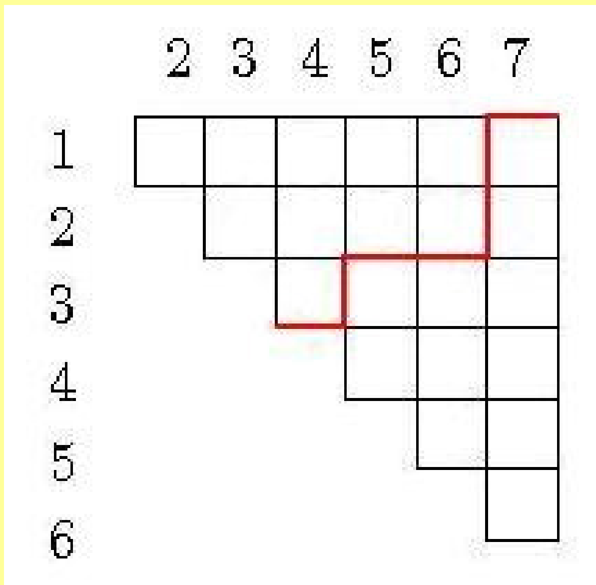
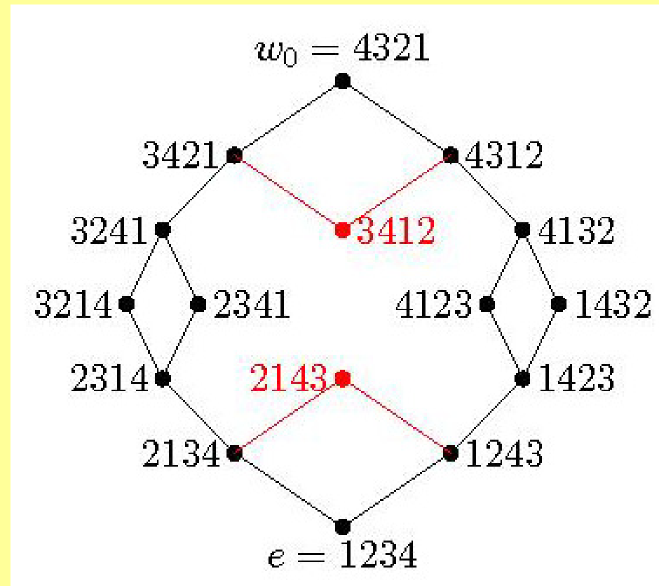
Arc Permutations





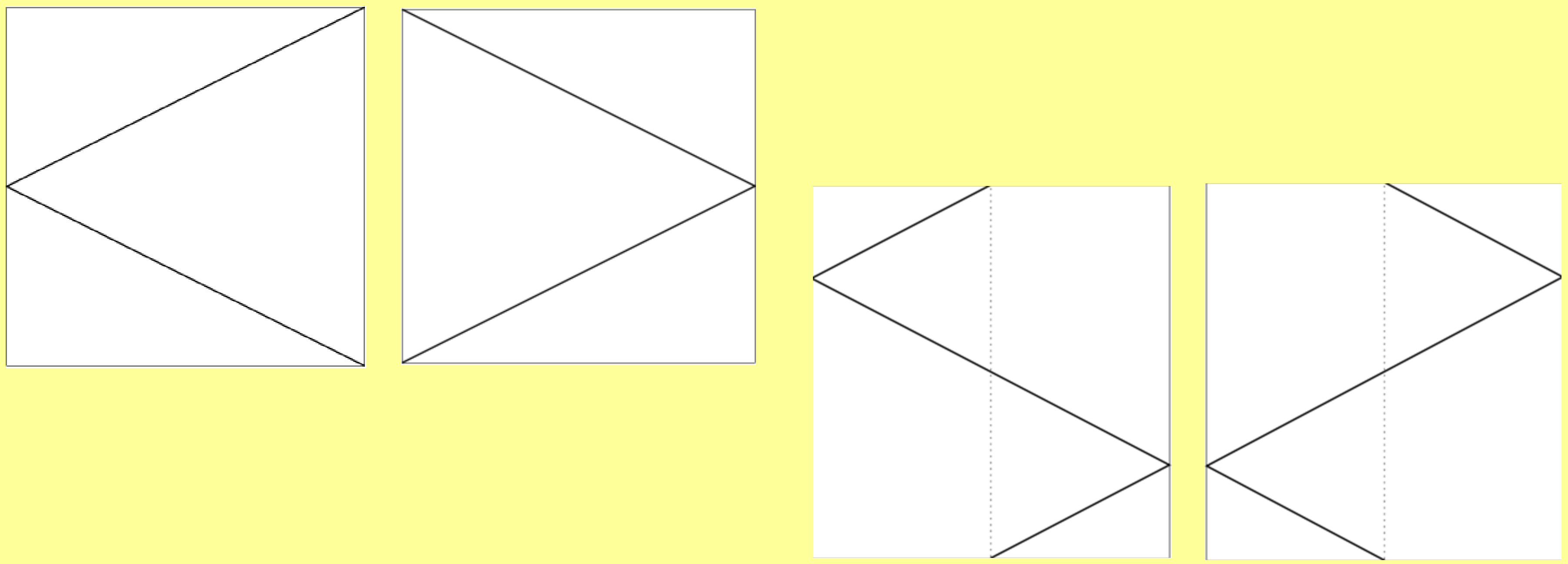
Sergi Elizalde and Yuval Roichman

Dartmouth College

Bar-Ilan University

Permutation Patterns Paris '13

A. Grid Classes



[a la Albert, Atkinson, Bouvel, Ruskuc, Vatter]

Baby case: Left Unimodal Permutations

A permutation $\pi \in S_n$ is left unimodal if its inverse is a unimodal sequence; namely, there exists j such that

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Example. $\pi = 435621 \in L_6$

$$\pi^{-1} = 652134$$

Left Unimodal Permutations

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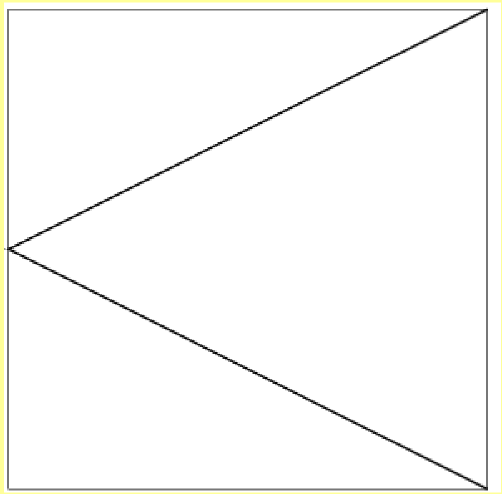

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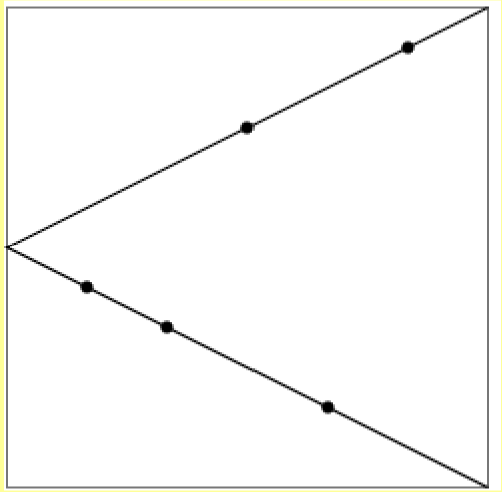

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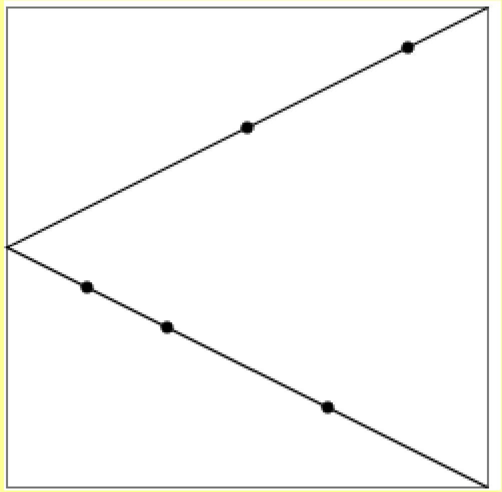

- (iv) $\pi \in Avoid\{132, 312\}$

Arc Permutations

A permutation $\pi \in S_n$ is an arc permutation if every prefix of π forms an interval in $\mathbb{Z}_n$

A_n - the set of arc permutations in S_n

Example.

$$\pi = 435612 \in A_6$$

Arc Permutations

Prop. TFAE

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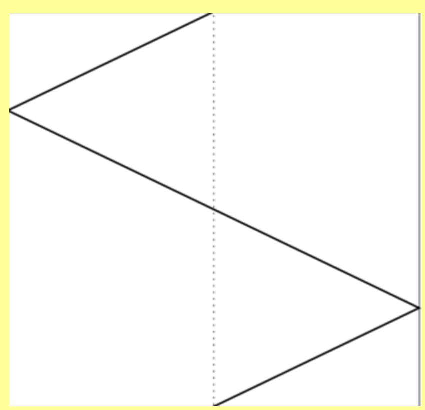

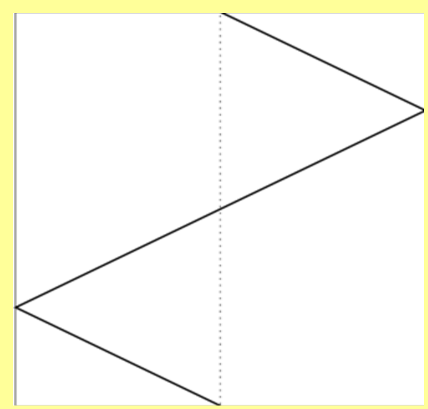

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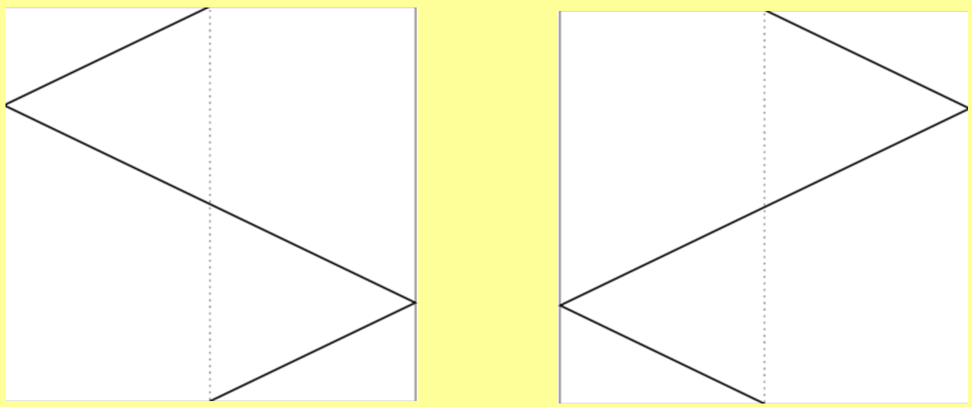
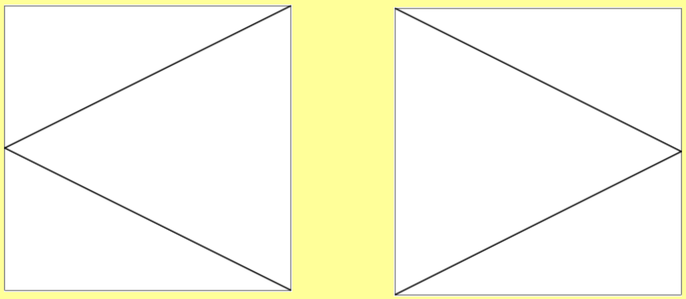


- (iii) $\pi \in Avoid\{\sigma \in S_4 : |\sigma(1) - \sigma(2)| = 2\}$

Unimodal Permutations

Prop. TFAE

- (i) π may be drawn on one of the grids



Unimodal Permutations

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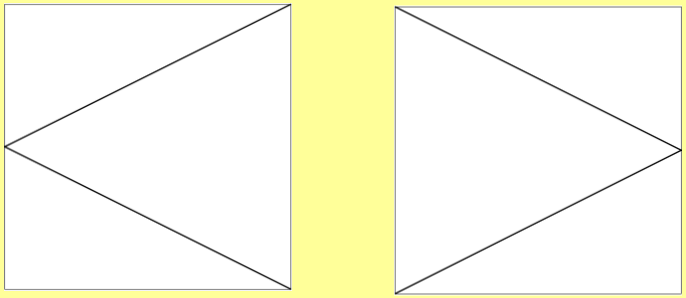
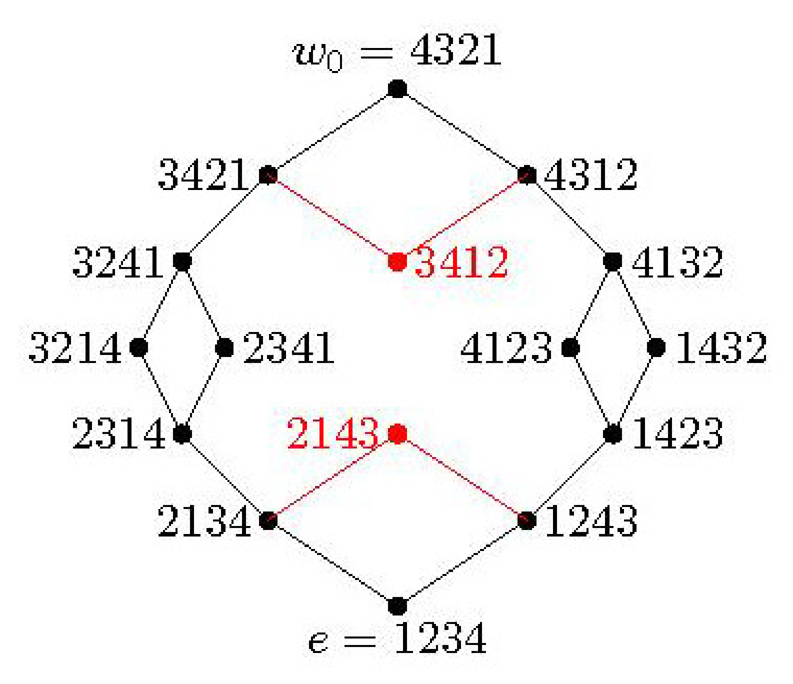


- (ii)

$$\pi \in Avoid\{\{\sigma \in S_4 : |\sigma(1) - \sigma(2)| = 2\} \bigcup \{2143, 3412\}\}$$

B. Graph Structure



Γ_4

The Graph Γ_n

- Vertices: A_n
- Edges: $\{(\pi,\sigma) : \pi^{-1}\sigma \in S\}$ $S := \{(i,i+1) : 1 \le i < n\}$

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Theorem

$$Diameter(\Gamma_n) = Diameter(X(S_n, S)) = \binom{n}{2}$$

The graph $X(S_n, S)$

Antipodes $e = 1, 2, \ldots, n$ and $w_0 = n, n-1, \ldots, 1$

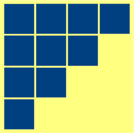
Theorem [Stanley]

a) # vertices in geodesics from e to w_0

$$= n!$$

b) # geodesics from e to w_0

$$= \binom{n}{2}! \prod_{i=0}^{n-2} \frac{1}{(2i+1)^{\binom{n}{2}-i-1}}$$

= # SYT 

The graph Γ_n

Antipodes $e = 1, 2, \ldots, n$ and $w_0 = n, n-1, \ldots, 1$

Theorem

a) # vertices in geodesics from e to w_0

$$= 2^n - 2$$

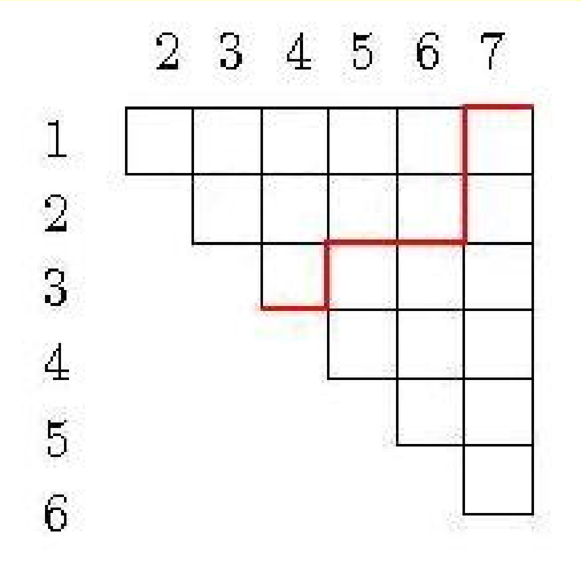
b) # geodesics from e to w_0

$$= 2\binom{n}{2}! \prod_{i=0}^{n-2} \frac{i!}{(2i+1)!}$$

= 2 # SYT 

Proof Idea

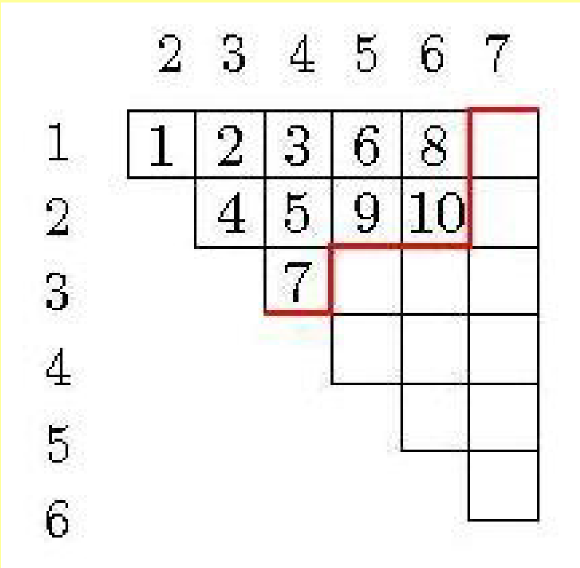
Step 1. Bijection from shifted shapes to L_n



$\rightarrow 4356217$

Proof Idea

Step 2. Bijection from SYT to subset of reduced words



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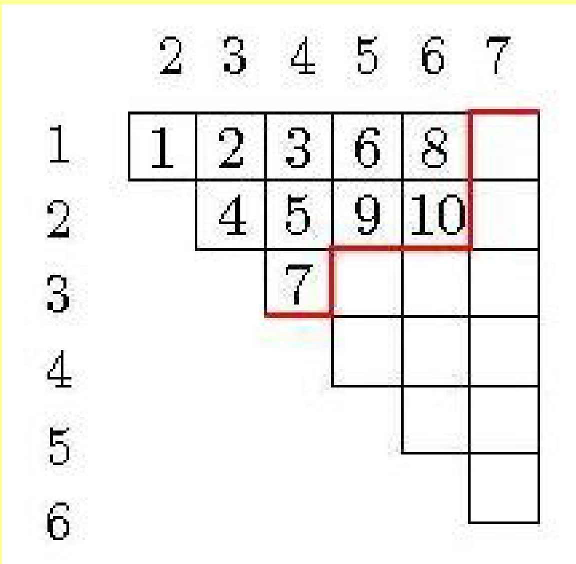

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$$= (4,5)(3,4)(5,6)(1,2)(4,5)(2,3)(1,2)(3,4)(2,3)(1,2)$$

Proof Idea

Cor. 1. (description of *Weak*(L_n))

$$\pi < \sigma \Leftrightarrow shape(\pi) \subset shape(\sigma)$$

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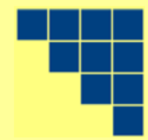
Cor 2. # maximal chains in *Weak*(L_n) = # SYT 

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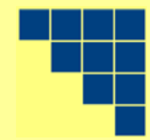
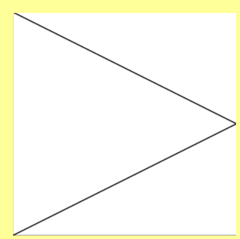
Cor 2. # maximal chains in *Weak*(L_n) = # SYT 

Lemma.

$$\pi \in [e, w_0] \Leftrightarrow \pi \in L_n \bigcup R_n$$

where

$R_n := w_0 L_n w_0 =$ grid class 

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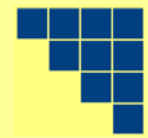
Cor 2. # maximal chains in *Weak*(L_n) = # SYT 

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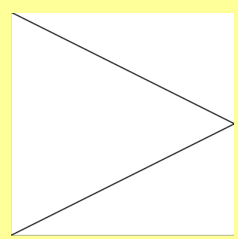
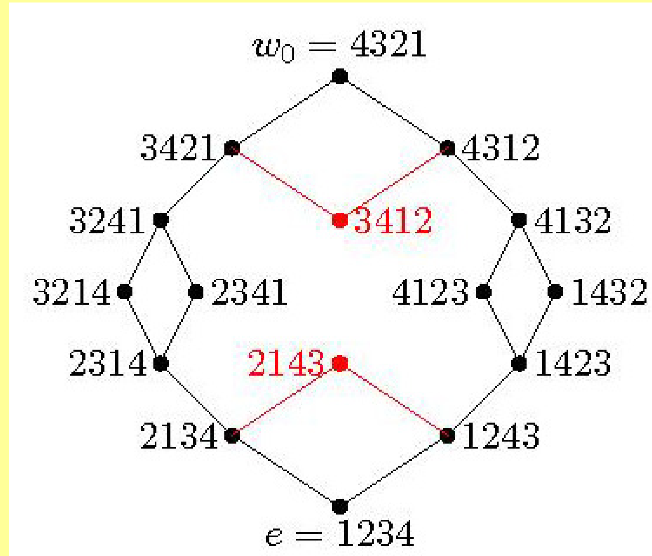
$$R_n := w_0 L_n w_0 = \text{ grid class}$$ 



C. Descents

Fact

$$\sum_{\pi \in L_n} X^{Des(\pi)} = \sum_{\pi \in R_n} X^{Des(\pi)} = \sum_{T \in hook} X^{Des(T)}$$

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Theorem

$$\sum_{\pi \in A_n \setminus (L_n \cup R_n)} X^{Des(\pi)} = \sum_{T \in hook+box} X^{Des(T)}$$

C. Descents - Applications

- Proof of Regev's conjectured character formula for induced exterior algebras

- Schur positivity of certain quasi-symmetric functions (a la Gessel-Reutenauer).

Future …

- Multivariate statistics
- Other subsets of permutations
  whose descent set is equidistributed with SYT of given shapes
- Other grid classes
- Other Coxeter types
- Characters and spectra

MERCI

GRACIAS

THANK YOU

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