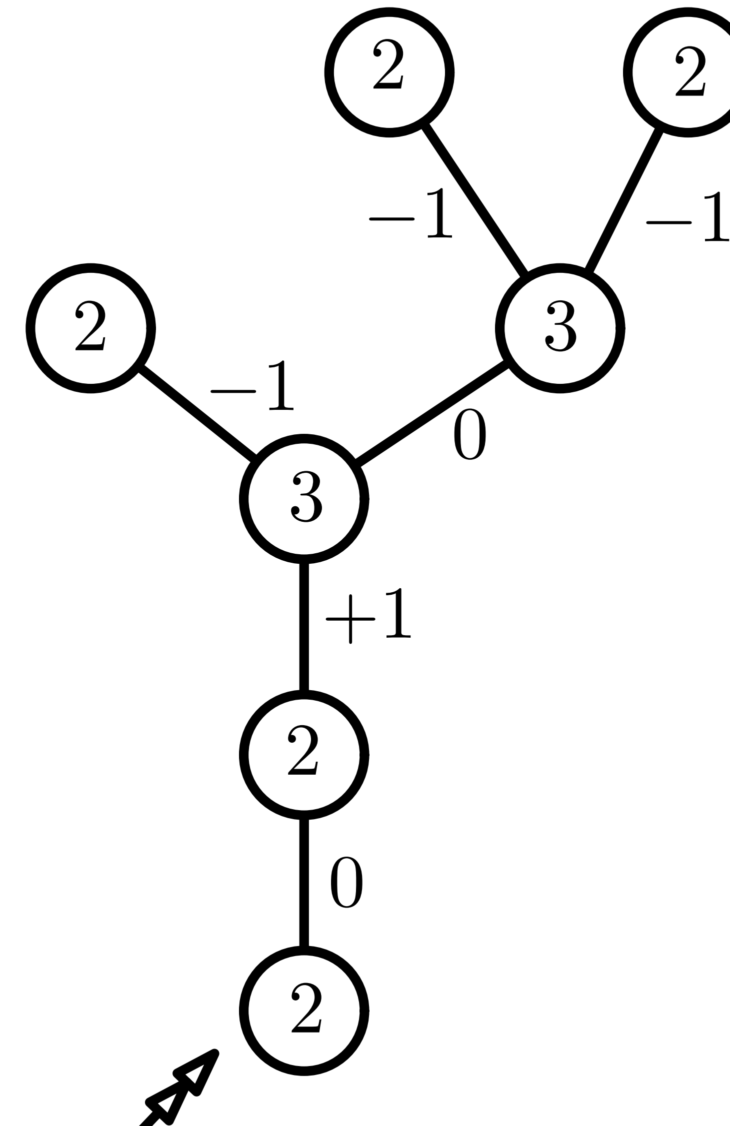


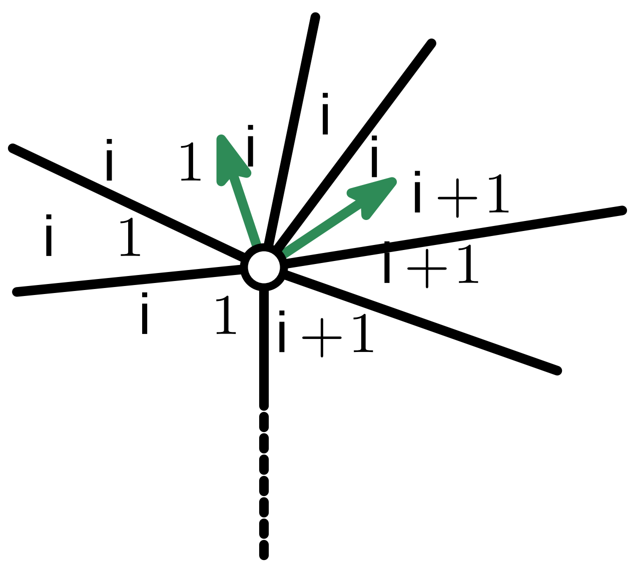
Generic vertex :

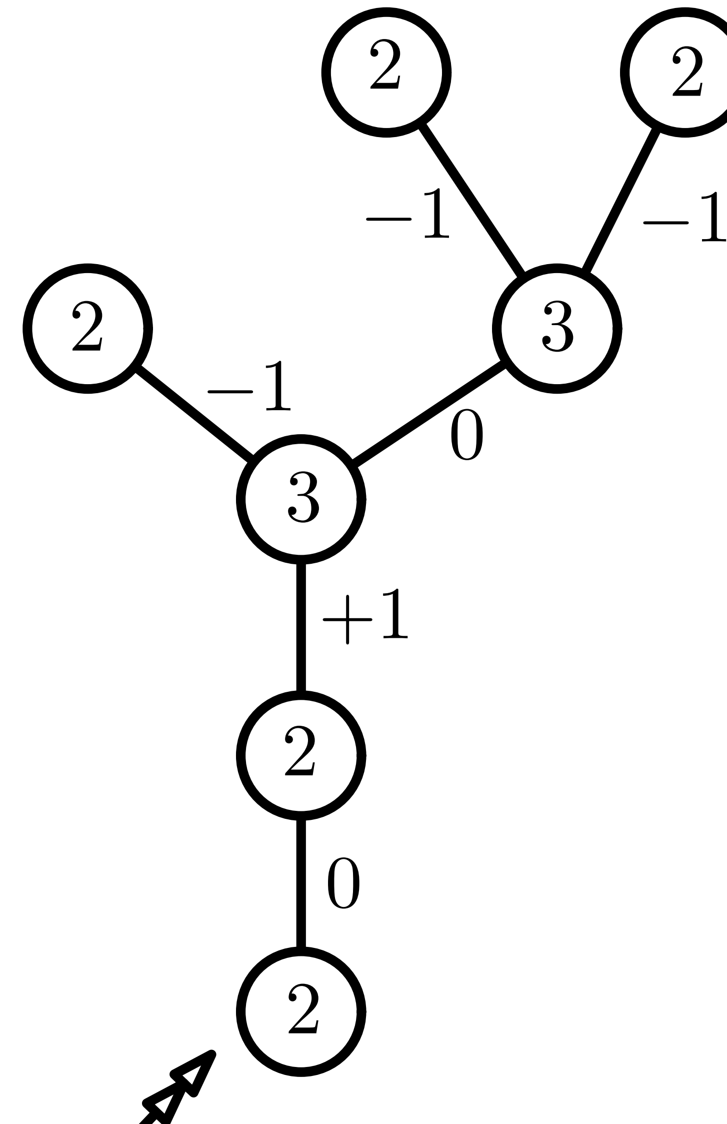




From blossoming trees to labeled trees

Generic vertex :





- Can retrieve the blossoming tree from the labeled tree.
- Labeled tree = GW trees + random displacements on edges uniform on

$$\{(-1; -1; :::; -1; 0; 0; :::; 0; 1; 1 :::; 1)\}.$$

almost the setting of [Janson-Marckert] and [Marckert-Miermont] but r.v are not "locally centered" ⇒ symmetrization required

Convergence of labeled trees

Theorem : [Addario-Berry, A.]
For a sequence of simple random triangulations (M_n), the contour and label process of the associated labeled tree satis e:

$$\left((3n)^{-1/2}C_{\lfloor nt\rfloor}; (4n=3)^{-1/4}\tilde{Z}_{\lfloor nt\rfloor}\right)_{0\leq t\leq 1} \xrightarrow[n\to\infty]{(d)} (e_t; Z_t)_{0\leq t\leq 1};$$

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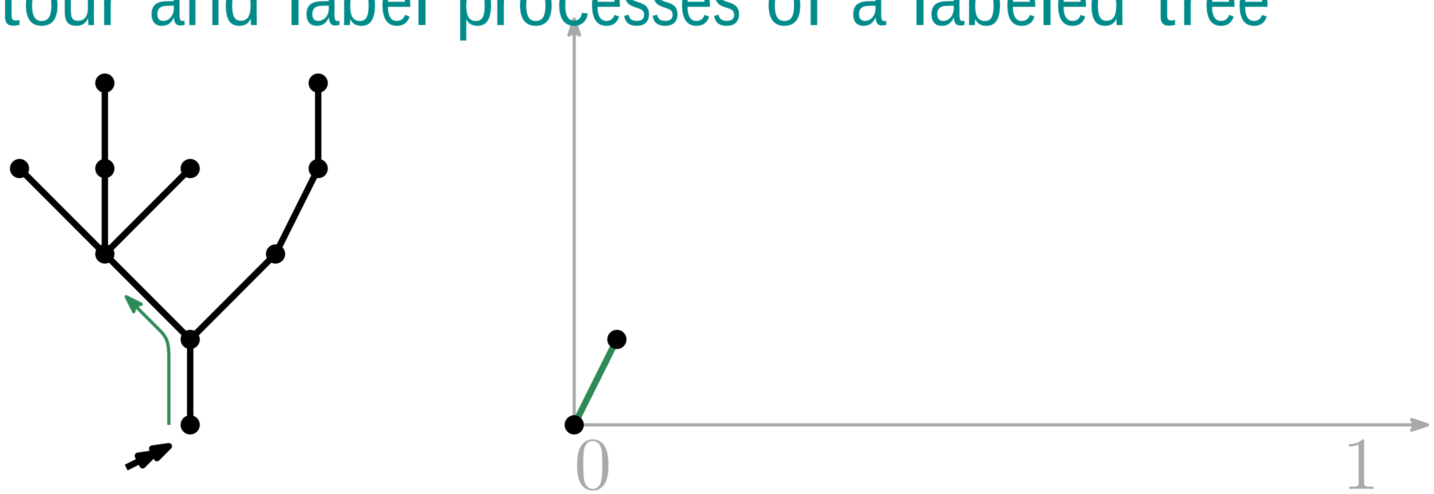

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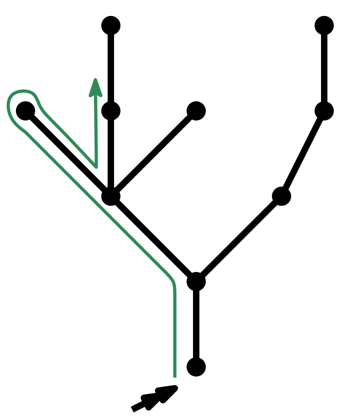

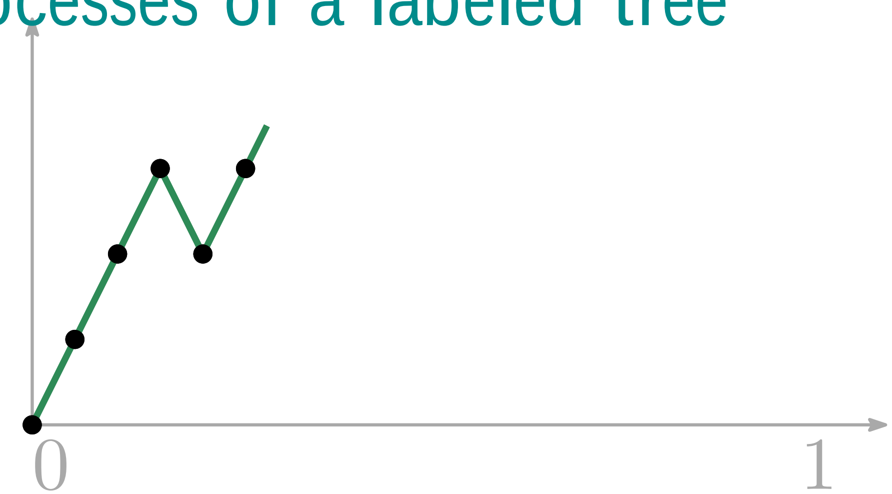

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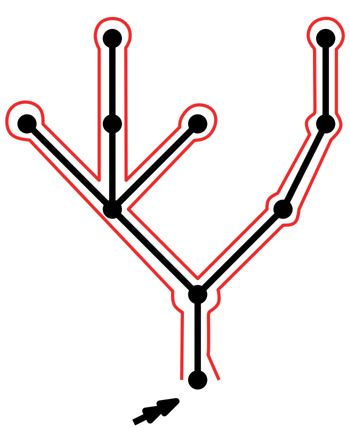

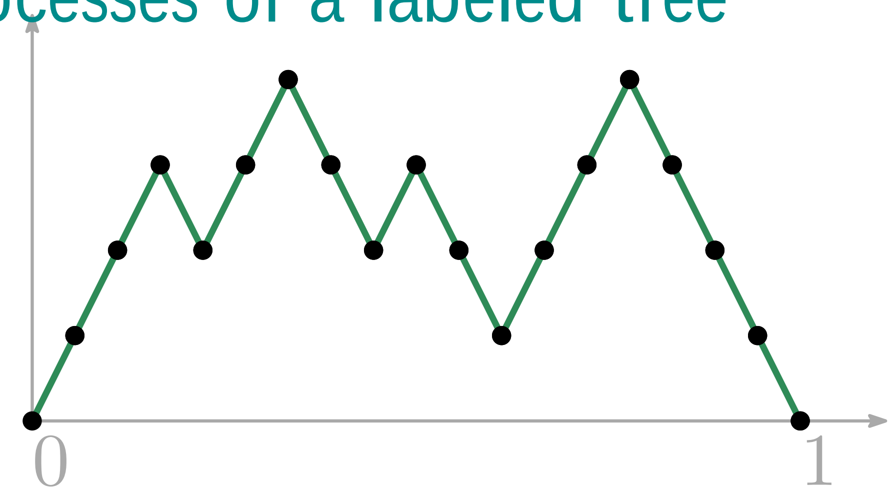

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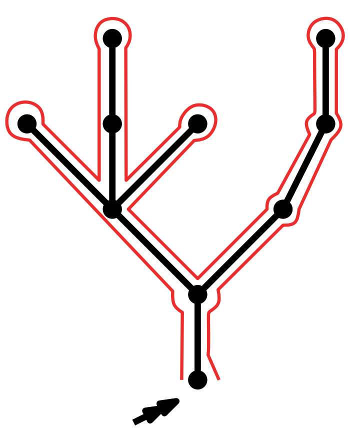

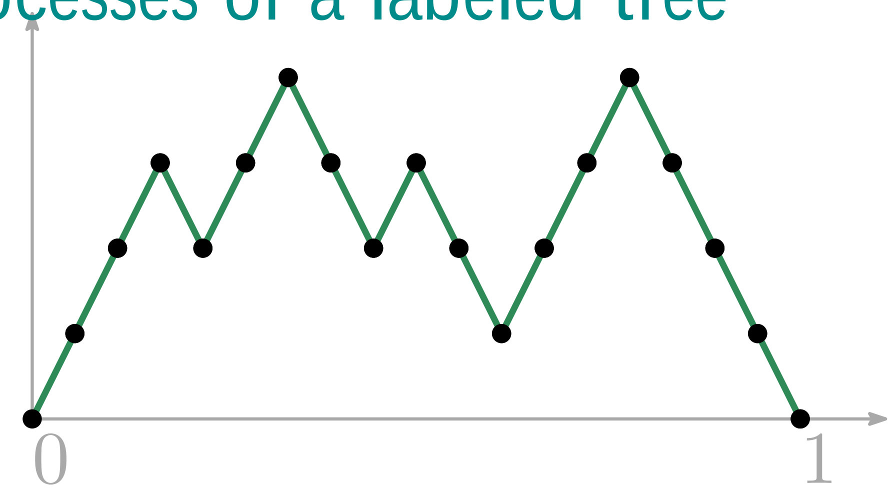

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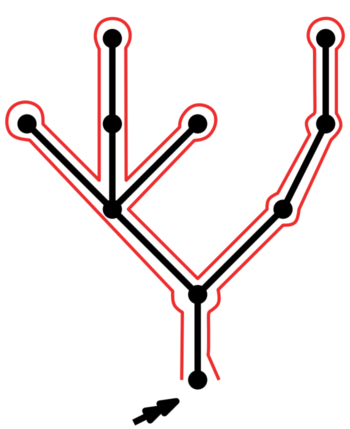

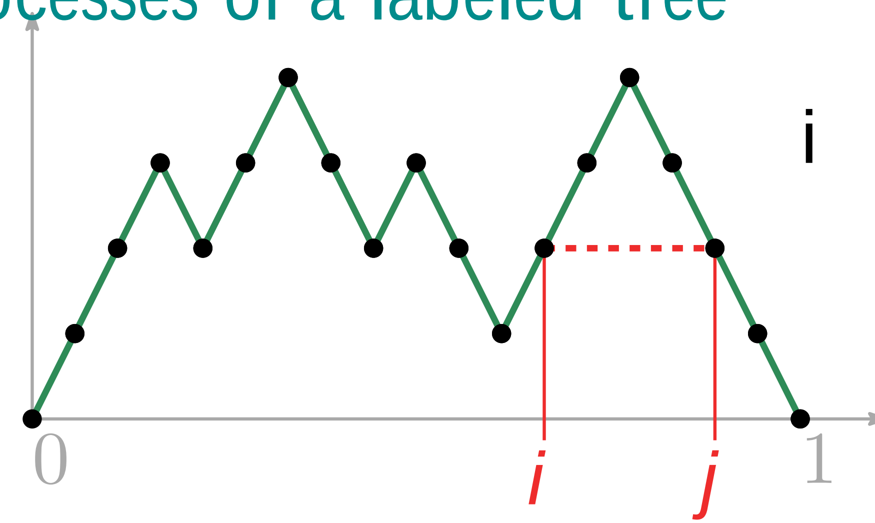

i and j = same vertex of T

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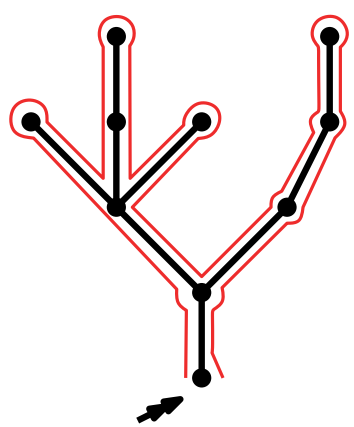

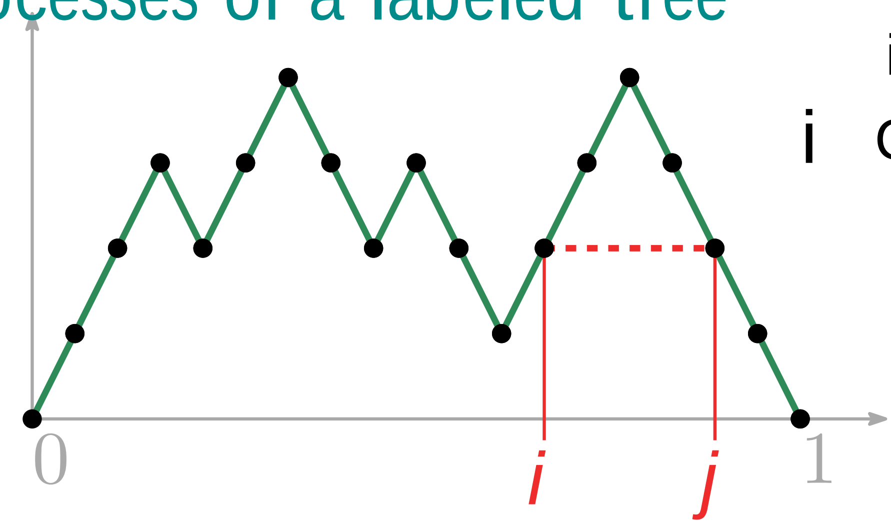

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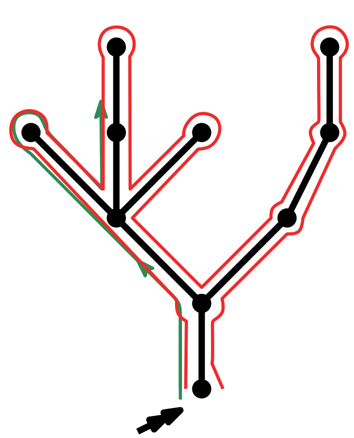

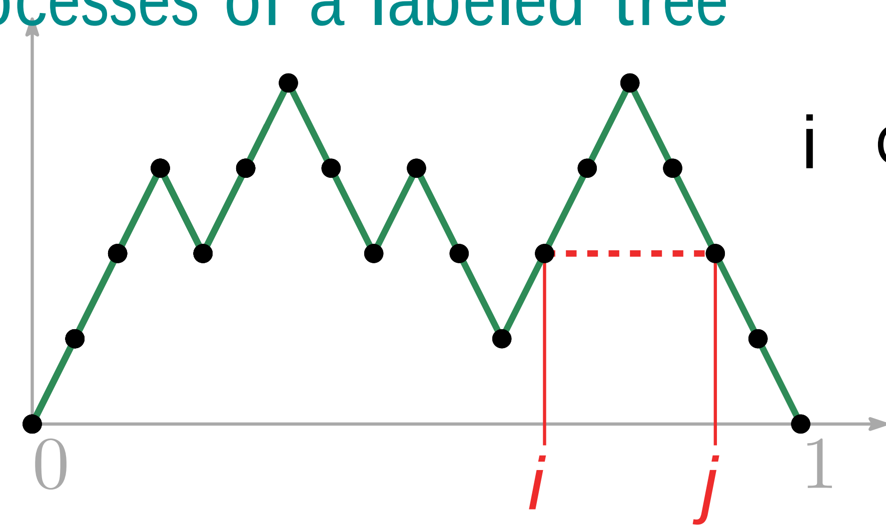

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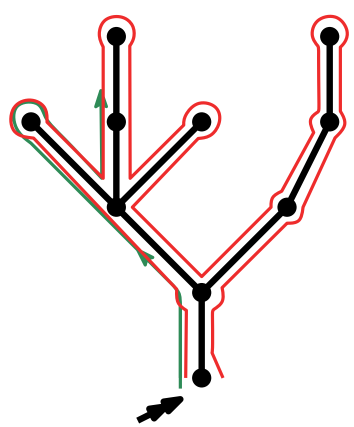
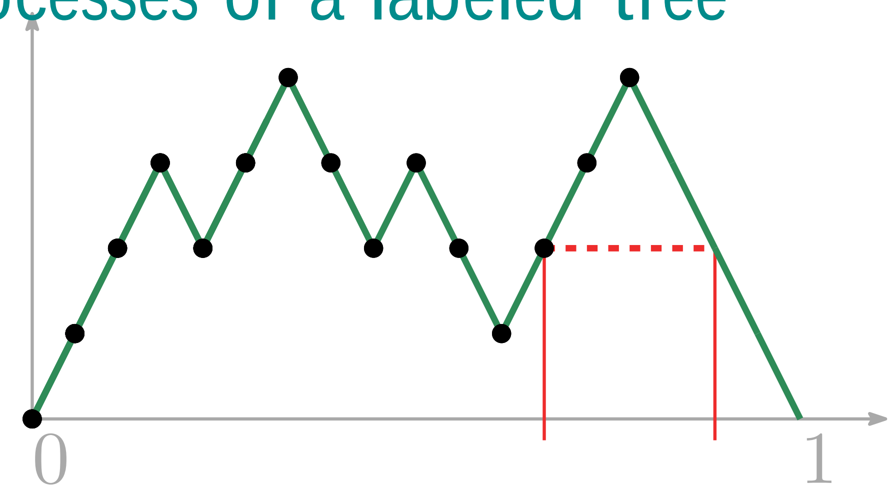
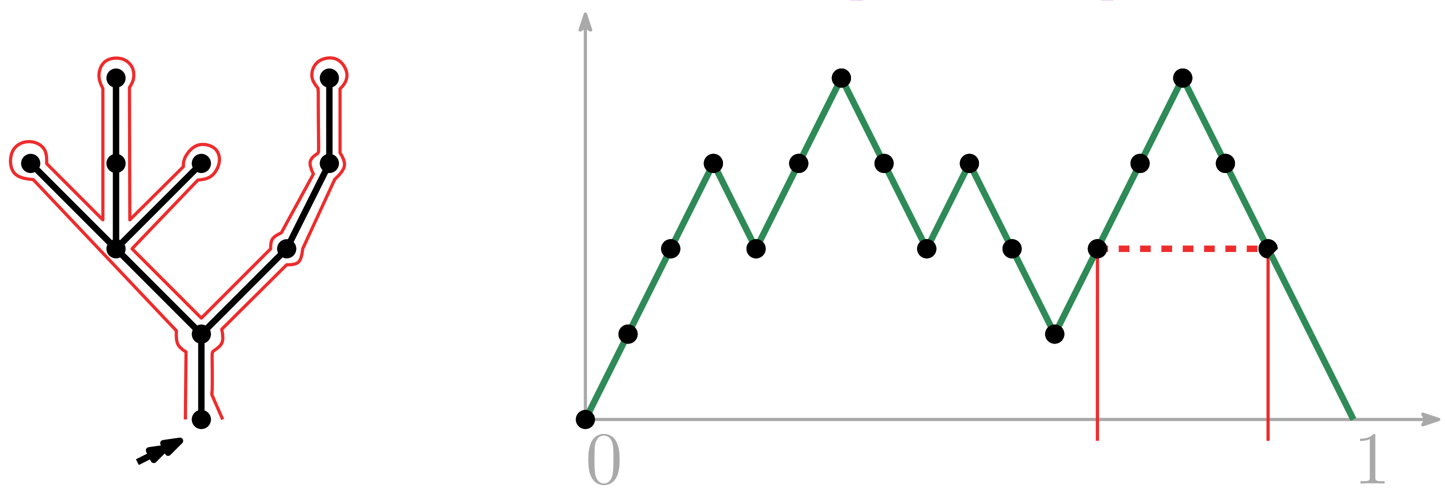




Brownian snake $(e_t, Z_t)_{0 \leq t \leq 1}$

1st step : the Brownian tree [Aldous]



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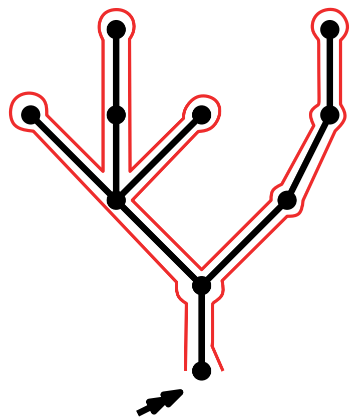

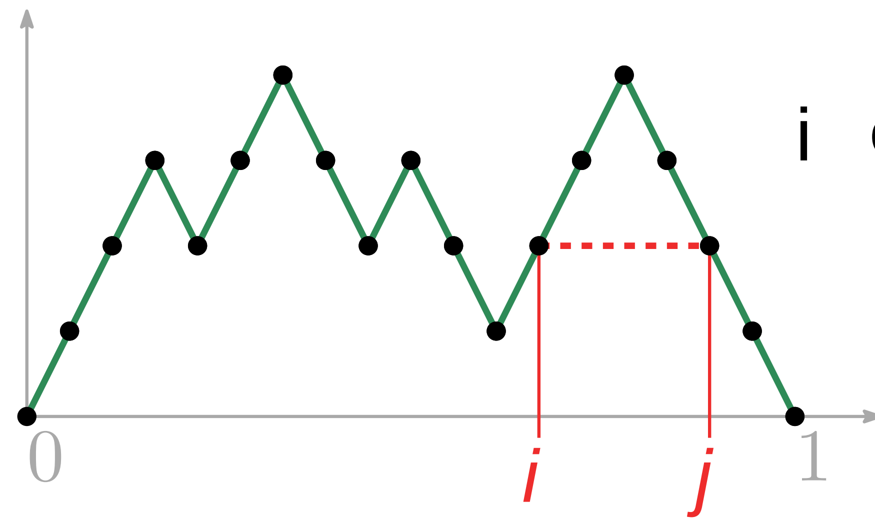

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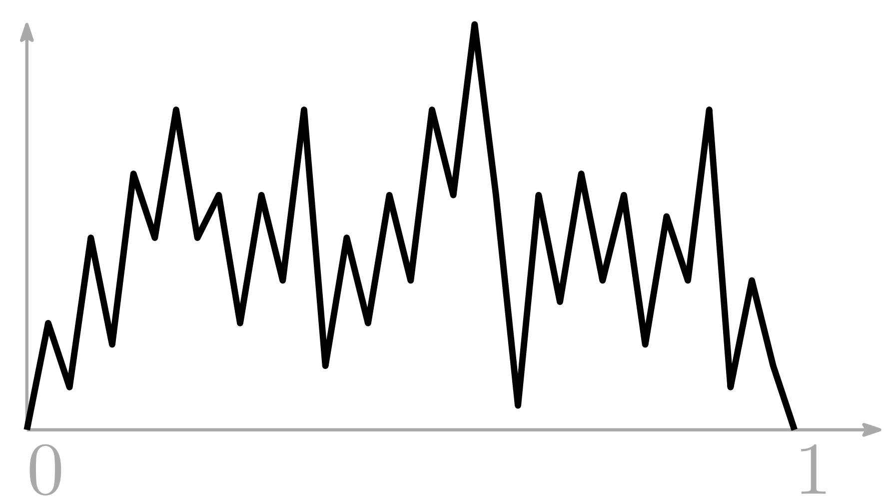

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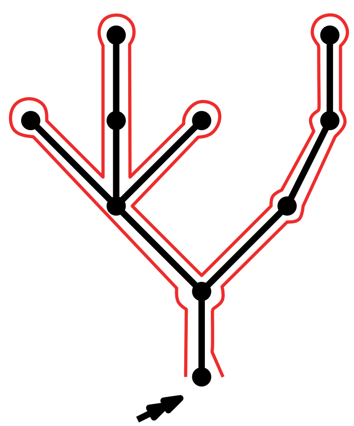

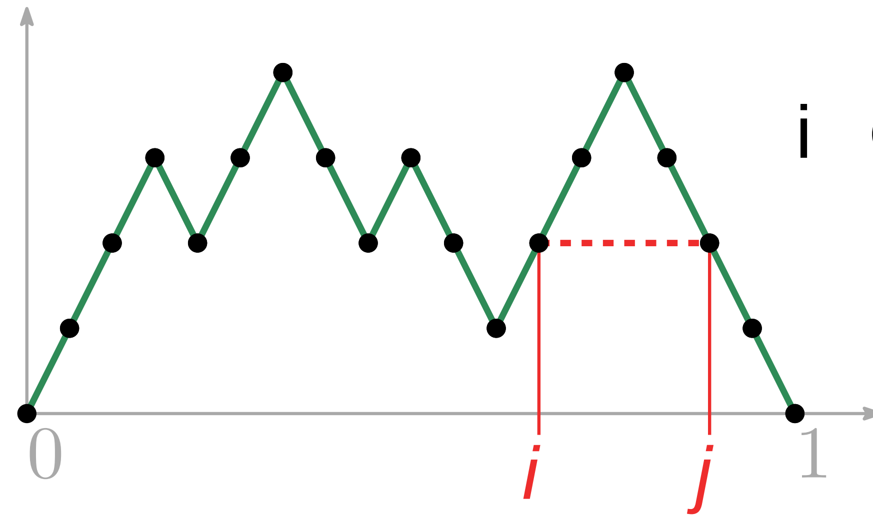

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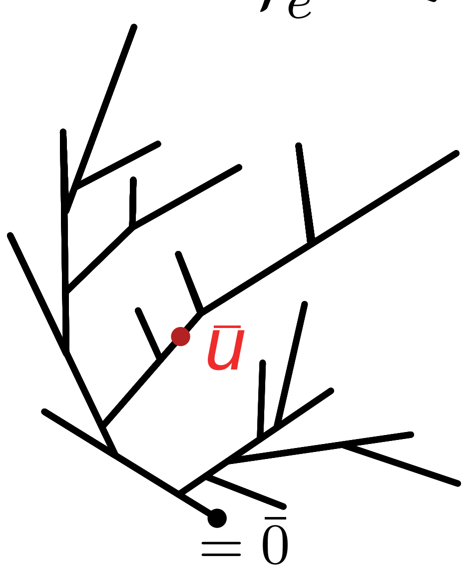

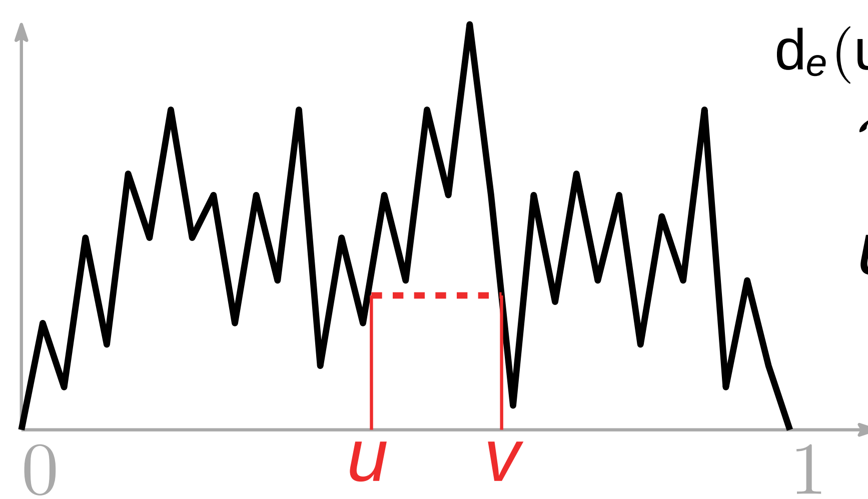

$d_e(u;v) = e_u + e_v \quad 2\min_{u\ \ s\ \ v} e_s$

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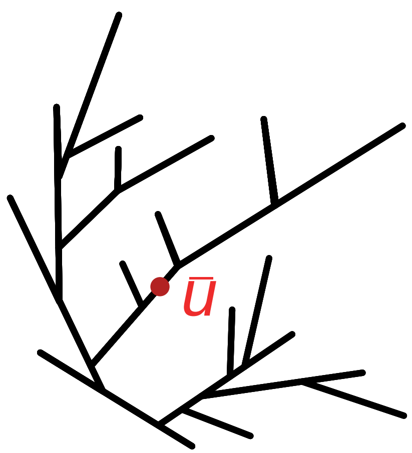

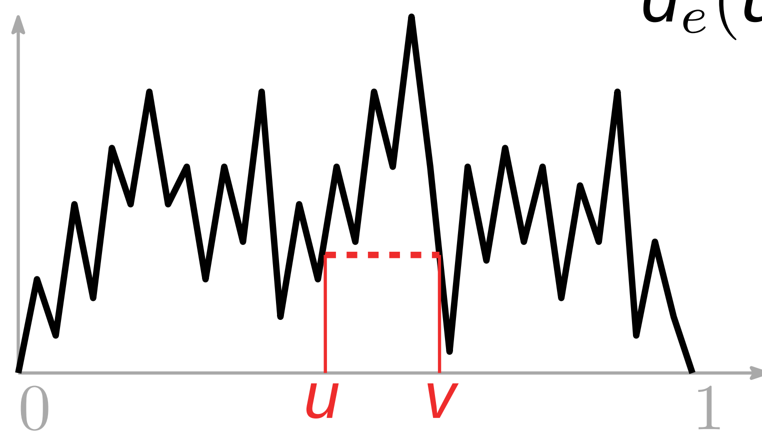

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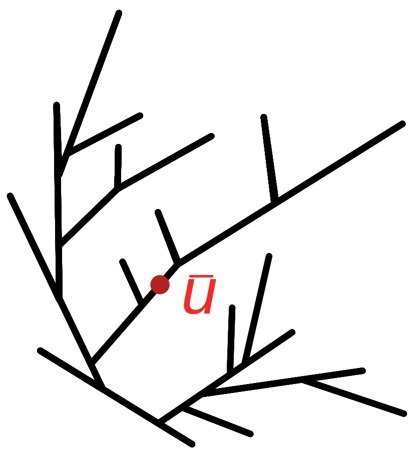

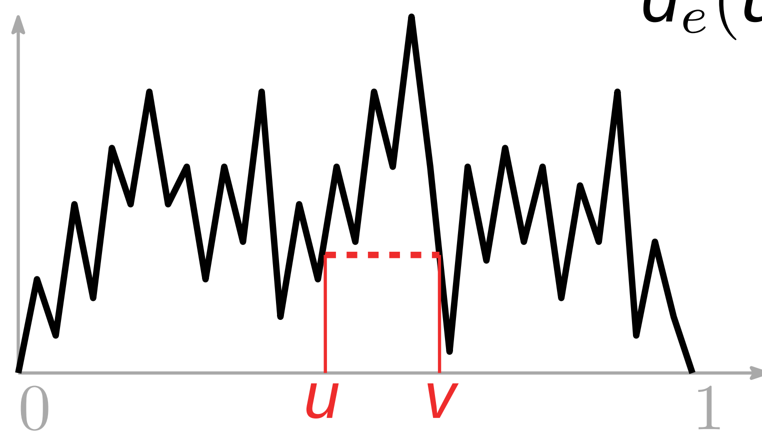

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Conditional on $\mathcal{T}_e$, Z a centered Gaussian process with $Z_\rho = 0$ and $E\left[(Z_s - Z_t)^2\right] = d_e(s;t)$

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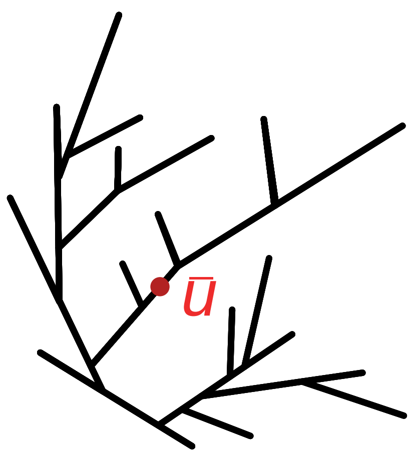

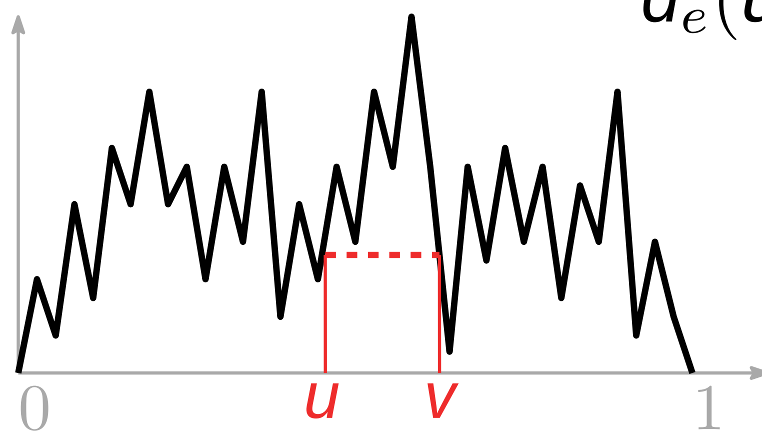

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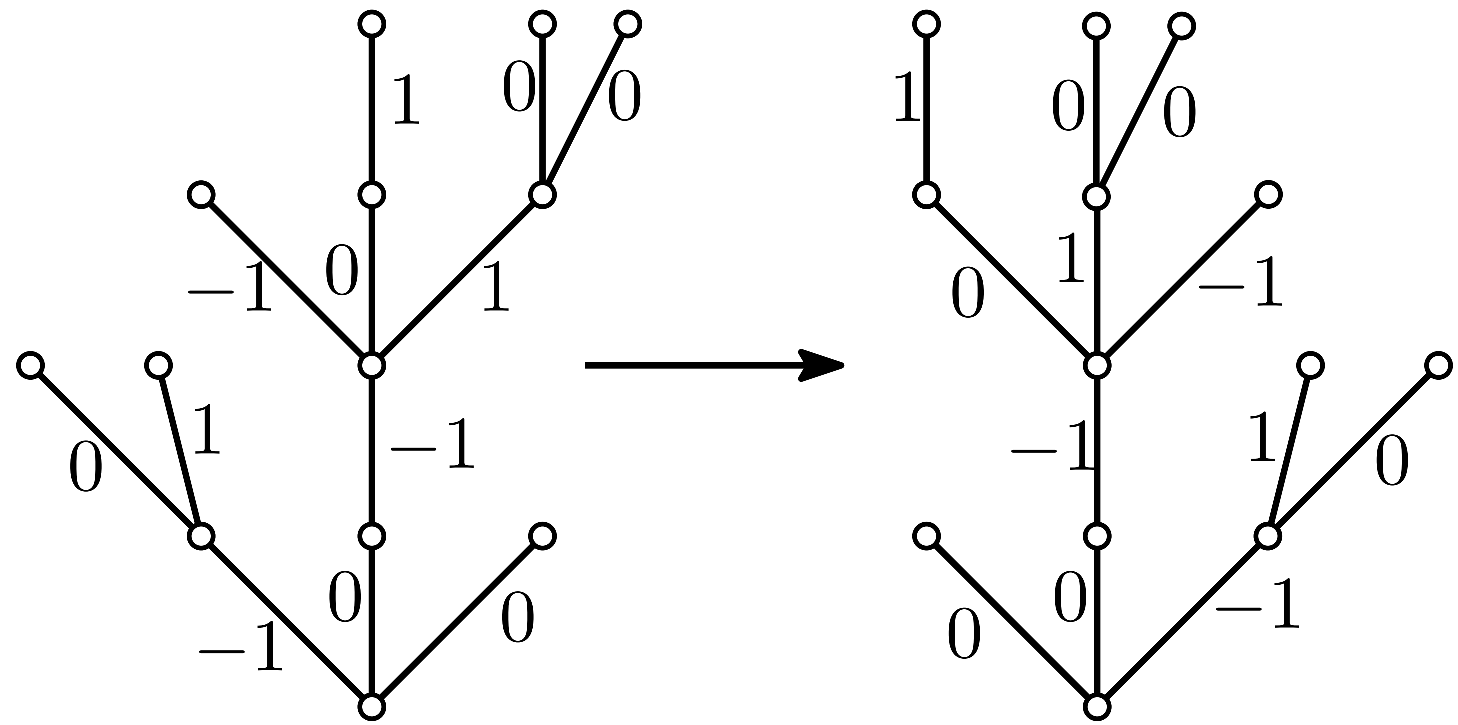
Start with one of \our" tree and apply a random permutation at each vertex



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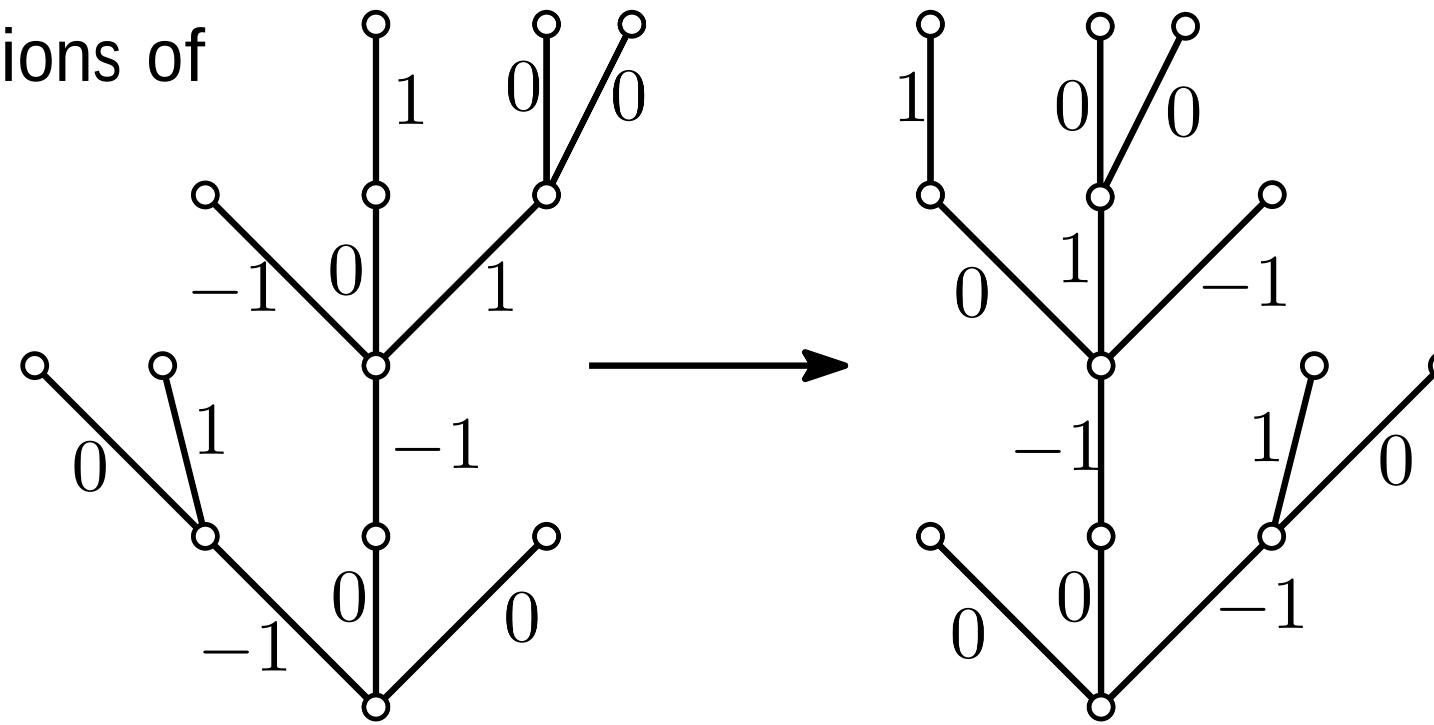

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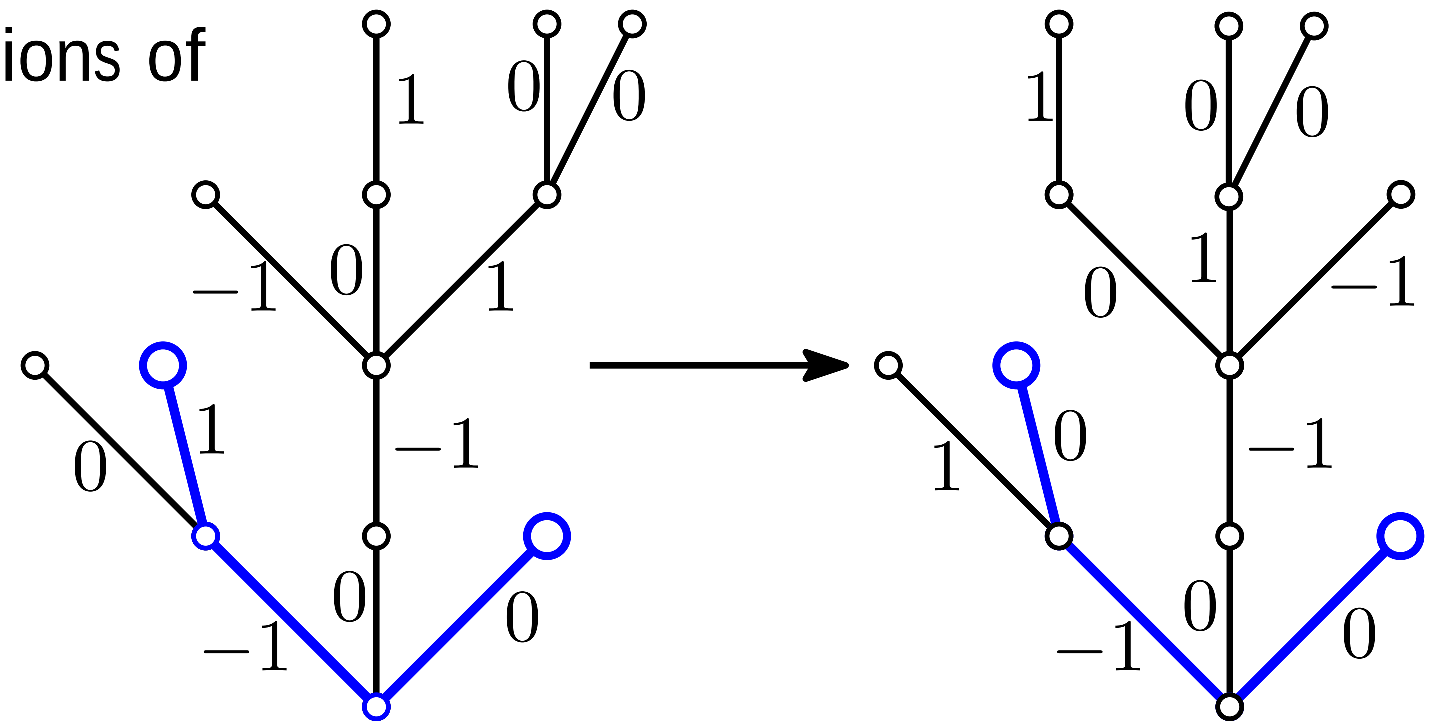

Solution:

- consider subtree $T\langle k\rangle$ spanned by k random vertices
- permute displacements and edges only outside $\langle T\rangle$.
- permute only displacements on $\langle T\rangle$.

Gives a coupling between "our" model and the fully permuted model: sufficient control to prove convergence for the true model.

Distances in simple triangulations

M_n = simple triangulation

$(C_{\lfloor nt \rfloor}; \tilde{Z}_{\lfloor nt \rfloor})$ = contour and label process of the associated tree

$Z_{\lfloor nt \rfloor}$ = distance **in the map** between vertex "$\lfloor nt \rfloor$" and the root.

Theorem : [Addario-Berry, A.]
M_n = random simple triangulation, then for all ″ > 0:

$$\mathbb{P}\left(\sup_{0 \le t \le 1} \left\{ \left| \tilde{Z}_{\lfloor nt \rfloor} - Z_{\lfloor nt \rfloor} \right| \right\} \ge {}'' n^{1/4} \right) \to 0:$$

i.e. the label process of the tree gives the distance to the root in the map.

Distances in simple triangulations

Claim 1: $3d_{M_n}(root; u_i) \geq L_n(u_i)$

First observation : In the tree, the labels of two adjacent vertices di er by at most 1. **What can go wrong with closures ?**

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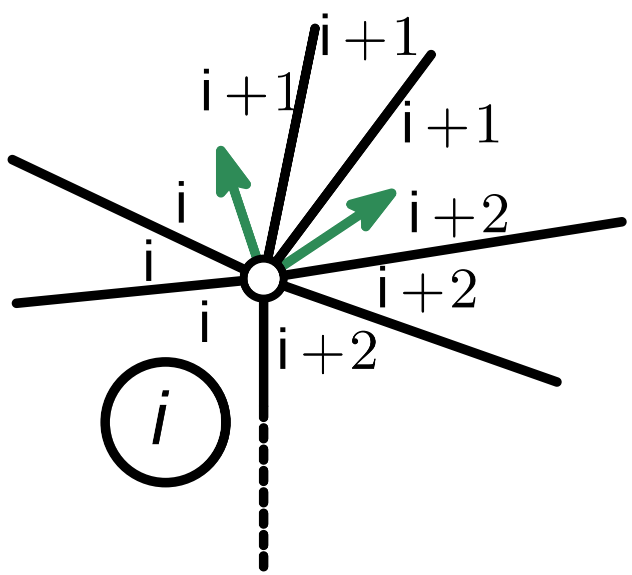

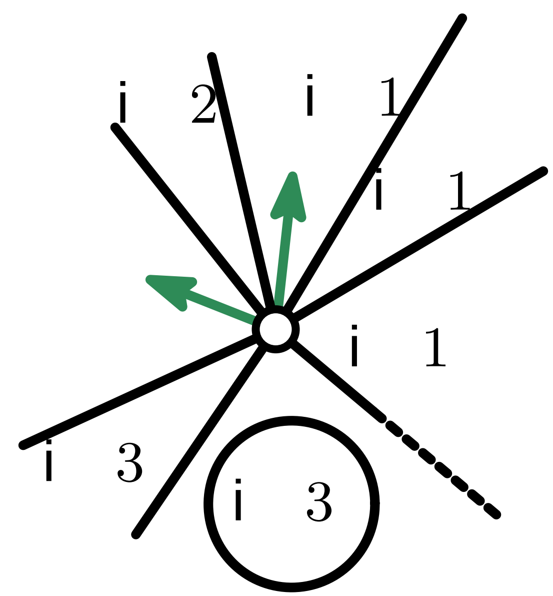

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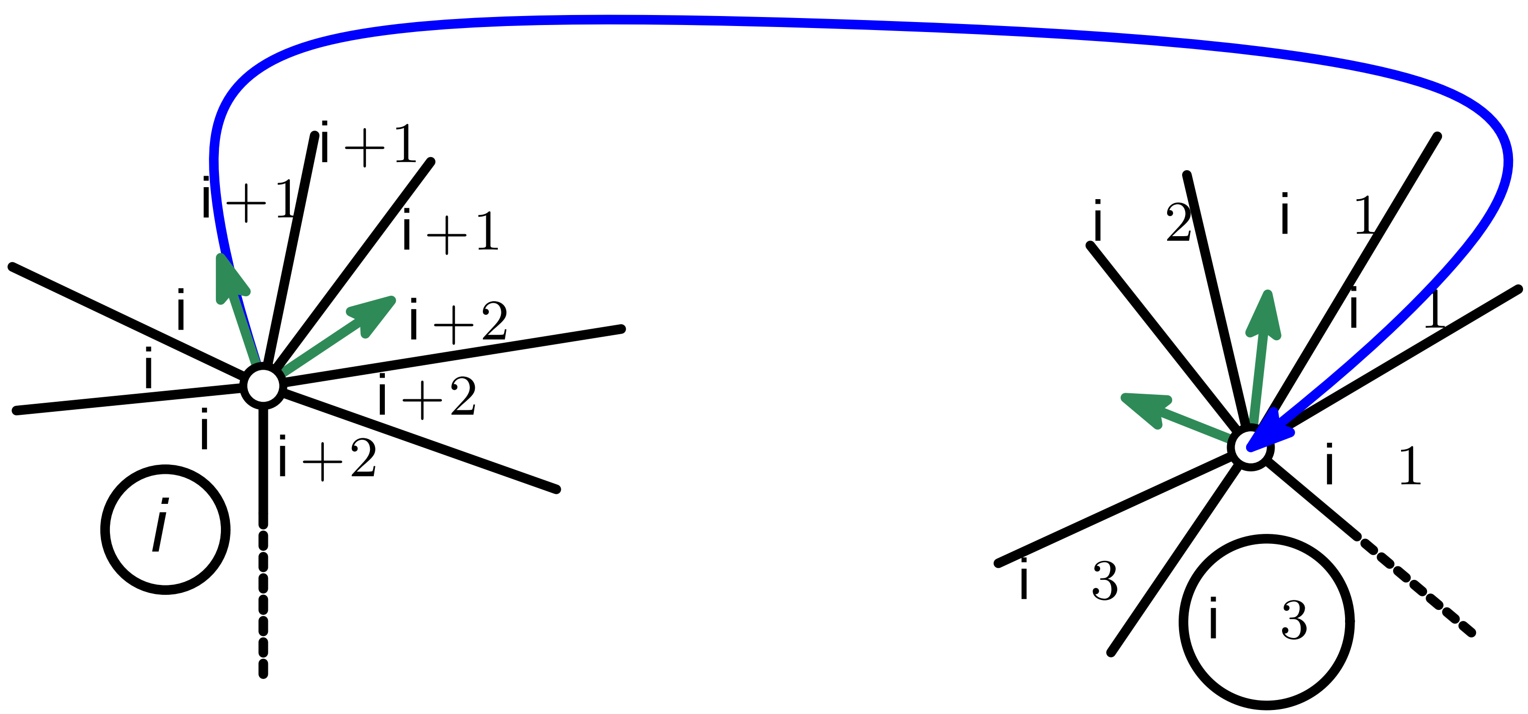

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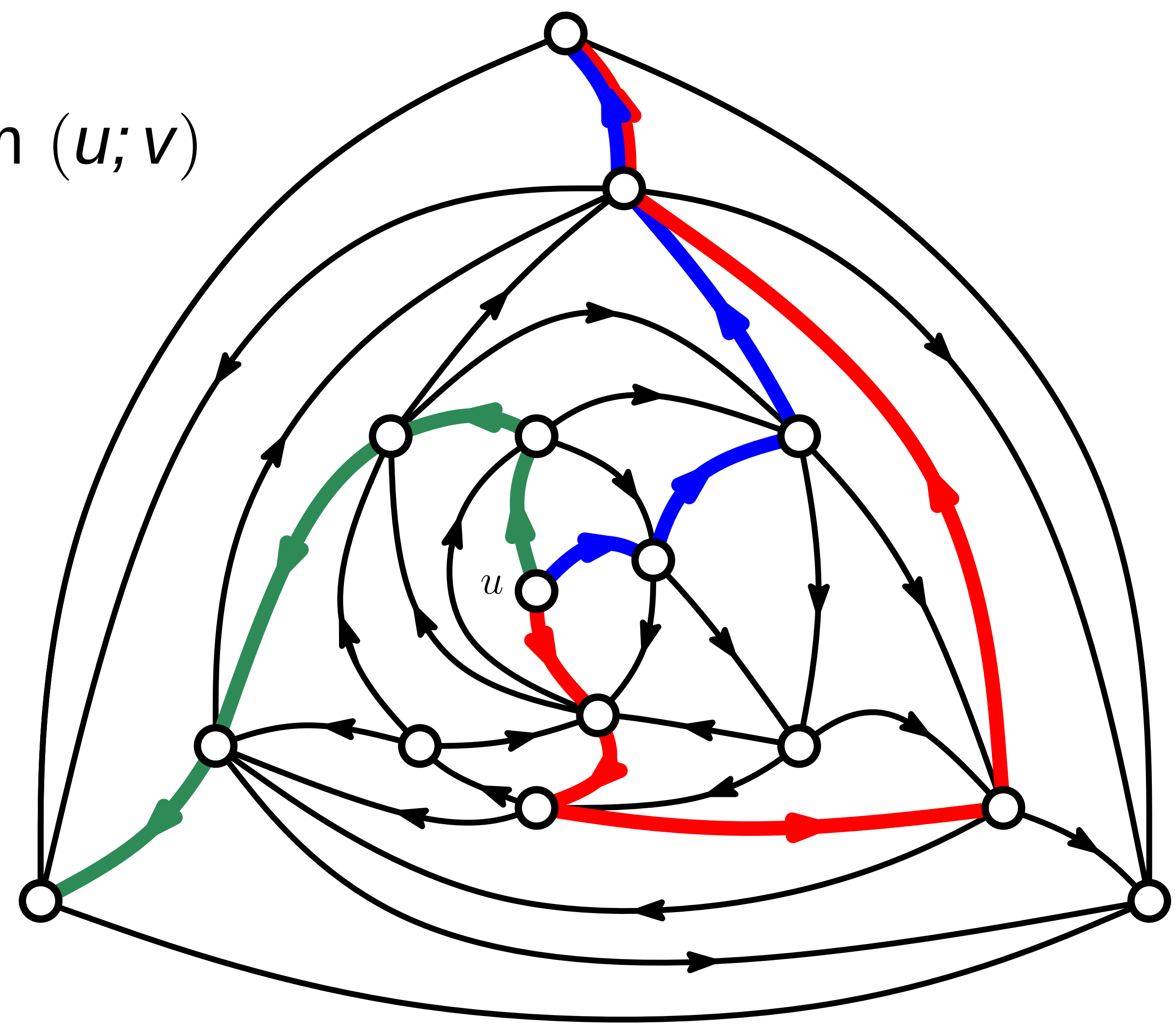
- For each inner vertex : 3 LMP



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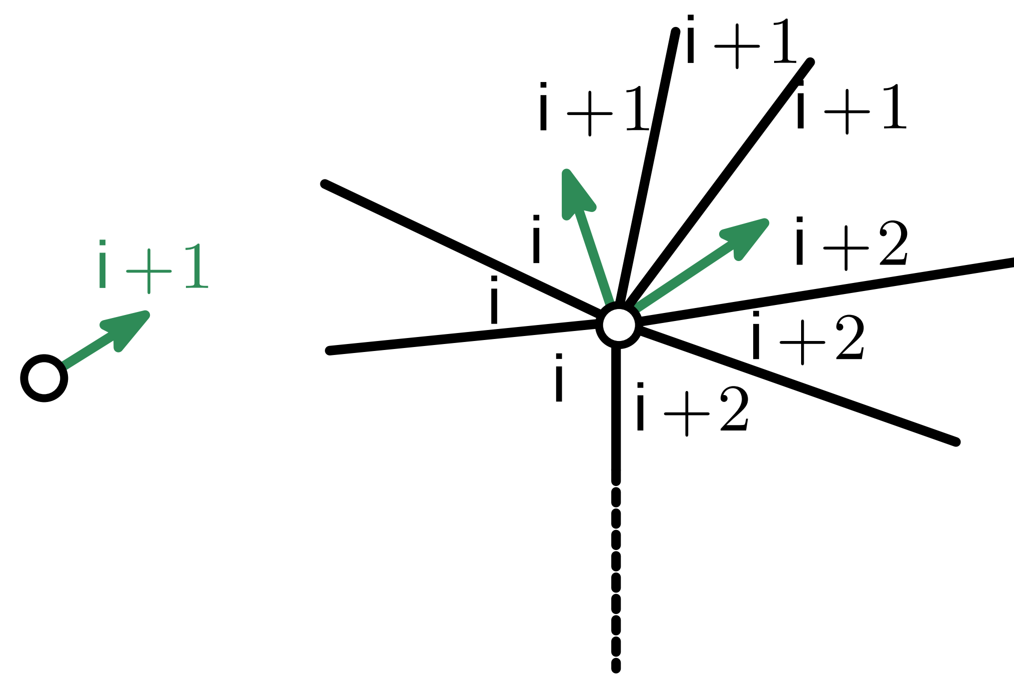

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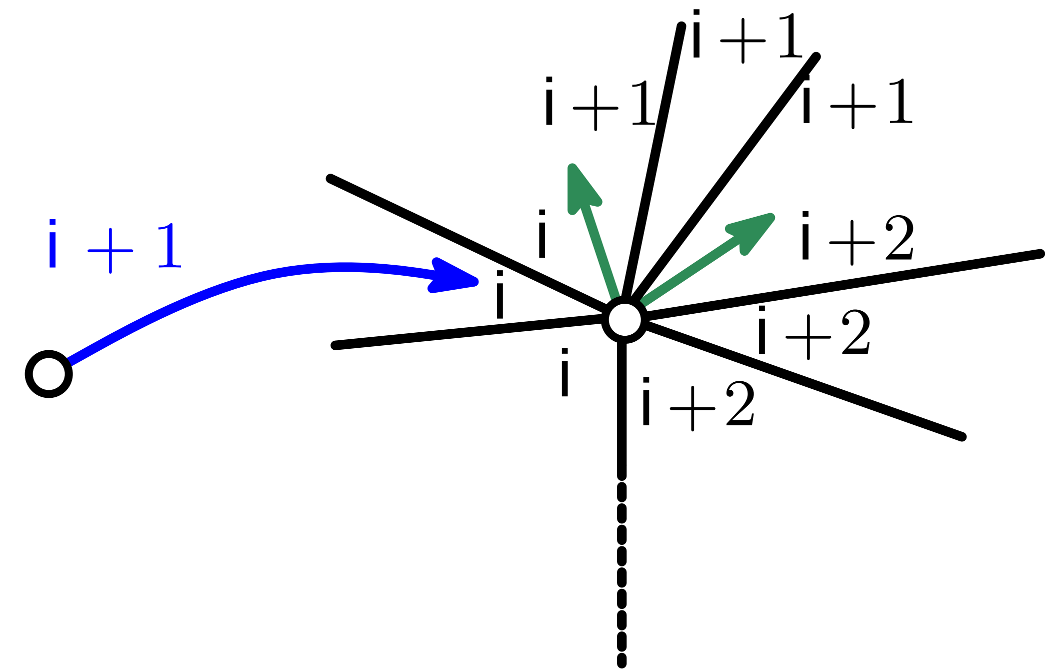

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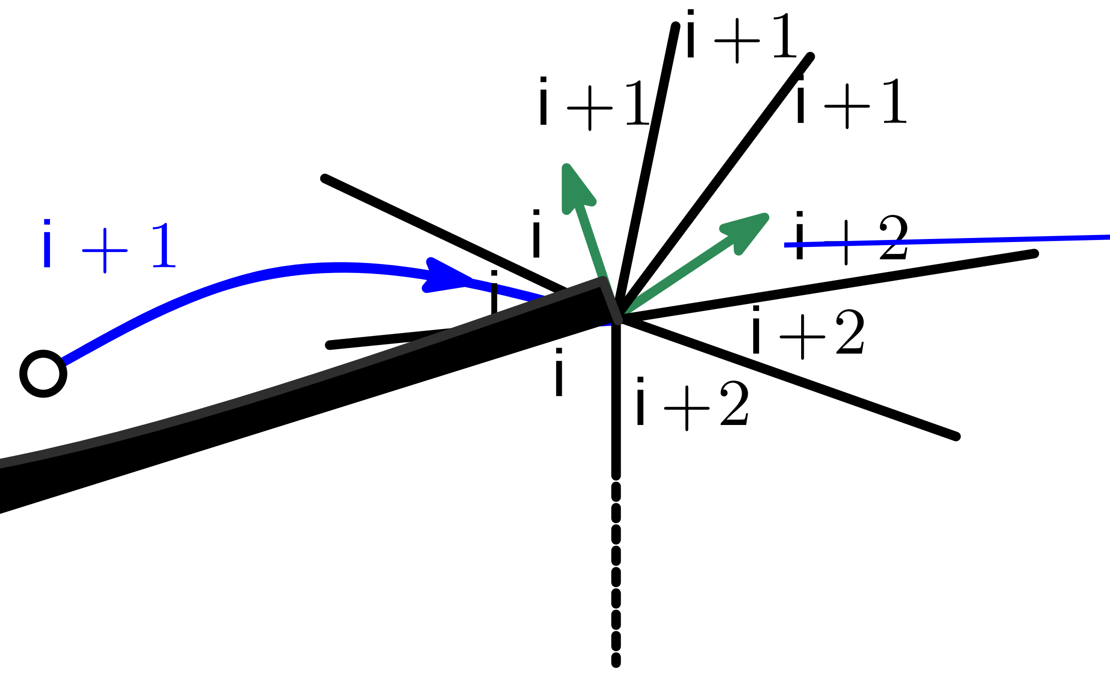
Left Most Path from $(u; v)$

e.

rtex : 3 LMP

ersecting

face

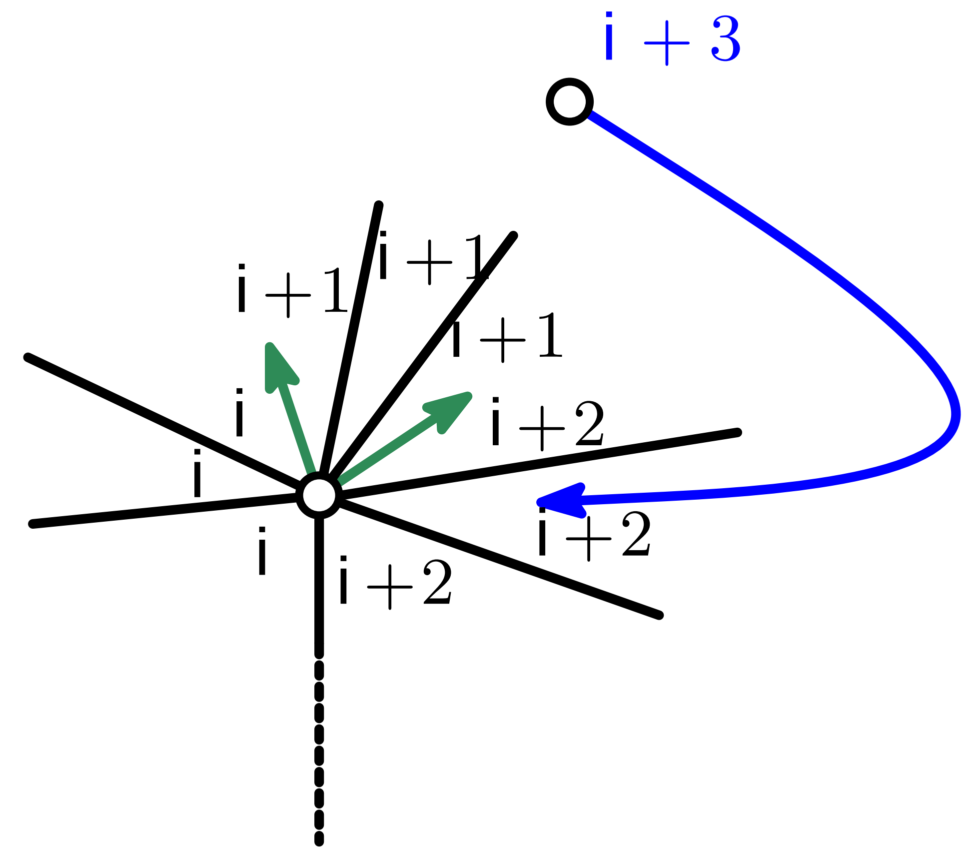


Distances in simple triangulations

Claim 1: $3d_{M_n}(root; u_i) \geq L_n(u_i)$

Claim 2 : $d_{M_n}(root; u_i) \leq L_n(u_i)$

- Consider the **Left Most Path** from $(u; v)$ to the root face.
- For each inner vertex : 3 LMP
- LMP are not self-intersecting $\Rightarrow$ they reach the outer face
- On the left of a LMP, corner labels decrease exactly by one.



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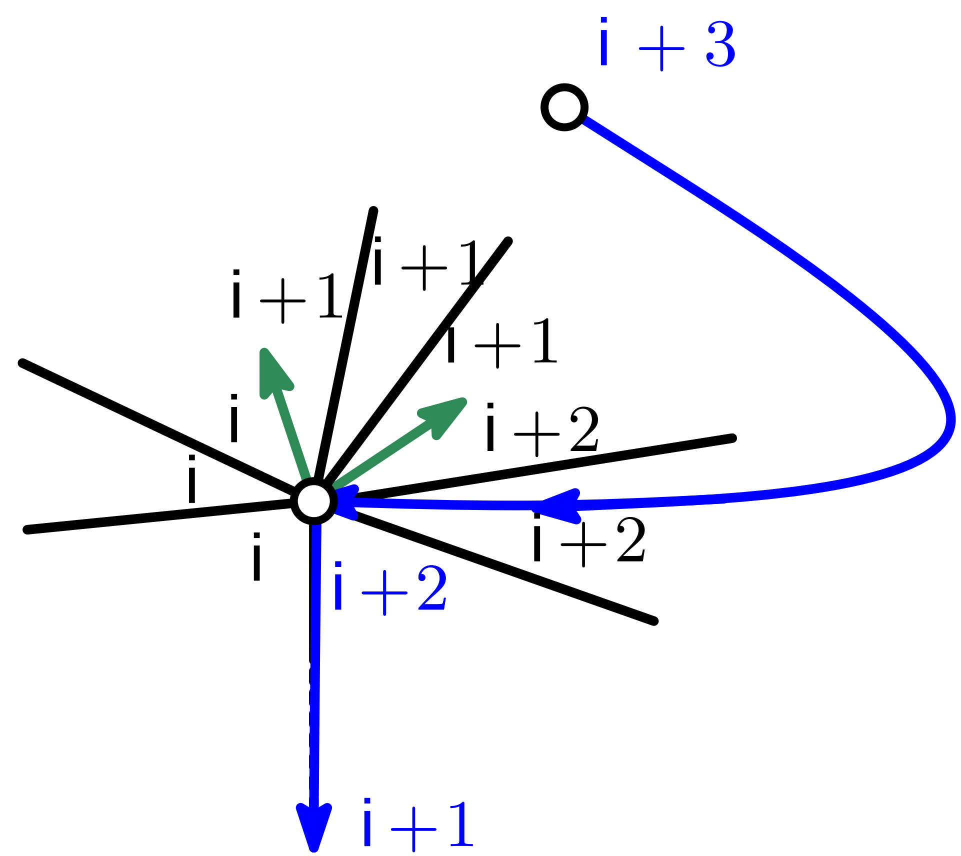
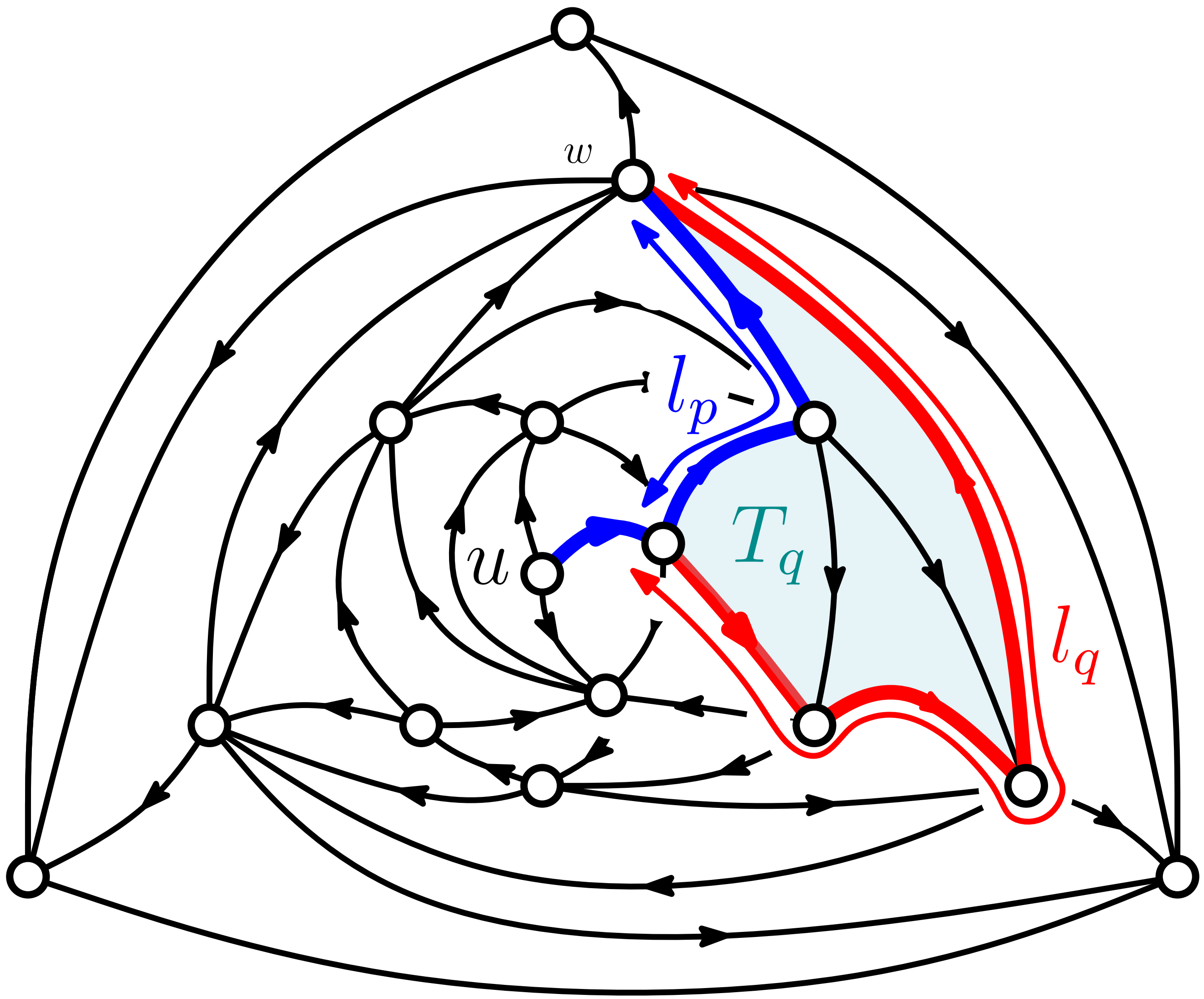


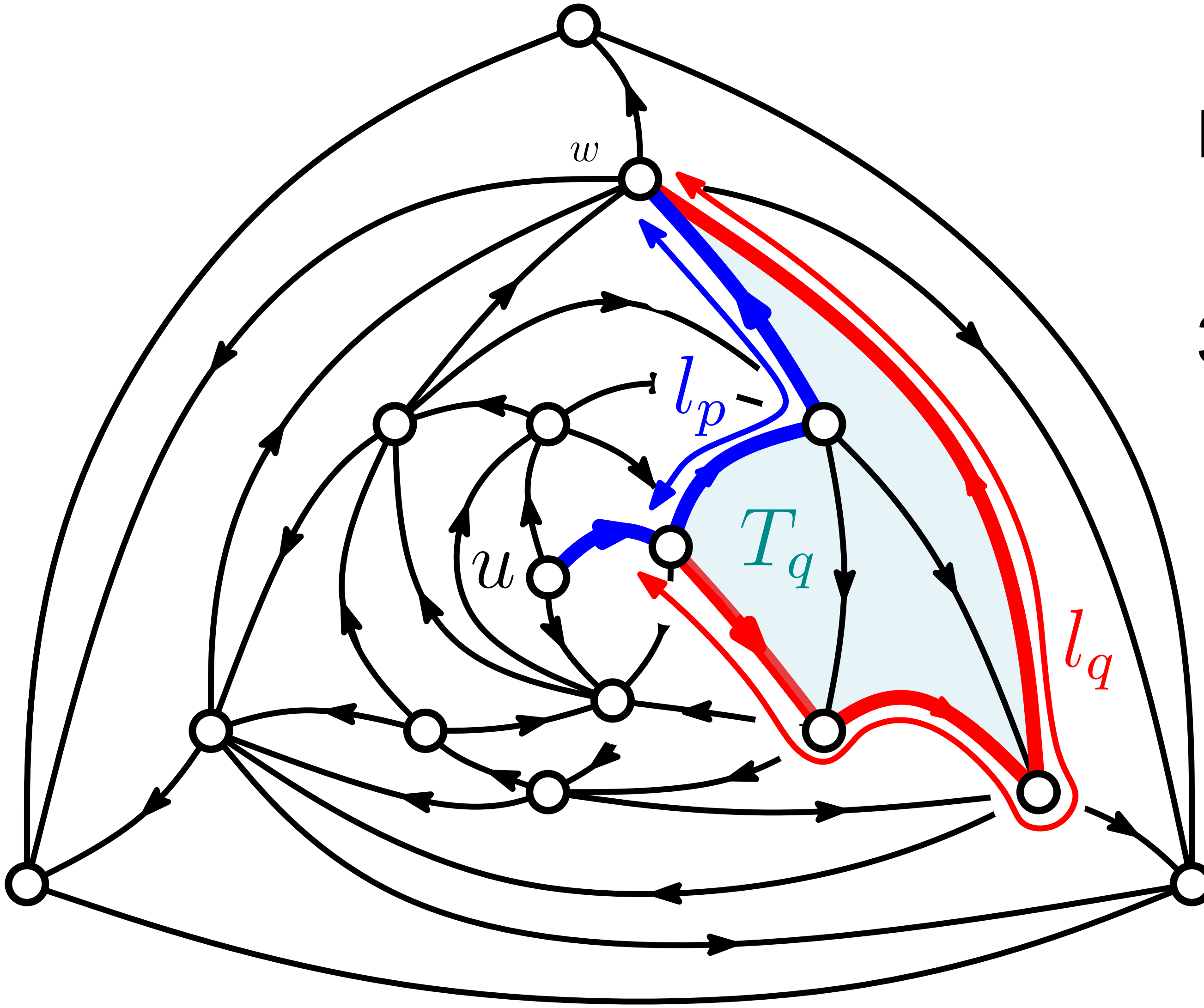
LMP are almost geodesic



Leftmost path

Another path: can it be shorter ?

LMP are almost geodesic



Euler Formula :

$|E(T_q)| = 3|V(T_q)| - 3 - (\ell_p + \ell_q)$

3-orientation + LMP :

$|E(T_q)| \geq 3|V(T_q)| - 2\ell_q - 2$

$\Longrightarrow \ell_q \geq \ell_p + 1$

Leftmost path

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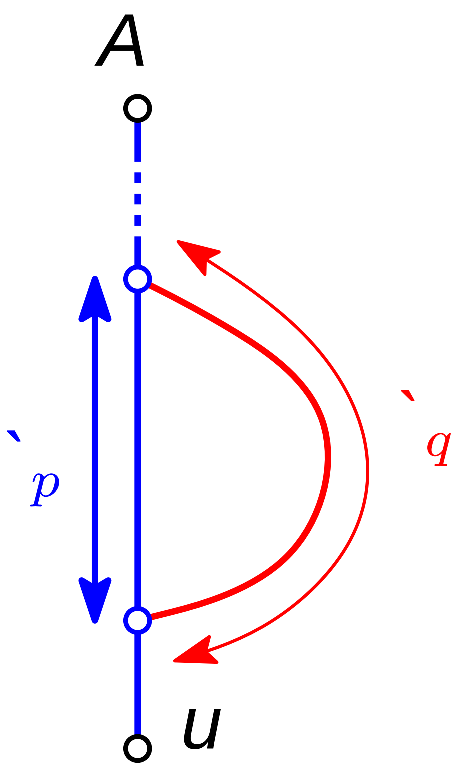

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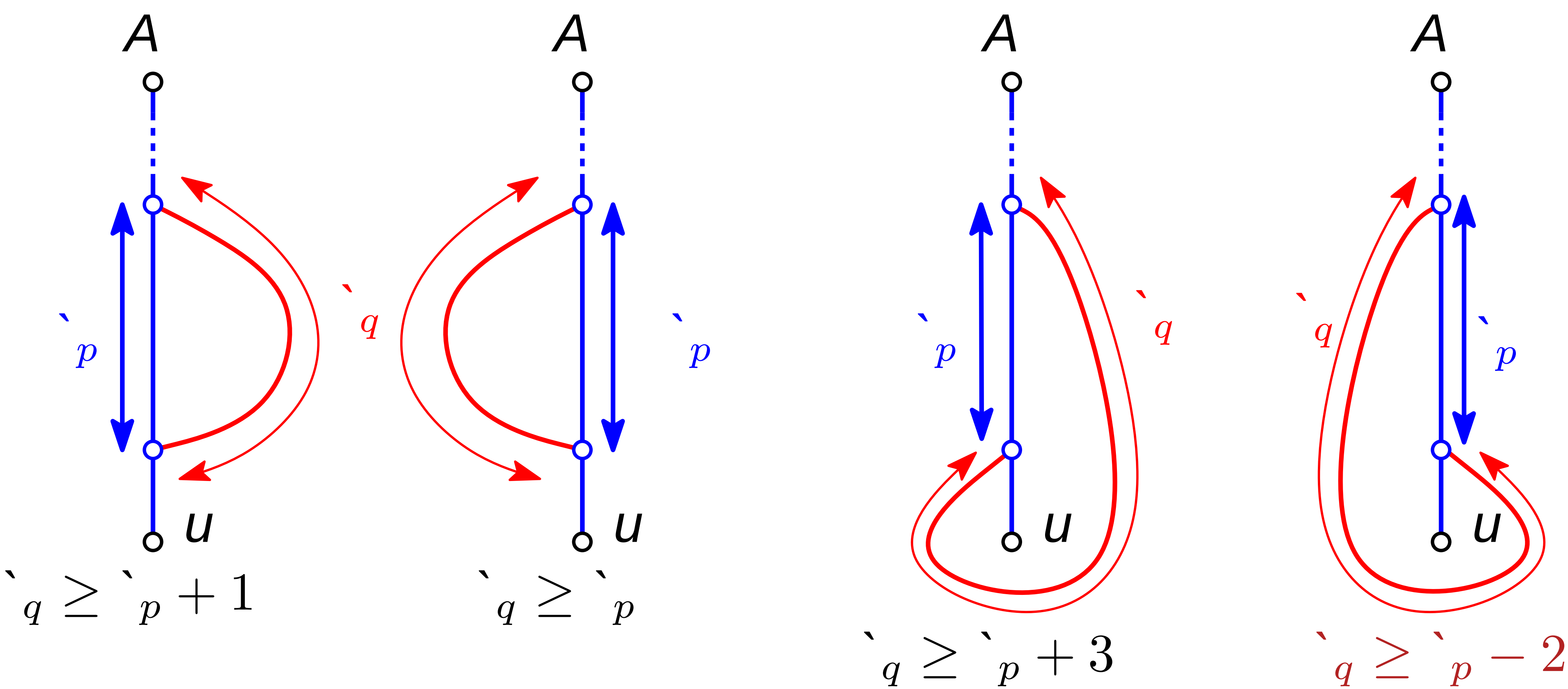
Leftmost path

Another path: can it be shorter ? YES



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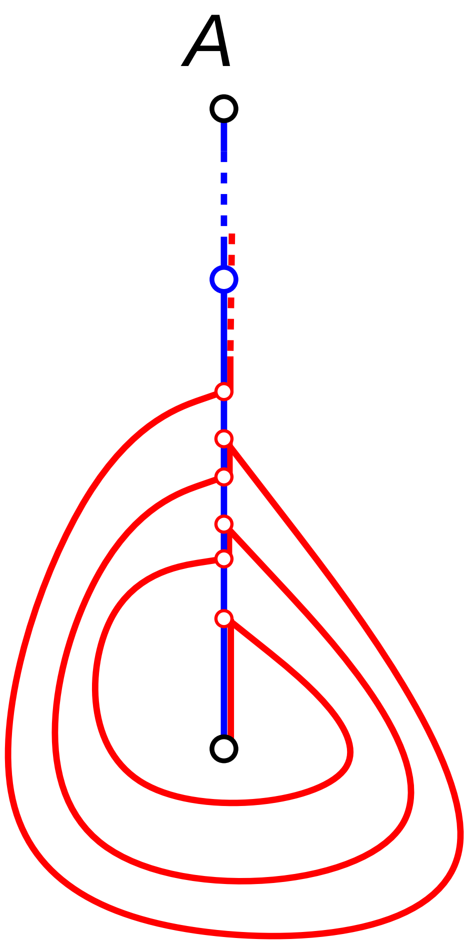
Another path: can it be shorter ? YES ... but not too often



Bad con guration =

too many **windings** around the LMP

But w.h.p a winding cannot be too short.

$\Longrightarrow$ w.h.p the number of windings is $o(n^{1/4})$.

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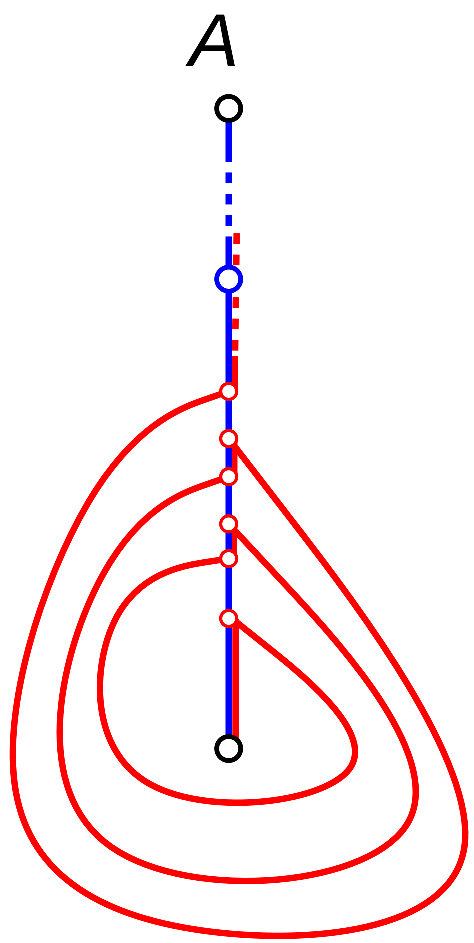

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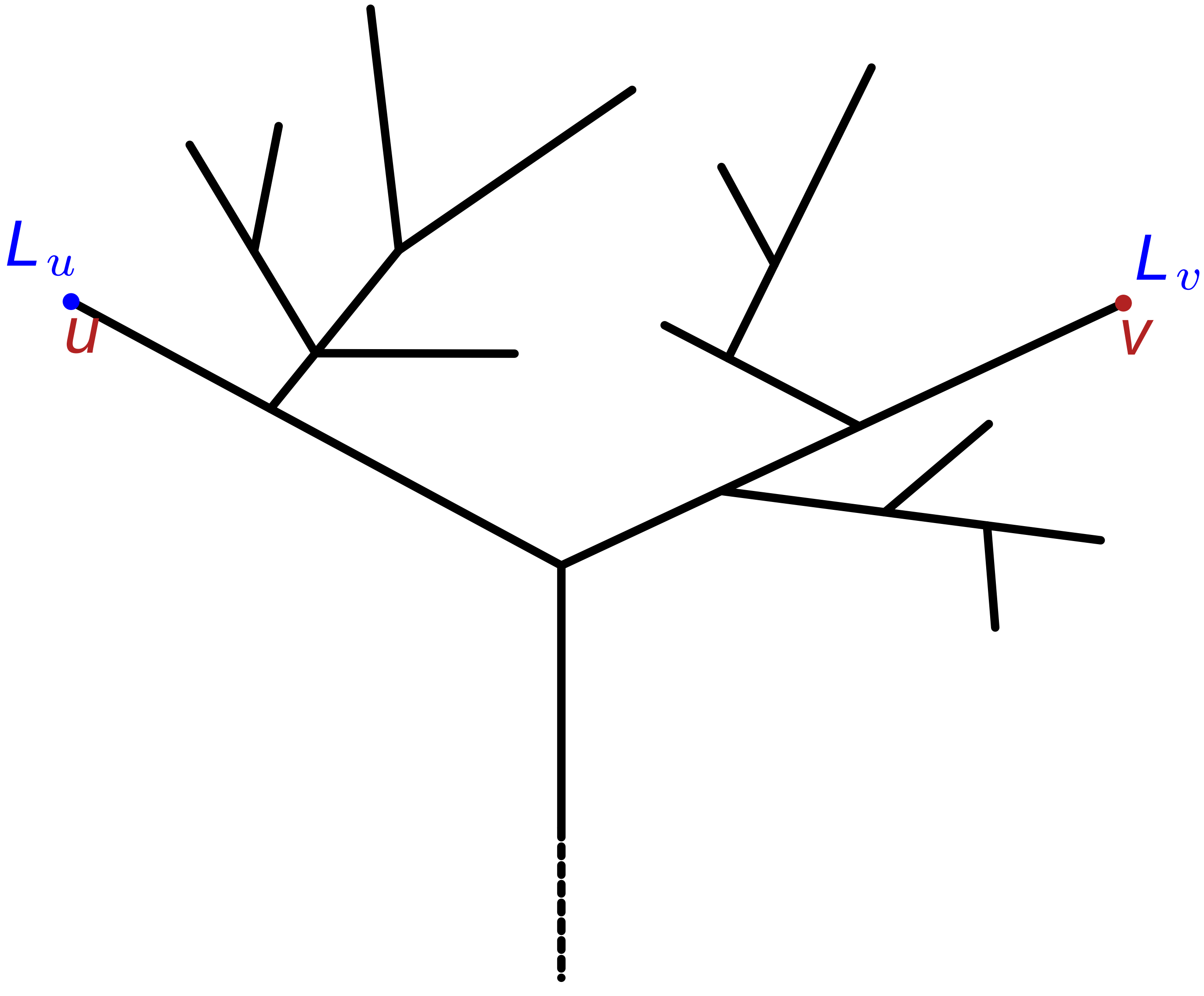
Proposition:

For ″ > 0, let $A_{n,\varepsilon}$ be the event that there exists $u \in M_n$ such that $L_n(u) \geq d_{M_n}(u; root) +$ ″$n^{1/4}$. Then under the uniform law on $\mathcal{M}_n$, for all ″ > 0:

$$\mathbb{P}\left(A_{n,\varepsilon}\right) \to 0:$$

Distances are tight



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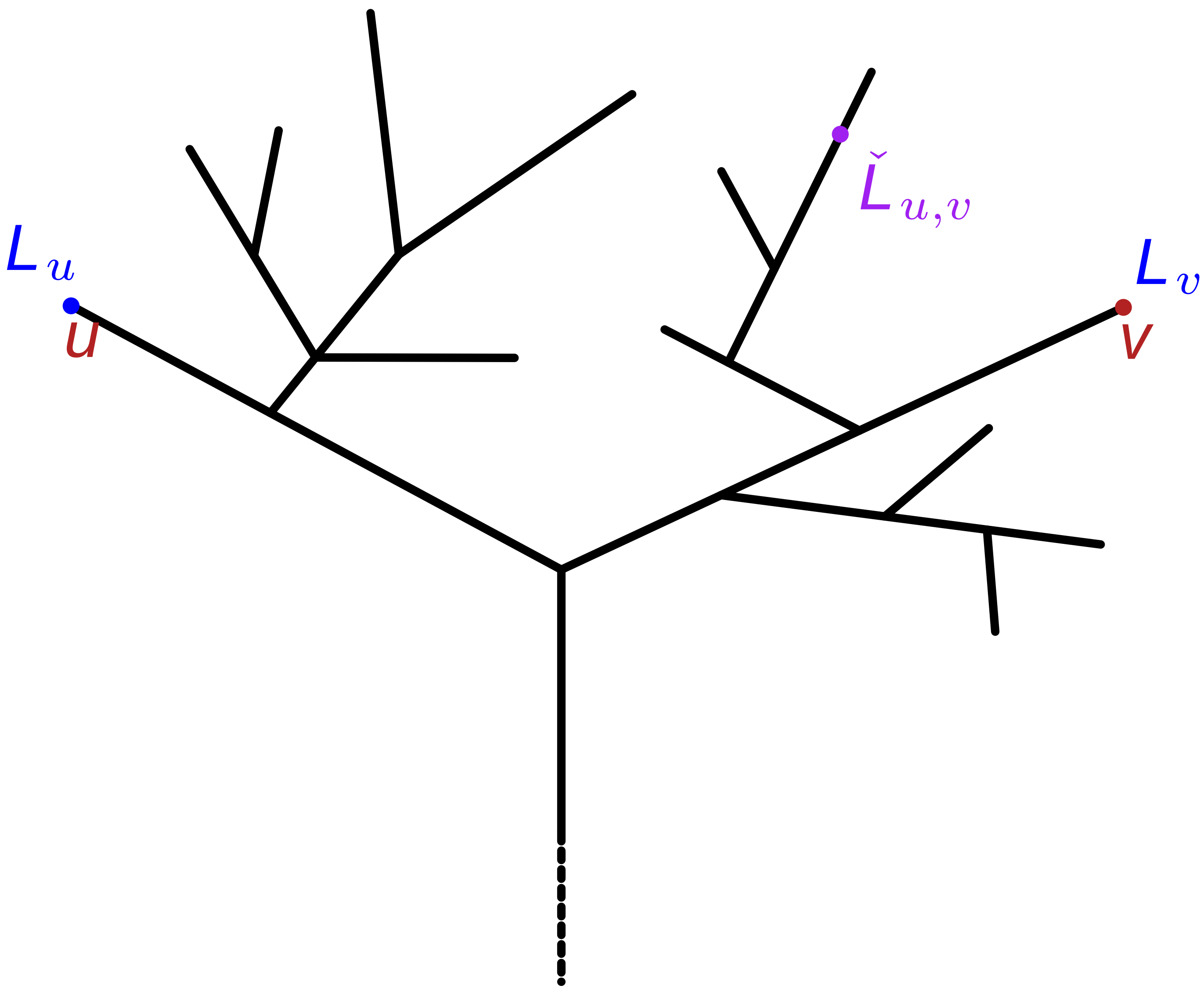
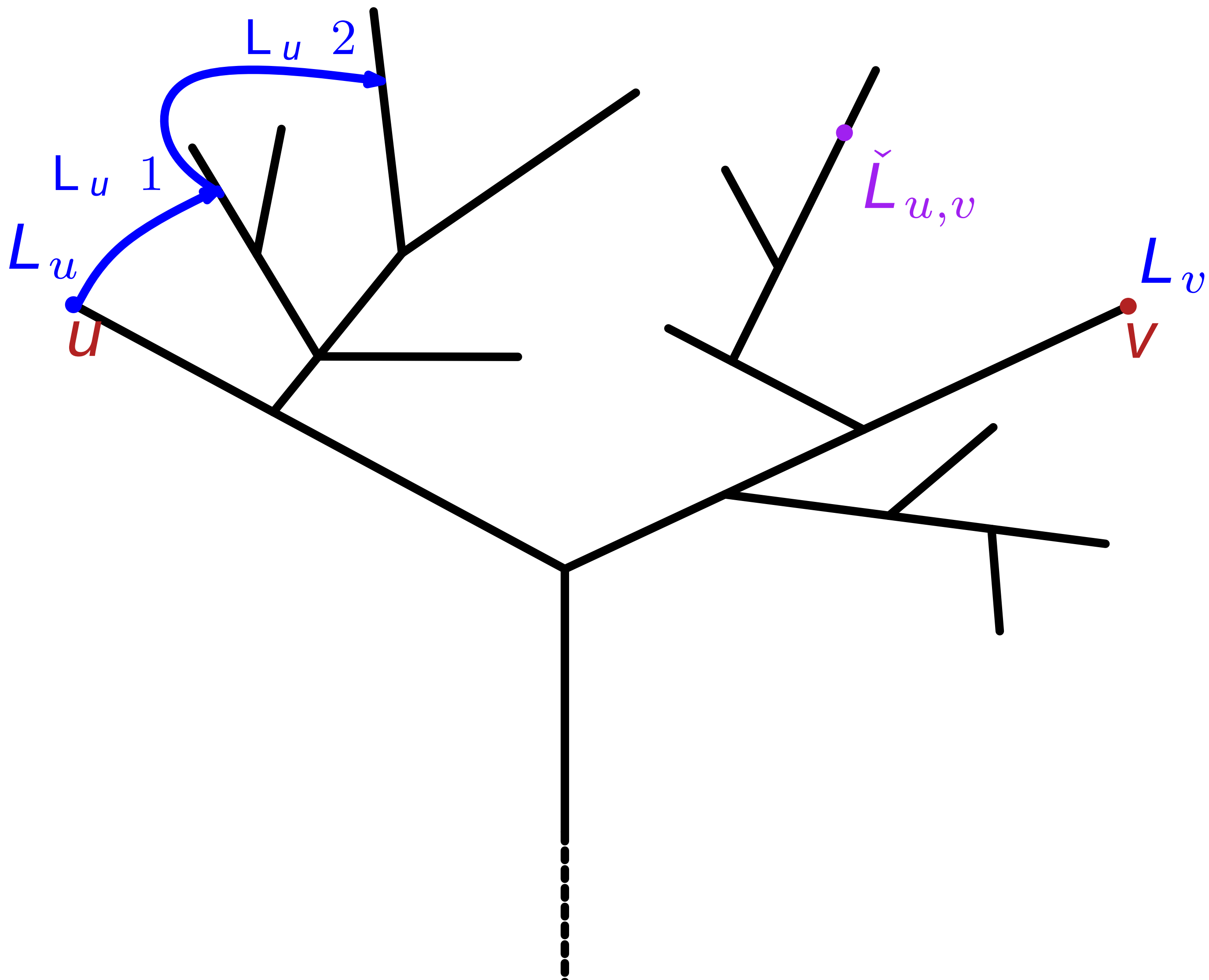


$$\check{L}_{u,v} = \min\{L_s; u \leq s \leq v\}$$

Distances are tight



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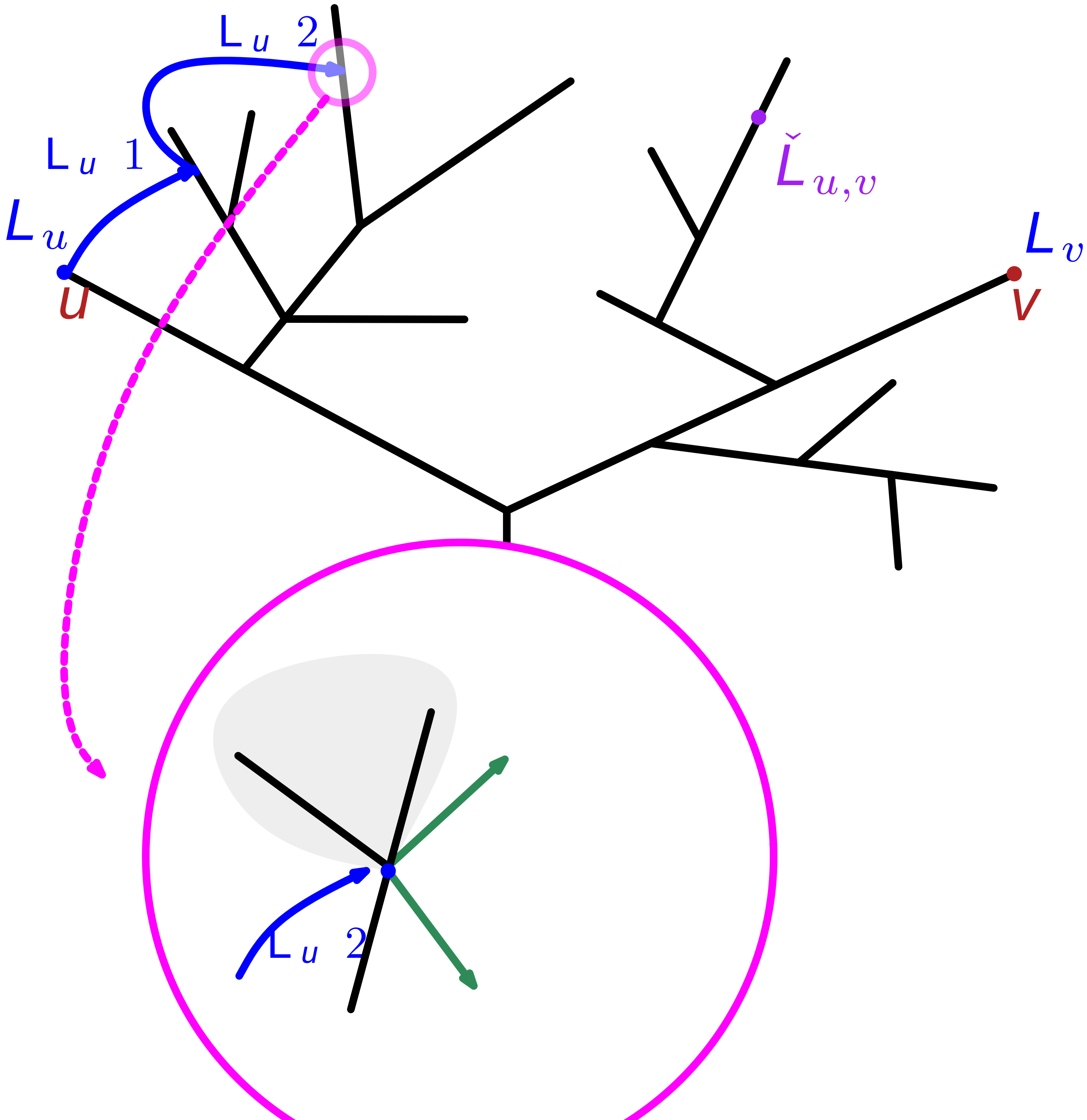

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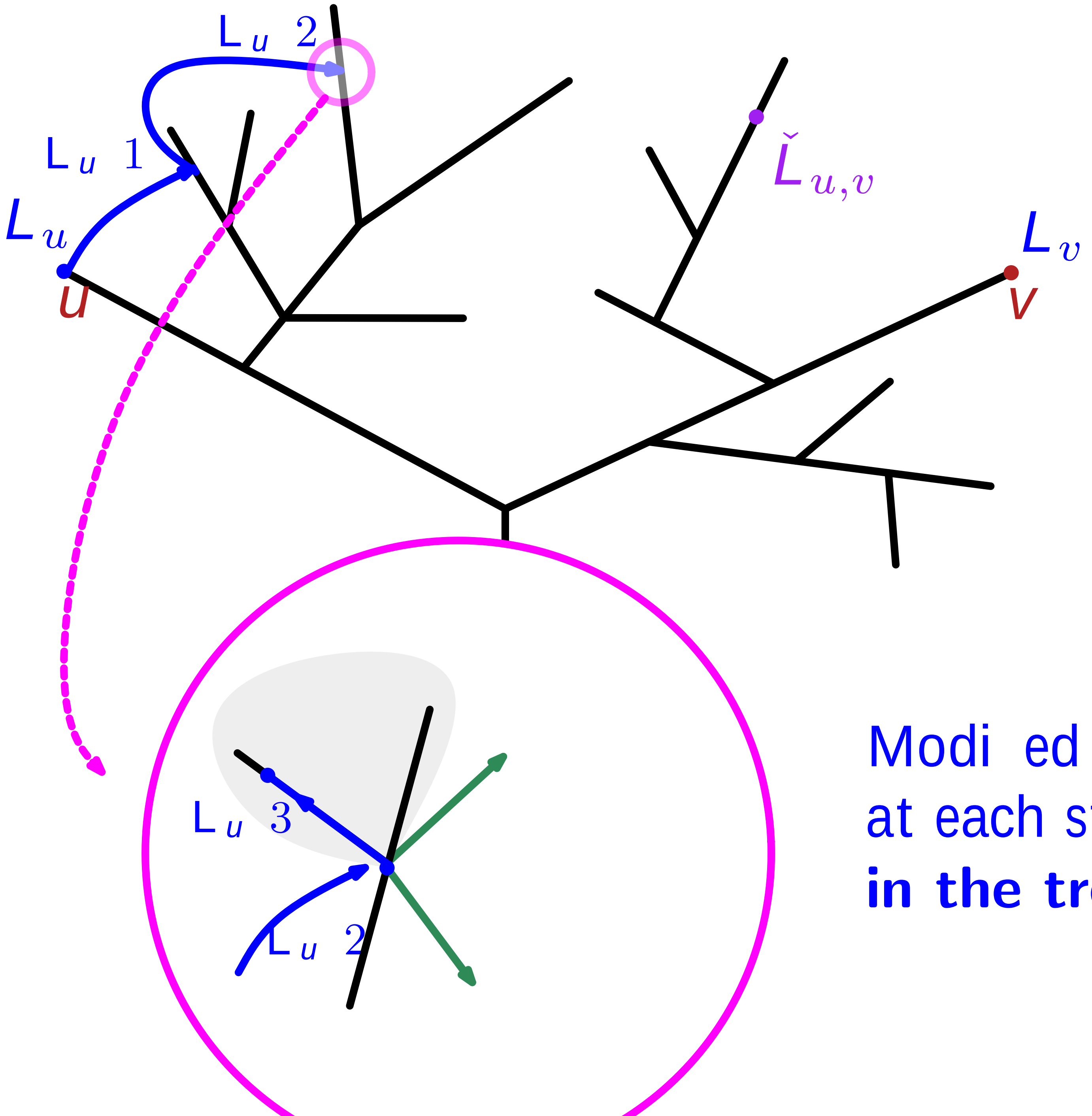

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Modi ed LMP:
at each step, we take the rst edge **in the tree**

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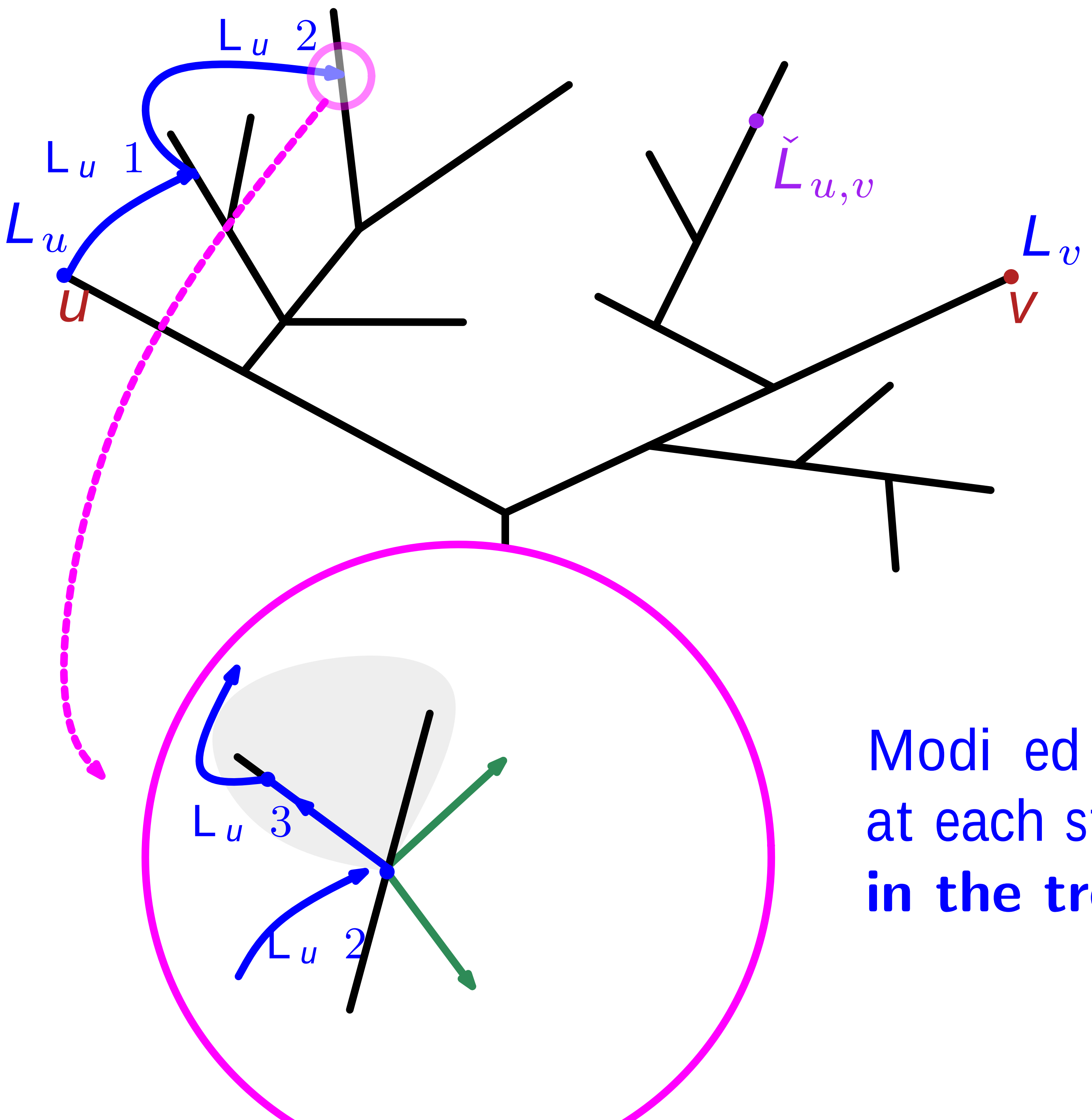

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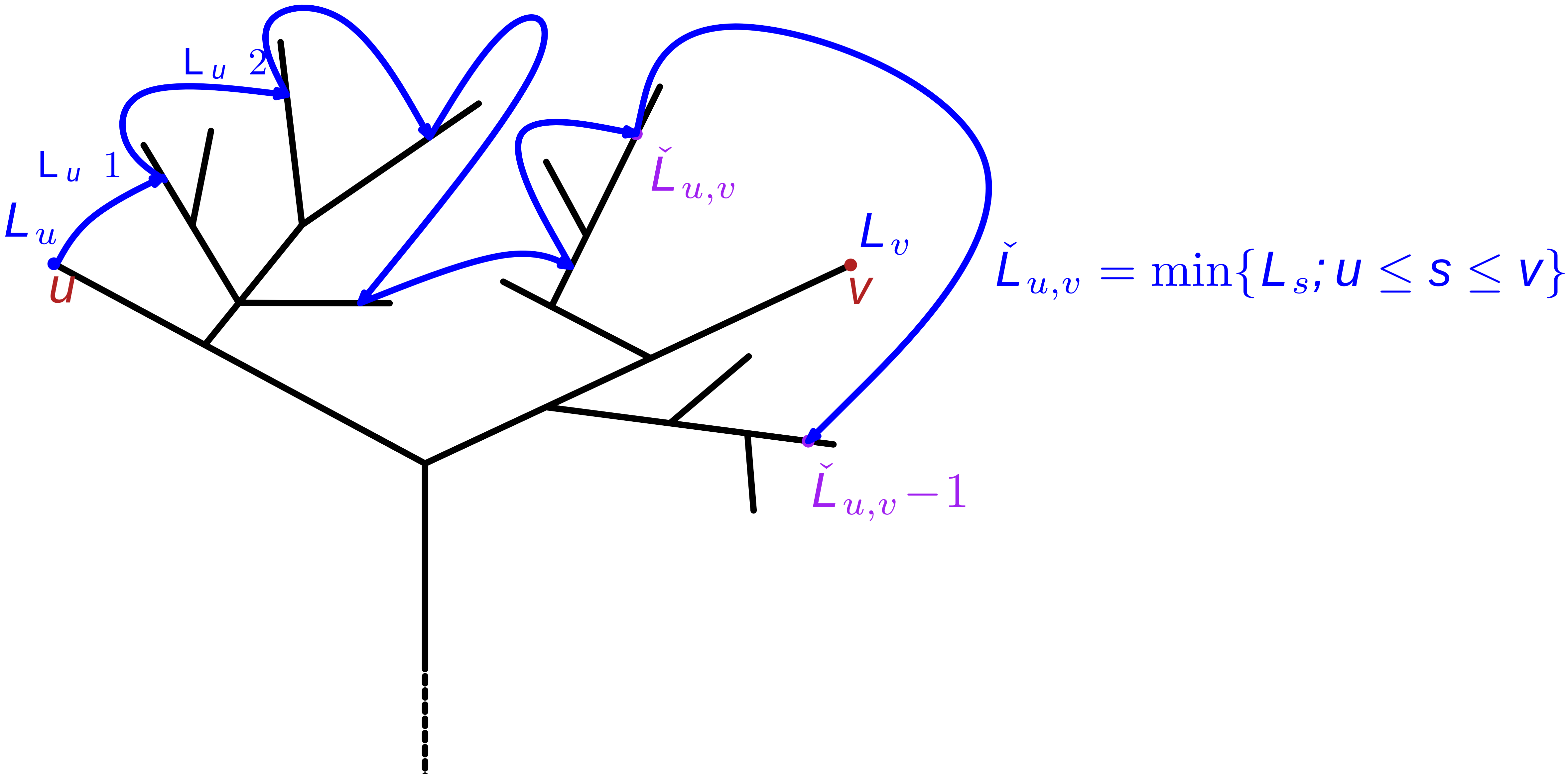

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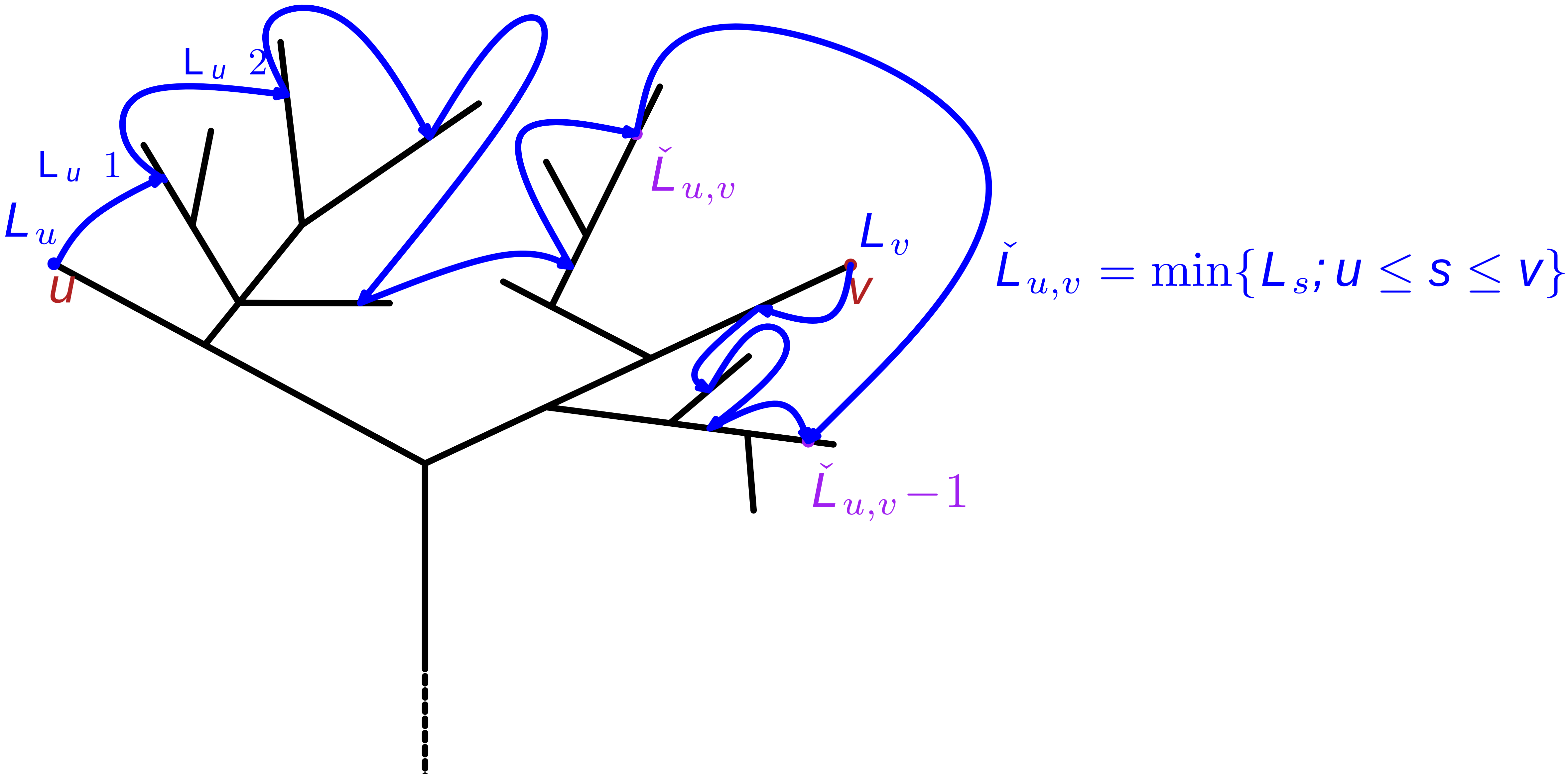

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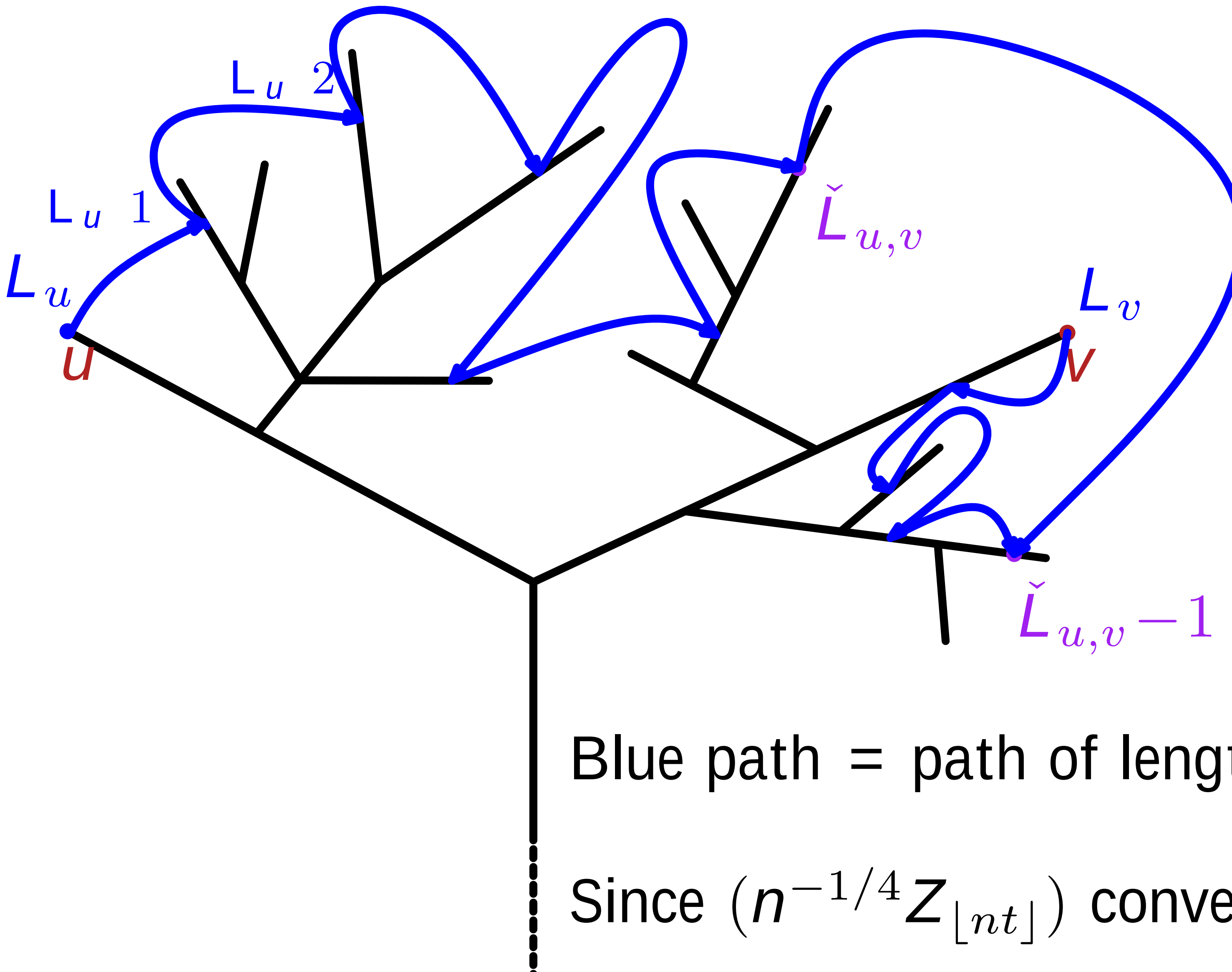


$\check{L}_{u,v} = \min\{L_s; u \leq s \leq v\}$

Blue path = path of length $L_u + L_v - 2\check{L}_{u,v} + 2$

Since $(n^{-1/4} Z_{\lfloor nt \rfloor})$ converges $\Rightarrow$ (d_n) tight

The result for the last time

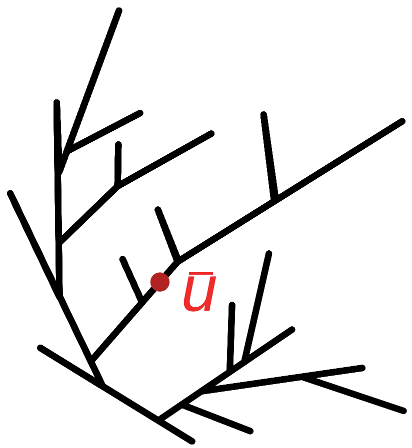
Theorem : [Addario-Berry, A.]
(M_n) = sequence of random **simple** triangulations, then:

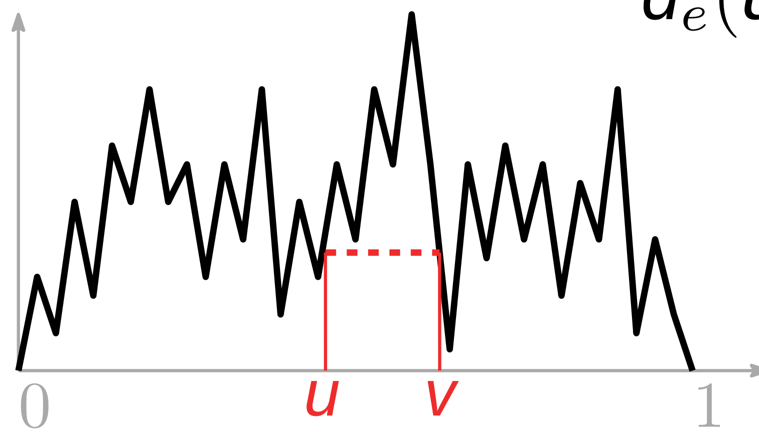
$$\left(M_n; \left(\frac{3}{4n}\right)^{1/4} d_{M_n} \right) \xrightarrow{(d)} (M; D^{\star});$$

for the distance of Gromov-Hausdor on the isometry classes of compact metric spaces.

The Brownian Map ??

The Brownian map





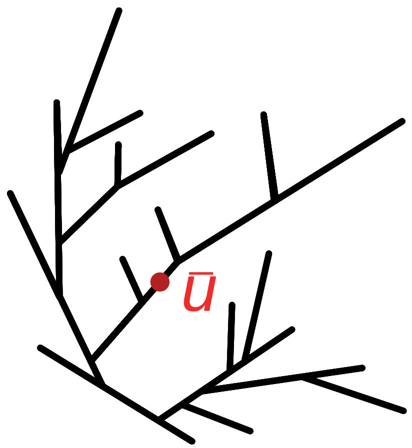
$$d_e(u;v) = e_u + e_v - 2\min_{u \le s \le v} e_s$$

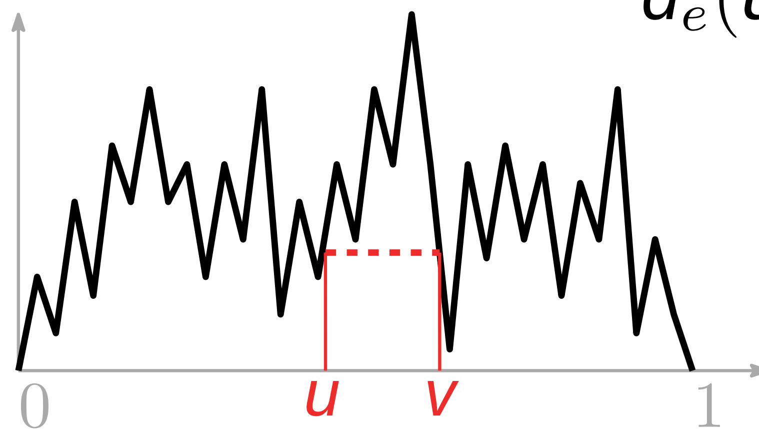
$$\mathcal{T}_e = [0;1]=\sim_e$$

$$u \sim_e v \text{ i } \quad d_e(u;v) = 0$$

Conditional on $\mathcal{T}_e$, Z a centered Gaussian process with $Z_\rho = 0$ and $E[(Z_s - Z_t)^2] = d_e(s;t)$

$Z \sim$ Brownian motion on the tree





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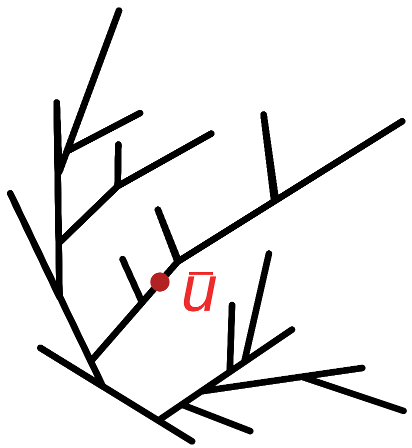

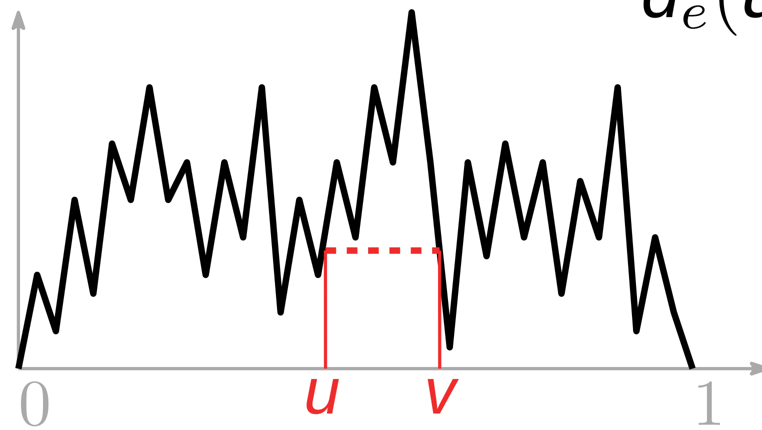

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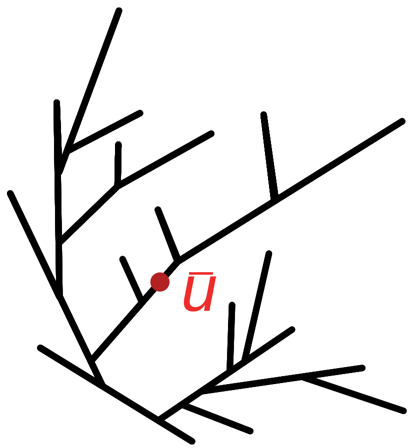
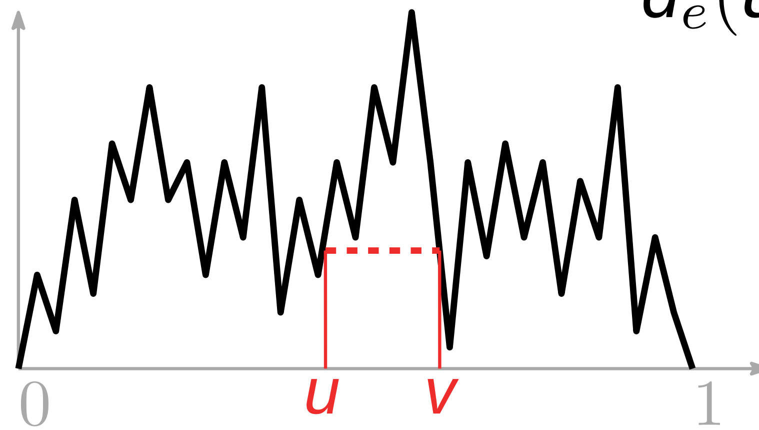
$$D^\circ(s;t) = Z_s + Z_t - 2\max\Big(\inf_{s \le u \le t} Z_u; \inf_{t \le u \le s} Z_u\Big); \quad s;t \in [0;1]:$$

$$D^*(a;b) = \inf\Big\{\sum_{i=1}^{k-1} D^\circ(a_i;a_{i+1}) : k \ge 1; a = a_1; a_2; :::; a_{k-1}; a_k = b\Big\};$$

The Brownian map





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Then $M = (\mathcal{T}_e = \sim_{D^?}; D^*)$ is the **Brownian map**.

Perspectives

Same approach works also for simple quadrangulations.

Can it be generalized to other families of maps ?

- Generic bijection between blossoming trees and maps [Bernardi, Fusy] [A.,Poulalhon].
  Can we say something about distances ?
- Convergence of Hurwitz maps: bijection also with blossoming trees [Duchi, Poulalhon, Schae er].

Can we say something about the embedding of the Brownian map in the sphere via circle packing ?

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Thank you !

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