

# Bijjective combinatorics of positive braids

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## Abstract

We give a new and bijective proof for the formula of the growth function of the positive braid monoid with respect to Artin generators.

*Keywords:* Bijective combinatorics, Braid monoids, Growth function

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## 1 Introduction

For  $n \in \mathbb{N}^*$ , we consider the monoid generated by  $\Sigma = \{\sigma_1, \dots, \sigma_n\}$  and by the relations:  $\sigma_i \sigma_j = \sigma_j \sigma_i$  for  $|i - j| \geq 2$  and  $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$  for  $1 \leq i \leq n - 1$ . This monoid is the positive braid monoid on  $n + 1$  strands, we denote it  $\mathbb{P}$ . For  $u \in \mathbb{P}$ , we denote  $|u|$  the length of  $u$  as a word of  $\Sigma$ . The growth function of the positive braid monoid with respect to  $\Sigma$  is defined by:

$$F(t) = \sum_{b \in \mathbb{P}} t^{|b|_\Sigma}$$

Different ideas have been used to compute the growth function of the positive braid monoid. In [2], Brazil use the fact that, to each braid is associated a

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unique decomposition usually called its normal form (or its Garside’s normal form or Thurston’s normal form). This set of normal forms is regular, meaning that it is recognized by a finite state automaton. From this automaton’s adjacency matrix, we can obtain directly the growth function. But it is not a very efficient way to compute it, as for braids on  $n$  strands, this automaton has  $n!$  states and then it takes a time superexponential in the number of strands to get the formula. In [6], Dehornoy gives a method to reduce from  $n!$  to  $p(n)$  (where  $p(n)$  is the number of partitions of  $n$ ) the number of states of this automaton.

Bronfman (see [4]) and Krammer (see chapter 17 of [7]) give a new method to compute the growth function in quadratic time. They obtain the result of Theorem 2.1 and their proof is based on an inclusion-exclusion principle. We give here a different and bijective proof of this result.

To explain our point of view, let look at the history of results for trace monoids. Trace monoids (also called “heaps of pieces monoids”) denoted  $\mathbb{M}$  are defined by the following semigroup presentation:  $\mathbb{M} = \langle \Sigma \mid ab = ba \text{ if } (a, b) \in I \rangle$ , where  $\Sigma$  is a finite set of generators and  $I$  is a symmetric and antireflexive relation of  $\Sigma \times \Sigma$  called the commutation relation. In 1969, Cartier and Foata computed the growth function of these monoids by using an inclusion-exclusion principle to get a Möbius inversion formula (see [5]). In [8], Viennot gives a new and bijective way to compute the growth function of heaps of pieces monoids.

In the present paper, we show that Viennot’s proof can be extended to braid monoids, which gives a new point of view into the combinatorics of braids. Following Bronfman, we show that Viennot’s idea can actually be extended to a wider class of monoids which includes for instance Artin-Tits monoids [3] and Birman-Ko-Lee braid monoids [1] (see the full version of this article for more details: <http://www.liafa.jussieu.fr/~albenque/>).

## 2 Growth function of braid monoids

### 2.1 Presentation of braid monoids

We denote by  $\Sigma$  the set  $\{\sigma_1, \dots, \sigma_n\}$  and by  $\Sigma^*$  the free monoid on  $\Sigma$ . That is to say  $\Sigma^*$  is the set of finite words on the alphabet  $\Sigma$  with concatenation as monoid law. We denote by 1 the empty word. The *positive braid monoid*  $\mathbb{P}$  on  $n + 1$  strands has the following semigroup presentation:

$$\mathbb{P} = \langle \sigma_1, \dots, \sigma_n \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ and } \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| \leq 2 \rangle$$

We denote by  $\prec$  divisibility on the left in  $\mathbb{P}$  (that is, for  $a, b \in \mathbb{P}$ , we have  $a \prec b$  if and only if there exists  $c \in \mathbb{P}$  such that  $ac = b$ ) and denote similarly by  $\succ$  divisibility on the right. For  $u \in \mathbb{P}$ , set  $\text{left}(u) = \{\sigma \in \Sigma \text{ such that } \sigma \prec u\}$  and  $\text{right}(u) = \{\sigma \in \Sigma \text{ such that } u \succ \sigma\}$ .

For  $u$  a word of  $\Sigma^*$  such that  $u = v \cdot w$  with  $v, w \in \Sigma^*$ , we set:  $v^{-1} \cdot u = w$  and  $u \cdot w^{-1} = v$ .

## 2.2 Growth function for braid monoids

We define a new set  $\mathcal{G}$  of generators of  $\mathbb{P}$ . For all  $j, i \in \{1, \dots, n\}$  such that  $j + i \leq n + 1$ , set:  $\Delta_{\{j, j+1, \dots, j+i\}} = (\sigma_j)(\sigma_{j+1}\sigma_j) \cdots (\sigma_{j+i-1} \cdots \sigma_{j+1}\sigma_j)$ . The element  $\Delta_{\{j, \dots, j+i\}}$  is the half-turn of the strands  $j, j+1, \dots, j+i$ .

Set  $\mathcal{G}$  to be:

$$\mathcal{G} = \bigcup_{J_1 \cup \dots \cup J_p} (\Delta_{J_1} \cdots \Delta_{J_p}) \quad (1)$$

where  $J_1, \dots, J_p$  are disjoint subsets formed by consecutive integers of  $\{1, \dots, n+1\}$ , such that  $J_l < J_{l+1}$  for all  $l \in \{1, \dots, p-1\}$  (i.e.: if  $i_l \in J_l$  and  $i_{l+1} \in J_{l+1}$  then  $i_l < i_{l+1}$ ) and such that  $|J_l| \geq 2$  for all  $l \in \{1, \dots, p\}$ .

For  $g \in \mathcal{G}$ , let  $\tau(g)$  be the number of different Artin generators which appear in a representative of  $g$ . In other words:

$$\tau(g) = \sum_{i=1}^p (|J_i| - 1) \text{ if } g = \Delta_{J_1} \cdots \Delta_{J_p}. \quad (2)$$

We can now state our main result:

**Theorem 2.1** *In  $\mathbb{Z}\langle\langle \mathbb{P} \rangle\rangle$ , the following identity holds:*

$$\left( \sum_{g \in \mathcal{G}} (-1)^{\tau(g)} g \right) \cdot \left( \sum_{b \in \mathbb{P}} b \right) = 1 \quad (3)$$

**Corollary 2.2** *The growth function of the positive braid monoid is equal to:*

$$F(t) = \sum_{b \in \mathbb{P}} t^{|b|_\Sigma} = \left[ \sum_{g \in \mathcal{G}} (-1)^{\tau(g)} t^{|g|_\Sigma} \right]^{-1} \quad (4)$$

**Example 2.3** [Explicit formula on 4 strands] On Figure 1, we can read the value of  $\tau$  and the length of all elements of  $\mathcal{G}$  on 4 strands. The growth function is then:

$$F_4(t) = \frac{1}{1 - 3t + t^2 + 2t^3 - t^6}$$

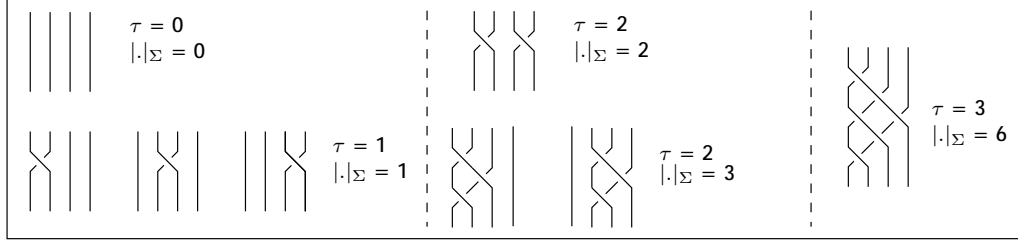


Fig. 1. The set  $\mathcal{G}$  on 4 strands

### 2.3 Definition of the involution of $(\mathcal{G} \times \mathbb{P})$

We construct an involution from  $(\mathcal{G} \times \mathbb{P})$  to itself with only  $(1, 1)$  as fixed point. This gives us a natural way to pair elements of  $(\mathcal{G} \times \mathbb{P}) \setminus (1, 1)$  and makes easy the counting of positive braids.

To construct the involution, given a couple  $(g, b) \in (\mathcal{G} \times \mathbb{P})$  we want to move “pieces” of  $g$  (respectively  $b$ ) from  $g$  to  $b$  (respectively from  $b$  to  $g$ ). Let us make it more precise; we define the set  $\mathcal{E}$  of elements which will be allowed to move:

$$\mathcal{E} = \{(\sigma_j \dots \sigma_{i+1} \sigma_i), \forall 1 \leq i \leq j \leq n\} \quad (5)$$

**Definition 2.4** Let  $(g, b)$  in  $(\mathcal{G} \times \mathbb{P})$  and  $u \in \mathcal{E}$  such that  $u \prec b$  (resp  $g \succ u$ ). We then set  $g' = g \cdot u$  and  $b' = u^{-1} \cdot b$  (resp.  $g' = g \cdot u^{-1}$  and  $b' = u \cdot b$ ). Now, if  $g'$  belongs to  $\mathcal{G}$  and  $\tau(g') = \tau(g) \pm 1$  then  $u$  is called an *eligible moving part* of  $(g, b)$ .

For  $(g, b) \neq (1, 1)$ , it exists at least one eligible moving part of  $(g, b)$ . In this case we consider the eligible moving part maximal for the lexicographic ordering induced by the ordering  $\sigma_1 < \sigma_2 < \dots < \sigma_n$  on the generators. We call it the *moving part* of  $(g, b)$  and denote it  $k$ . Since  $k$  cannot be both from  $g$  to  $b$  and from  $b$  to  $g$ , we can set:

$$\Psi(g, b) = \begin{cases} (1, 1) & \text{if } (g, b) = (1, 1) \\ (gk, k^{-1}b) & \text{if } k \text{ is from } b \text{ to } g \\ (gk^{-1}, kb) & \text{if } k \text{ is from } g \text{ to } b \end{cases}$$

**Lemma 2.5** *The function  $\Psi$  is an involution from  $(\mathcal{G} \times \mathbb{P})$  into itself whose unique fixed point is  $(1, 1)$ .*

### 3 Generalisation

We follow the work of Bronfman (see [4]) to extend our result to a larger class of monoids. We can indeed quite easily adapt our results and proof to monoids  $\mathbb{M}$  defined by  $\mathbb{M} = \langle \mathcal{S} | \mathcal{R} \rangle$  (where  $\mathcal{S}$  is a finite generating set and  $\mathcal{R}$  a finite set of relations) and such that the following properties hold:

- (i)  $\mathbb{M}$  is homogeneous (ie: all relations are  $\mathcal{R}$  are length-preserving)
- (ii)  $\mathbb{M}$  is left-cancellative, ie: if  $a, u, v \in \mathbb{M}$  are such that  $au = av$  then  $u = v$ .
- (iii) If a subset  $\{s_j | j \in J\}$  of the generating set  $\mathcal{S}$  has a common multiple, then this subset has a least common multiple.

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