Frame Patterns in Words

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The pattern μ

- Avgustinovich, Kitaev, and Valyuzhenich studied avoidance in permutations.
- Jones, Kitaev, and Remmel studied distribution in cycles of permutations.
- Goal: Study distribution in words over $[k] = \{1, 2, \ldots, k\}$.

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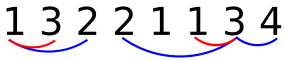
$$w = w_1 w_2 \ldots w_n$$

Definition

The pair (w_i, w_j) is an occurrence of the frame pattern μ in w if $i < j$, $w_i < w_j$, and there is no $i < l < j$ such that $w_i \leq w_l \leq w_j$.

1 3 2 2 1 1 3 4

Generating functions



$$A_k(x, y, t) = \sum_{w \in [k]^*} x^{triv(w)} y^{nontriv(w)} t^{|w|}$$

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Theorem

$$[x^a y^b t^n] A_k(x,y,t) = \sum_{s=1}^{n} \binom{n-1}{s-1} [x^a y^b t^s] N_k(x,y,t)$$

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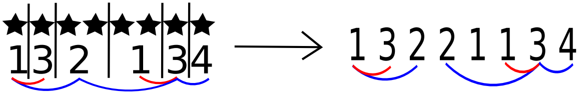
example: k=4, a=3, b=2, n=8, s=6



Recurrence for $N_k(x, y, t)$

- classify words based on some finite initial segment
- use this information to account for all matches involving w_1
- reduce to $w_2 \dots w_n$

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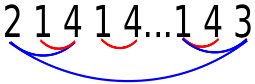
When $k \geq 4$, it is not possible to use this technique to find a recurrence for $N_k(x, y, t)$ because of words like this:



State matrices

Read the word one letter at a time, recording potential matches.

1 3 2 1 3 4

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} x & 0 & 0 & 0 \\ x & x & 0 & 0 \\ x & x & x & 0 \\ x & x & x & x \end{pmatrix} \end{array}$$

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\hline
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3. In a nonzero row, the first zero entry indicates a match that is present in the word. Changing from 1 to 0 indicates completion of a match.
4. A nonzero state matrix completely determines the last letter w_n of the associated word.

Counting valid state matrices

$S_k = \#\{\text{state matrices for words over } [k]\}$
$S_{k,j} = \#\{\text{state matrices for words over } [k] \text{ ending in } j\}$

Goal: Find $S_k = \sum_{j=1}^{k} S_{k,j}$.

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k=6, j=4

	1	2	3	4	5	6
1	x			0	0	0
2	x	x		0	0	0
3	x	x	x	0	0	0
4	x	x	x	x	1	1
5	x	x	x	x	x	
6	x	x	x	x	x	x

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x & & & 0 & 0 & 0 \\
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$S_{6,4} = S_3 S_2$

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$S_{6,4} = \color{red}{S_3}\color{blue}{S_2}$
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Theorem

$S_k = C_k = \frac{1}{k+1}\binom{2k}{k}$

S_k satisfies the Catalan recurrence.

Transition diagram

To go from one state to another, append one of $k-1$ letters to avoid consecutive repeats. Record completed μ-matches and the change in length as the edge weight.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} x & 1 & 0 & 0 \\ x & x & 0 & 0 \\ x & x & x & 0 \\ x & x & x & x \end{pmatrix} \end{array}$$

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add 1

t

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add 1 (t) ↙ add 2 (xt) ↓

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add 3, t →

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Finding $N_k(x, y, t)$

The transition diagram gives rise to a system of equations where each state is a variable. Solving the system involves inverting a sparse symbolic matrix of size $C_k \times C_k$, which is only possible for small values of k.

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$$A_{i,j} = \text{weight of the edge from state } j \text{ to state } i$$

For a given value of p, however, it is not difficult to find the terms from $N_k(x, y, t)$ with $deg(t) = p$:

$$t^p[t^p]N_k(x, y, t) = \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix} A^p \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Here, the 1 is in the position corresponding to the start state.

$N_2(x, y, t)$

$$2\ 1\ 2\ 1\ 2 \ldots 1\ 2$$

$$N_2(x, y, t) = \frac{(1+t)^2}{1 - xt^2}$$

$N_3(x, y, t)$

$$N_3(x, y, t) = \frac{(1+t)^2(-1-t+t^3(-1+x)^2y)}{-1+2t^4(-1+x)xy+t^3x^2(1+y)+t^2(2x+y)}$$

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Expanded as a power series in t,

$$\begin{aligned} N_3(x, y, t) \quad = \quad & 1+ \\ & 3t+ \\ & (3+2x+y)t^2+ \\ & (1+6x+x^2+2y+2xy)t^3+ \\ & (6x+7x^2+y+6xy+3x^2y+y^2)t^4+ \\ & (2x+15x^2+4x^3+6xy+14x^2y+2x^3y+2y^2+2xy^2+x^2y^2)t^5+ \\ & \vdots \end{aligned}$$

Sequences

Can uncover many known sequences with the following substitutions, among others:

- $N_k(x, y, t)$ full distribution
- $N_k(x, 1, t)$ trivial matches only
- $N_k(1, y, t)$ nontrivial matches only
- $N_k(x, x, t)$ all matches
- $N_k(1, -1, t)$ even number of nontrivial matches minus odd number of nontrivial matches

Also, we can algebraically manipulate $N_k(x, y, t)$ to find generating functions for sequences of coefficients.

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Powers of 2

s–1 is the lowest power of x that appears in $[t^{2s+1}]N_3(x,x,t)$ and

$$[x^{s-1}t^{2s+1}]N_3(x,x,t) = 2^{s-1}$$

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$s = 2$ 32321 32121 1 match

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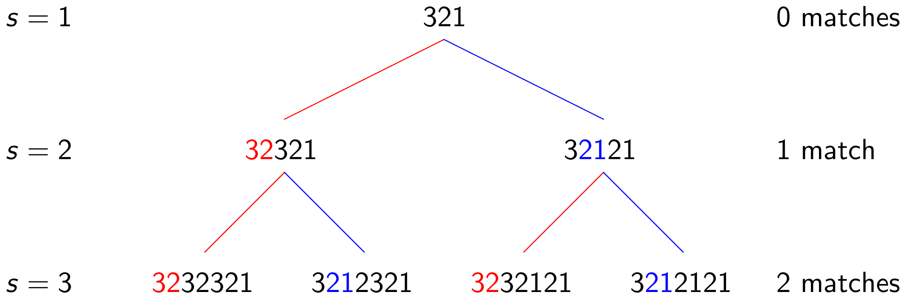

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$$\begin{aligned} N_3(x,y,t) \quad = \quad & 1+ \\ & 3t+ \\ & (3+2x+y)t^2+ \\ & (1+6x+x^2+2y+2xy)t^3+ \\ & (6x+7x^2+y+6xy+3x^2y+y^2)t^4+ \\ & (2x+15x^2+4x^3+6xy+14x^2y+2x^3y+2y^2+2xy^2+x^2y^2)t^5+ \\ & \vdots \end{aligned}$$

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Fibonacci numbers

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$$A_1^{(n)} = A_1^{(n-1)} + A_1^{(n-2)}$$

Similarly, $A_2^{(n)} = A_2^{(n-1)} + A_2^{(n-2)}$

and $A_3^{(n)} = A_3^{(n-1)} + A_3^{(n-2)}$.

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$1 - - - - - 2$ $\qquad A_{1,2}^{(n)} = A_{1,1}^{(n-1)} + A_{1,3}^{(n-1)}$

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$$A_1^{(n)} = A_1^{(n-1)} + A_1^{(n-2)}$$

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Therefore, $\qquad A^{(n)} = A^{(n-1)} + A^{(n-2)}.$

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$1 - - - - - 3 \qquad A^{(n)}_{1,3} = A^{(n-1)}_{1,2}$

$$A^{(n)}_{1} = A^{(n-1)}_{1} + A^{(n-2)}_{1}$$

Similarly, $A^{(n)}_{2} = A^{(n-1)}_{2} + A^{(n-2)}_{2}$

and $A^{(n)}_{3} = A^{(n-1)}_{3} + A^{(n-2)}_{3}$.

Therefore, $A^{(n)} = A^{(n-1)} + A^{(n-2)}$.

length 1: $1, 2, 3 \rightarrow A^{(1)} = 3$

length 2: $12, 21, 23, 31, 32 \rightarrow A^{(2)} = 5$

Further refinements

$$\begin{array}{c} \\ 1\\2\\3\\4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} & x & y & y \\ & & x & y \\ & & & x \\ & & & \end{pmatrix} \end{array}$$

trivial vs. nontrivial

$$\begin{array}{c} \\ 1\\2\\3\\4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} & x_1 & x_2 & x_3 \\ & & x_1 & x_2 \\ & & & x_1 \\ & & & \end{pmatrix} \end{array}$$

classify by the difference

$$\begin{array}{c} \\ 1\\2\\3\\4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} & x_1 & x_1 & x_1 \\ & & x_2 & x_2 \\ & & & x_3 \\ & & & \end{pmatrix} \end{array}$$

classify by first number

Thank you!