About half permutations

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Basic definitions

A *permutomino* of size $n$ is a polyomino (with no holes) having $n$ rows and $n$ columns, such that for each abscissa (ordinate) between 1 and $n + 1$ there is exactly one vertical (horizontal) bond in the boundary of $P$ with that coordinate.
Basic definitions

A permutomino $P$ of size $n$ is uniquely defined by a pair of permutations of length $n + 1$, denoted by $\pi_1(P)$ and $\pi_2(P)$, called the first and the second components of $P$, respectively.

$$\pi_1 = (6, 3, 9, 8, 12, 11, 13, 1, 5, 10, 4, 7, 2)$$
A permutomino $P$ of size $n$ is uniquely defined by a pair of permutations of length $n + 1$, denoted by $\pi_1(P)$ and $\pi_2(P)$, called the first and the second components of $P$, respectively.

$\pi_2 = (9, 6, 8, 13, 11, 10, 12, 3, 4, 7, 2, 5, 1)$
Directed column-convex permutominoes

Definition
A permutomino $P$ is said to be column-convex if all its columns are connected.

Definition
A permutomino $P$ is said to be directed column-convex if it is a column-convex permutomino and all its cells can be reached from a distinguished cell – called source – by means of a path, internal to the permutomino, and using only north and east unit steps.
Proposition (Beaton, Disanto, Guttman, Rinaldi, 2010)

The number of directed column-convex permutominoes of size $n$ is $\frac{(n+1)!}{2}$.
Directed column-convex permutominoes

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Remark

The authors prove this result analytically.
We present a bijective proof that the number of directed column-convex permutominoes of size $n$ is $\frac{(n+1)!}{2}$.
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We prove that:

- every directed column-convex permutomino $P$ is uniquely determined by its second component $\pi_2(P)$;
- the set

$$\{\pi_2(P) : P \text{ is a directed column-convex permutomino of size } n \}$$

is in bijective correspondence with its complement in $S_{n+1}$, where $S_{n+1}$ denotes the set of permutations of length $n + 1$. 

S.Rinaldi, S.Socci (Università di Siena)
Proposition

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Proof.

Let $\pi = \pi_2(P)$ for some directed column-convex $P$. 
Enumeration of directed column-convex permutominoes

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Let $\pi = \pi_2(P)$ for some directed column-convex $P$.

- $\pi(1)$ is connected with $\pi(i) = 1$ (directed);
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Let $\pi = \pi_2(P)$ for some directed column-convex $P$.

- $\pi(1)$ is connected with $\pi(i) = 1$ (directed);
- the right-to-left minima of $\pi$ have to be connected in sequence (directed);
**Proposition**

A directed column-convex permutomino $P$ is uniquely determined by its second component $\pi_2(P)$.

**Proof.**

Let $\pi = \pi_2(P)$ for some directed column-convex $P$.

- $\pi(1)$ is connected with $\pi(i) = 1$ (directed);
- the right-to-left minima of $\pi$ have to be connected in sequence (directed);
- the remaining entries of $\pi$ have to be connected in sequence (column-convex).
Definition

We define

\[ P''_n = \{ \pi : \pi = \pi_2(P) \text{ for some } P \in D_{n-1} \}. \]

The permutations of \( P''_n \) will be called \textit{dcc-permutations}. 
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We provide

- a characterization of dcc-permutations of size \( n \);
- a bijective correspondence between dcc-permutations of length \( n \) and non dcc-permutations of length \( n \).
**Definition**

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The permutations of $P''_n$ will be called *dcc-permutations*.

We provide

- a characterization of dcc-permutations of size $n$;
- a bijective correspondence between dcc-permutations of length $n$ and non dcc-permutations of length $n$.

And so we prove in a **bijective** way that

$$|D_{n-1}| = \frac{n!}{2}.$$
Characterization of dcc-permutations

Definition

- $\mathcal{R}(\pi)$: right-to-left minima of $\pi$;
- $\overline{\mathcal{R}}(\pi)$: $(\pi(j - 1), \pi(j), \ldots, \pi(n))$ of $\pi$ minus the points of $\mathcal{R}(\pi)$, where $\pi \in S_n$ ($n > 1$) with $\pi(1) \neq 1$ and $\pi(j) = 1$;
- $L(\pi)$: the rightmost element of $\overline{\mathcal{R}}(\pi)$.
Characterization of dcc-permutations

Definition

Let $\pi \in S_n$ such that $\pi(1) \neq 1$, for each $X \in \overline{R} - \{L\}$,
- $Y$: the leftmost point of $\overline{R}$ on the right of $X$;
- $Z$: the leftmost point of $R$ on the right of $Y$.

We set $C_X = (X, Y, Z)$. 
Theorem

A permutation \( \pi \in S_n \) is a dcc-permutation if and only if the following properties hold:

i) \( \pi(1) \neq 1 \);

ii) \( \forall X \in \mathcal{R}(\pi) - \{L\}, \ C_X = (X, Y, Z), \ we \ have \ X > Z \);

iii) \( L > \pi(n) \).
Characterization of dcc-permutations

Theorem

A permutation $\pi \in S_n$ is a dcc-permutation if and only if the following properties hold:

i) $\pi(1) \neq 1$;

ii) $\forall X \in \overline{R}(\pi) - \{L\}$, $C_X = (X, Y, Z)$, we have $X > Z$;

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The conditions ii) and iii) express formally when the boundary of the permutomino crosses itself.
Characterization of dcc-permutations

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Theorem

The number of dcc-permutations of length $n$ is $\frac{n!}{2}$. 
A bijection for dcc-permutations

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Proof.

We determine a bijective correspondence $\phi : S_n \setminus \mathcal{P}_n'' \rightarrow \mathcal{P}_n''$. 

case 1) $\pi(1) = 1$

\[\pi = (1, 6, 8, 7, 4, 2, 10, 12, 5, 11, 3, 9)\]
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\begin{align*}
\pi &= (1, 6, 8, 7, 4, 2, 10, 12, 5, 11, 3, 9) \\
\phi(\pi) &= (6, 8, 7, 4, 2, 10, 12, 5, 11, 3, 9, 1)
\end{align*}
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  \item [case 2)] $\pi$ satisfies i) but not ii)
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\[ \pi = (6, 12, 8, 1, 4, 2, 3, 5, 9, 7, 10, 11) \]

Let $X$ be the leftmost of the elements which do not satisfy ii).

We exchange $X$ with $\pi(n)$.
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case 2) $\pi$ satisfies i) but not ii)

Let $U$ be the rightmost right-to-left minimum of $\phi(\pi)$ different from $\phi(\pi)(n)$.

Let $V$ be the rightmost element in $\overline{R}(\phi(\pi))$ on the left of $U$.

We exchange $V$ with $\phi(\pi)(n)$.

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A bijection for dcc-permutations

**Theorem**

*The number of dcc-permutations of length $n$ is $\frac{n!}{2}$.*

**Proof.**

We determine a bijective correspondence $\phi : S_n \setminus \mathcal{P}_n'' \rightarrow \mathcal{P}_n''$. 

**case 3)** $\pi$ satisfies i) and ii) but not iii)

$$\pi = (4, 7, 2, 5, 1, 11, 12, 8, 3, 6, 9, 10)$$
A bijection for dcc-permutations

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The number of dcc-permutations of length $n$ is $\frac{n!}{2}$.

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We determine a bijective correspondence $\phi : S_n \setminus \mathcal{P}_n'' \to \mathcal{P}_n''$. 

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\pi = (4, 7, 2, 5, 1, 11, 12, 8, 3, 6, 9, 10)
\]
\[
\phi(\pi) = (4, 7, 2, 5, 1, 11, 12, 10, 3, 6, 9, 8)
\]
Combinatorial characterizations of dcc-permutations

\( \pi \) decomposable: there is an index \( i < n \) s.t. \((\pi(1), \ldots, \pi(i))\) is a permutation.

\( \pi \) m-decomposable: if its mirror image \( \pi^M \) is decomposable.

\( \pi = (4, 3, 5, 1, 2, 8, 7, 6) \) decomposable

\( \pi = (5, 6, 8, 7, 4, 3, 1, 2) \) m-decomposable
Let $P$ be a *column-convex* permutomino of size $n$, let $\pi_1$ be the first component of $P$, and let $U_i = (i, \pi_1(i))$, $1 \leq i \leq n + 1$, be the points of the graphical representation of $\pi_1$.

We call *upper* (resp. *lower*) path of $P$ the part of the boundary of $P$ running from $U_1$ to $U_{n+1}$ and starting with a north step (resp. east step).
Combinatorial characterizations of dcc-permutations

We define a valuation $\nu$ on the points of a permutation $\pi = \pi_1(P)$ for some column-convex permutomino $P$ of size $n$ in this way:

- $\nu(U_i) = 1$ iff $U_i$ belongs to the upper path or $i = n + 1$;
- $\nu(U_i) = 0$ iff $U_i$ belongs to the lower path or $i = 1$;

Remark

A column-convex permutomino $P$ of size $n$ is uniquely determined by $\pi_1(P)$, and by the array $\nu(\pi_1) = (\nu(U_1), \ldots, \nu(U_{n+1}))$.

\[
\pi_1 = (2, 6, 3, 5, 1, 7, 4)
\]

\[
\nu(\pi_1) = (0, 1, 0, 1, 0, 1, 1)
\]
Definition

The pair \((U_i, U_j)\) forms an *inversion* if and only if \(i < j\) and \(\pi(i) > \pi(j)\).

The array \([U_i, U_j] = (U_i, U_{i+1}, \ldots, U_j)\) is a *locally decomposable (m-decomposable)* permutation if the normalization of \((\pi(i), \pi(i+1), \ldots, \pi(j))\) is a decomposable \((m\text{-decomposable})\) permutation.
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\[\text{[U1, U2] locally decomposable permutation}\]
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\[\begin{align*}
[U_1, U_2] \text{ locally decomposable permutation} \\
[U_2, U_5] \text{ locally } m\text{-decomposable permutation}
\end{align*}\]
Combinatorial characterizations of dcc-permutations

Given $\pi \in S_n$ we define a set of logic implication formulas $\mathcal{F}(\pi)$ on the variables $\mathcal{U} = \{U_1, \ldots, U_n\}$ in this way:

**Definition**

For any pair $U_i, U_j \in \mathcal{U}$ we have that $U_j \rightarrow U_i \in \mathcal{F}(\pi)$ if and only if

- $(U_i, U_j)$ is an inversion;
- the array $[U_i; U_j]$ is a locally $m$-decomposable permutation.

$$\pi = (2, 6, 3, 5, 1, 7, 4)$$

$$\mathcal{F}(\pi) = \{U_3 \rightarrow U_2, U_4 \rightarrow U_2, U_5 \rightarrow U_1U_2U_3U_4, U_7 \rightarrow U_6\}$$
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\[
\pi = (2, 6, 3, 5, 1, 7, 4)
\]

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\mathcal{F}(\pi) = \{U_3 \rightarrow U_2, U_4 \rightarrow U_2, U_5 \rightarrow U_1U_2U_3U_4, U_7 \rightarrow U_6\}
\]
We define:

\[ C'_n = \{ \pi_1(P) : P \text{ column-convex permutomino of size } n - 1 \} \]
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**Theorem**

A permutation \( \pi \in C'_n \) if and only if \( F(\pi) \) is satisfiable.
We define:

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**Theorem**

A permutation \( \pi \in C'_n \) if and only if \( \mathcal{F}(\pi) \) is satisfiable.

**Remark**

Each valuation \( v \) that satisfies \( \mathcal{F}(\pi) \) corresponds to a column-convex permutomino \( P \) of size \( n - 1 \) such that \( \pi = \pi_1(P) \).
Remark

Given a permutomino $P$, the first component of $P$ is just the mirror image of the second component of the polyomino $P^M$ obtained by reflecting $P$ with respect to the $y$-axis. Namely,

$$\pi_1(P) = (\pi_2(P^M))^M.$$
The valuation \( \hat{v} \) of \( \pi \) is defined as follows:

\[
\hat{v}(U_i) = 0 \text{ if and only if } U_i \text{ is a left-to-right minimum.}
\]
Combinatorial characterizations of dcc-permutations

The valuation $\hat{v}$ of $\pi$ is defined as follows:

$$\hat{v}(U_i) = 0 \text{ if and only if } U_i \text{ is a left-to-right minimum.}$$

Proposition

A permutation $\pi$ is a dcc-permutation if and only if the valuation $\hat{v}$ satisfies $\mathcal{F}(\pi^M)$. 
Theorem

A permutation $\pi$ of length $n$ is a dcc-permutation if and only if:

1. $\pi(1) \neq 1$,
2. $\mathcal{F}(\pi^M)$ is satisfiable,
3. for every implication $U_i \rightarrow U_1$ belonging to $\mathcal{F}(\pi^M)$, we have that $U_i$ is a left-to-right minimum.
Theorem

A permutation $\pi$ of length $n$ is a dcc-permutation if and only if:

- $\pi(1) \neq 1$,
- $\mathcal{F}(\pi^M)$ is satisfiable,
- for every implication $U_i \rightarrow U_1$ belonging to $\mathcal{F}(\pi^M)$, we have that $U_i$ is a left-to-right minimum.

Corollary

A permutation $\pi$ of length $n$ is a dcc-permutation if and only if $\pi(1) \neq 1$ and there is no point $U_i$ of $\pi$ such that $[U_i, U_n]$ is a locally decomposable permutation and $U_i$ is not a right-to-left minimum.
The previous result can be used to provide a characterization of the class of dcc-permutations in terms of \textit{mesh patterns}.

\textbf{Theorem}

A permutation $\pi$ is a dcc-permutation if and only if $\pi$ avoids the mesh patterns represented below.

\begin{itemize}
  \item \begin{tikzpicture}[scale=0.5]
    \draw[very thin, lightgray] (0,0) grid (3,3);
    \foreach \x in {0,1,2,3} {
      \foreach \y in {0,1,2,3} {
        \fill[black] (\x,\y) circle (0.1cm);
      }
    }
  \end{tikzpicture}
  \item \begin{tikzpicture}[scale=0.5]
    \draw[very thin, lightgray] (0,0) grid (2,2);
    \fill[black] (0,0) circle (0.1cm);
    \fill[black] (1,1) circle (0.1cm);
  \end{tikzpicture}
\end{itemize}
Combinatorial characterizations of dcc-permutations

Theorem

A permutation \( \pi \in S_n \) is a dcc-permutation if and only if \( \pi \) avoids the mesh patterns represented below.
The class $B_n$ and its enumeration

Let $B_n$ be the class of permutations avoiding the mesh pattern

\[
\begin{array}{cc}
\cdot & \cdot \\
\cdot & \\
\cdot & \\
\end{array}
\]
The class $\mathcal{B}_n$ and its enumeration

Let $\mathcal{B}_n$ be the class of permutations avoiding the mesh pattern

![Mesh Pattern Diagram]

**Proposition**

*We have that:*

$$|\mathcal{B}_n| = 1 + \sum_{i=2}^{n} \frac{i!}{2}.$$
Further works

Enumeration of directed column-convex permutominoes according to the semi-perimeter

Let $P$ be directed column-convex permutomino of size $n$.

$$\deg(P) = sp(P) - 2n.$$

$\mathcal{D}_{n,k}$: directed column-convex permutominoes of size $n$ and degree $k$. 

![Diagram of directed column-convex permutominoes]
Further works

- $D_{n,0}$: directed convex permutominoes of size $n$, whose number is given by $\binom{2n-1}{n}$ (Disanto, Duchi, Pinzani, Rinaldi, 2012).
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- We have proved that $|\mathcal{D}_{n,1}| = \frac{(2n-3)(n-2)}{n} \binom{2n-4}{n-2}$.
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- We have proved that $|D_{n,1}| = \binom{2n-3}{n-2} \binom{2n-4}{n-2}$.

Open problem

Enumerate $D_{n,k}$ for $k > 1$. 
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Thank you!!