Wilf Equivalence of Interval Embeddings

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Permutation Patterns 2013
Université Paris Diderot
July 3, 2013

Outline

1. Definitions & Background

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2. Generalized Interval Embeddings
3. Results & Future Work

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  - "Wilf equivalence for g.f.o. modulo k," preprint, by Langley, Liese, and Remmel

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We say that $u \in \mathbb{N}^*$ is a *factor* of $v \in \mathbb{N}^*$ if there exist $w_1, w_2 \in \mathbb{N}^*$ such that $v = w_1 u w_2$. If $w_1 = \epsilon$ ($w_2 = \epsilon$), then we say that u is a prefix (suffix) of v.

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Given any poset $P = (\mathbb{N}, \leq_P)$ and two words $u, w \in \mathbb{N}^*$, we say that there is an *embedding u into w* if there exists a factor $z = z_1 z_2 \cdots z_k$ of w such that for every $1 \leq i \leq k$, $u_i \leq_P z_i$.

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$$\mathcal{S}^P(u, x, t) = \mathcal{S}^P(v, x, t).$$

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- The functions $\mathcal{E}^P, \mathcal{A}^P, \mathcal{S}^P$ are rational for any choice of u.
- $u \sim u^r$
- If $u \sim v$, then $1u \sim 1v$ and $u^+ \sim v^+$.

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A separate conjecture from the first Langley, Liese, and Remmel paper:

- (Strong Rearrangement Conjecture) If $u \sim v$, then there is a weight-preserving bijection $f : \mathbb{N}^* \to \mathbb{N}^*$ such that if $w \in \mathcal{S}^P(u, x, t)$, then $f(w) \in \mathcal{S}^P(v, x, t)$ and $w, f(w)$ are rearrangements.

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1. $u_i \leq z_i$, and
2. $u_i \equiv z_i \mod m$.

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Define $\vec{U} = \{\mathcal{I}^{\mathcal{P}}_{m_1,n_1}, \mathcal{I}^{\mathcal{P}}_{m_2,n_2}, \ldots, \mathcal{I}^{\mathcal{P}}_{m_k,n_k}\}$, where for each $1 \leq i \leq k$, $m_i \leq_{\mathcal{P}} n_i$ with either $m_i, n_i \in \mathbb{N}$ or $m_i \in \mathbb{N}$ and $n_i = \infty$.

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We say that w contains an *interval embedding of* $\vec{U}$ *relative to* $\mathcal{P}$ if there is a factor z of w such that for every $1 \leq i \leq k$, $z_i \in \mathcal{I}^{\mathcal{P}}_{m_i,n_i}$.

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As an example, if $\mathcal{P} = (\mathbb{N}, \leq)$, then $w = 3\underline{396}2435112$ contains an interval embedding of $\vec{U} = \{[2,4],[7,12],[3,7]\}$, but avoids $\vec{V} = \{[4,4],[2,7],[6,9]\}$

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We say that $\vec{U}$ and $\vec{V}$ are *interval-Wilf equivalent* with respect to $\mathcal{P}$, denoted at $\vec{U} \sim_{\mathcal{P}} \vec{V}$, if

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Relation to Previous Models

If $\mathcal{P} = (\mathbb{N}, \leq)$ and $\vec{U} = \{\mathcal{I}^{\mathcal{P}}_{m_1,n_1}, \mathcal{I}^{\mathcal{P}}_{m_2,n_2}, \ldots, \mathcal{I}^{\mathcal{P}}_{m_k,n_k}\}$ with $n_i = \infty$ for all i, then this is the K.L.R.S. version of embedding the word $u = m_1 m_2 \cdots m_k$.

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If $\mathcal{P}^m = (\mathbb{N}, \leq)$ and $\vec{U} = \{B^{\mathcal{P}}_{m_1}, B^{\mathcal{P}}_{m_2}, \ldots, B^{\mathcal{P}}_{m_k}\}$ with $B_{m_j} = \{m_j + qm \mid q \in \{0\} \cup \mathbb{N}\}$ for each j, then this is the L.L.R. version of (modular) embedding the word $u = m_1 m_2 \cdots m_k$.

Interval-Wilf Equivalence

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As an example, suppose $\mathcal{P} = (\mathbb{N}, \leq)$, $\vec{U} = \{\mathcal{I}^{\mathcal{P}}_{3,8}, \mathcal{I}^{\mathcal{P}}_{1,8}, \mathcal{I}^{\mathcal{P}}_{2,8}\}$, and $\vec{V} = \{\mathcal{I}^{\mathcal{P}}_{2,8}, \mathcal{I}^{\mathcal{P}}_{1,8}, \mathcal{I}^{\mathcal{P}}_{3,8}\}$.

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Theorem

If $\mathcal{P} = (\mathbb{N}, \leq)$, then $\vec{U} \sim_{\mathcal{P}} \vec{U}^r$.

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Rationality of Generating Functions

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If $\mathcal{P} = (\mathbb{N}, \leq)$ *and* $\vec{U}$ *is a sequence of "continuous" intervals, then the functions* $\mathcal{S}^{\mathcal{P}}(\vec{U}, x, t)$, $\mathcal{E}^{\mathcal{P}}(\vec{U}, x, t)$, *and* $\mathcal{A}^{\mathcal{P}}(\vec{U}, x, t)$ *are all rational.*

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A counterexample in the case of "noncontinuous" intervals can be found in the modulo k L.L.R. paper.

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Rearrangment

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Let $\mathcal{P} = (\mathbb{N}, \leq)$ and fix $n \in \mathbb{N}$. If $\vec{U} \sim_{\mathcal{P}} \vec{V}$ with $\vec{U} = \{\mathcal{I}^{\mathcal{P}}_{m_1,n}, \mathcal{I}^{\mathcal{P}}_{m_2,n}, \ldots, \mathcal{I}^{\mathcal{P}}_{m_k,n}\}$ and $\vec{V} = \{\mathcal{I}^{\mathcal{P}}_{r_1,n}, \mathcal{I}^{\mathcal{P}}_{r_2,n}, \ldots, \mathcal{I}^{\mathcal{P}}_{r_\ell,n}\}$, then $\vec{U}$ and $\vec{V}$ are rearrangements of one another.

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This was another conjecture in the K.L.R.S. paper, and a proof was given at PP2012.

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Theorem

Let $B \subseteq A$ and suppose $s, t, n \geq 0$ with $s + t = n$. Then $\vec{U} = \{B^n, A\} \sim \vec{V} = \{B^s, A, B^t\}$.

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Let $\vec{U} = (U_1, U_2, \ldots, U_n)$, $\vec{V} = (V_1, V_2, \ldots, V_n)$ *with* $\vec{U} \sim \vec{V}$. *Then* $\vec{U}^+ \sim \vec{V}^+$, *where* $\vec{U}^+$ *is obtained by sending* $U_i = [a_i, b_i] \mapsto U_i^+ = [a_i + 1, b_i + 1]$.

Defintions

Given the sequence of intervals $\vec{U} = (U_1, U_2, \ldots, U_n)$, define $d(U_i)$ to be the number of elements in U_i if U_i is finite, and let $d(U_i) = \infty$ otherwise.

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Given two interval sequences $\vec{U}$ and $\vec{V}$ for which $d(\vec{U}) = d(\vec{V})$, let $\Delta(\vec{U}, \vec{V}) = (\delta_1, \delta_2, \ldots, \delta_n)$, where given $U_i = [a_i, b_i]$ and $V_i = [c_i, d_i]$, $\delta_i = c_i - a_i$.

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For example, in the case that $\vec{U} = ([1,2],[3,4],[7,9],[8,\infty))$ and $\vec{V} = ([2,3],[2,3],[6,8],[9,\infty])$, then

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Theorem

Let $\vec{U} = (U_1, U_2, \ldots, U_n)$ and $\vec{V} = (V_1, V_2, \ldots, V_n)$ be non-overlapping sequences and suppose there exists $\sigma \in \mathcal{S}_n$ such that

1. $d(\vec{U}) = \sigma(d(\vec{V}))$ *and*
2. $\Delta(\vec{U}, \sigma(\vec{V})) = 0.$

Then $\vec{U} \sim \vec{V}$.

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Example

Let $\vec{U} = ([1,1],[4,9],[5,10])$ and $\vec{V} = ([1,1],[2,7],[7,12])$, where $\Delta(\vec{V},\vec{U}) = (0,2,-2)$.

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Consider $w = 129912\overline{1}\overline{1}$, which avoids $\vec{U}$ but contains two occurrences of $\vec{V}$.

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We see that $\phi(w)$ avoids $\vec{V}$ but not $\vec{U}$.

Future Work

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2. Keep working on the rearrangement conjecture

Thank you.