Simple Frame Pattern Occurrences in Cycles

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Permutation Patterns 2013

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PLAN

1. Introduction

2. Frame Patterns (in cycles)

3. Non-trivial matches

4. Incontractible cycles

5. Integer partitions

6. Dyck paths

Permutations

Let S_n be the set of all permutations of length n.

One-Line Notation

A permutation of length n in *one-line notation* is an arrangement of the integers from 1 to n.

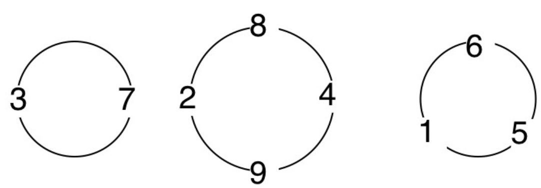
Example: An element of S_9

$$6\ 8\ 7\ 9\ 1\ 5\ 3\ 4\ 2$$

Cycle Notation

A permutation of length n can be written in cycle notation by...........

$$\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 6 & 8 & 7 & 9 & 1 & 5 & 3 & 4 & 2 \end{array}$$



$$(3, 7)(2, 8, 4, 9)(1, 6, 5)$$

Frame Patterns

Mesh patterns introduced by Brändén and Claesson in recent years to generalize classical, consecutive, vincular, bivincular patterns.

Later, Avgustinovich and Kitaev introduced a class of frame patterns called *boxed patterns* which later were called *frame patterns*.

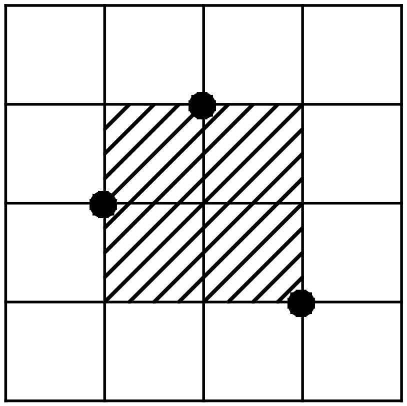


Figure 1: An example of a frame pattern

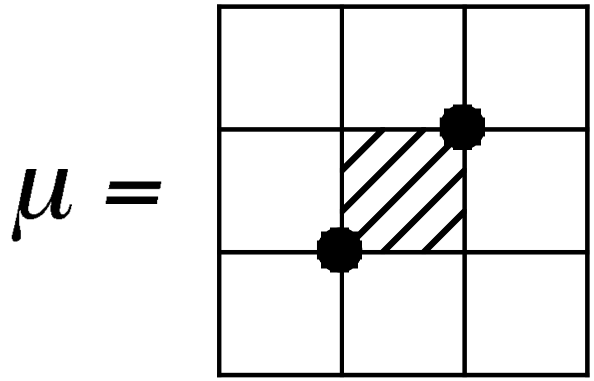


Figure 2: The mesh pattern μ

Consider the mesh pattern μ in Figure 2. Let σ be a permutation $\sigma = \sigma_1, \ldots \sigma_n$ then there is an occurrence of μ in σ if there is a pair (σ_i, σ_j) such that $i < j, \sigma_i < \sigma_j$ and if there is an integer k such that $i < k < j$ then either $\sigma_k < \sigma_i$ or $\sigma_j < \sigma_k$.

For example, If $\sigma = 4\ 8\ 2\ 6\ 1\ 3\ 5\ 7$ then the occurrences of μ are

$$(4, 8), (4, 6), (4, 5), (2, 6), (2, 3), (6, 7), (1, 3), (3, 5), (5, 7).$$

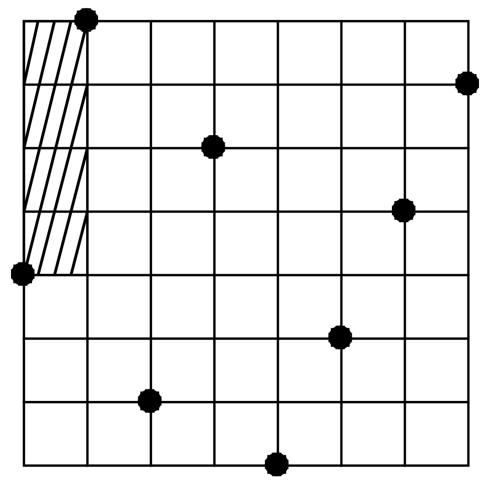


Figure 3: The graph of the permutation 48261357 with the occurrence $(4, 6)$ highlighted

For example, If $\sigma = 4\ 8\ 2\ 6\ 1\ 3\ 5\ 7$ then the occurrences of μ are

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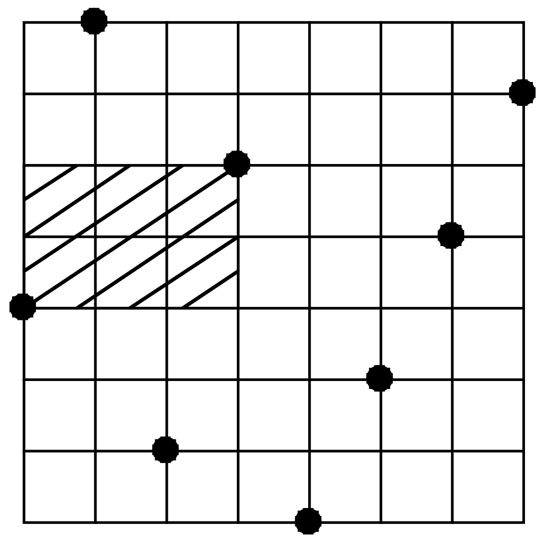

Figure 4: The graph of the permutation 48261357 with the occurrence $(4,6)$ highlighted

For example, If $\sigma = 4\ 8\ 2\ 6\ 1\ 3\ 5\ 7$ then the occurrences of μ are

$$(4,8),(4,6),(4,5),(2,6),(2,3),(6,7),(1,3),(3,5),(5,7).$$

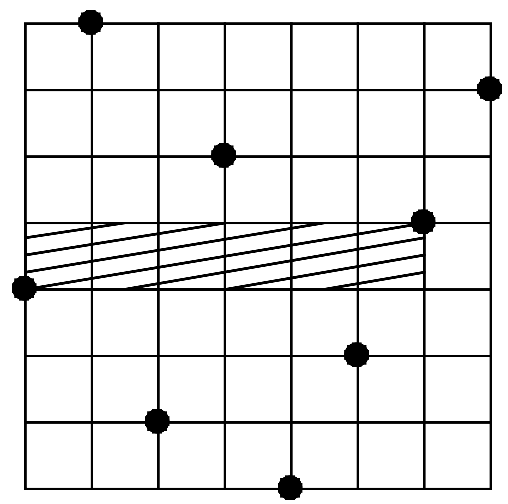

Figure 5: The graph of the permutation 48261357 with the occurrence $(4,6)$ highlighted

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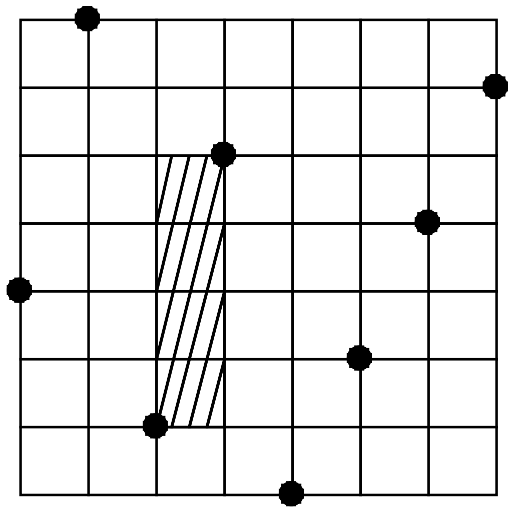

Figure 6: The graph of the permutation 48261357 with the occurrence $(4, 6)$ highlighted

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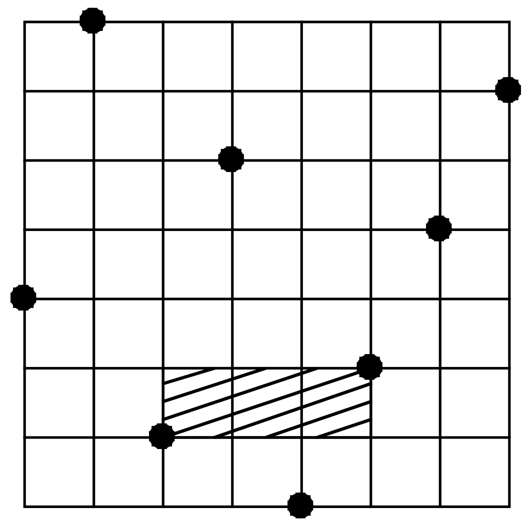

Figure 7: The graph of the permutation 48261357 with the occurrence $(4,6)$ highlighted

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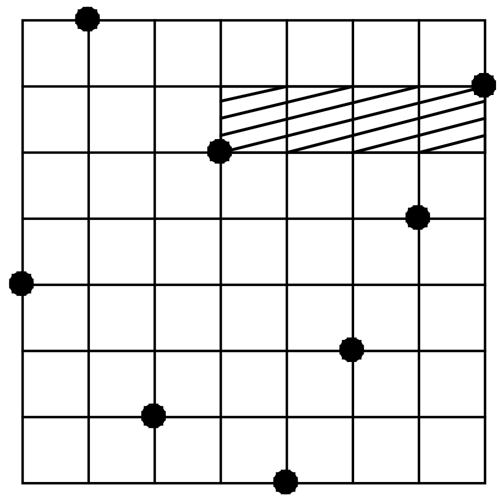

Figure 8: The graph of the permutation 48261357 with the occurrence $(4,6)$ highlighted

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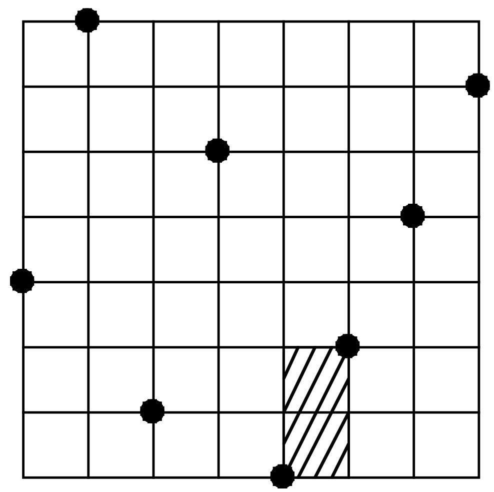

Figure 9: The graph of the permutation 48261357 with the occurrence $(4,6)$ highlighted

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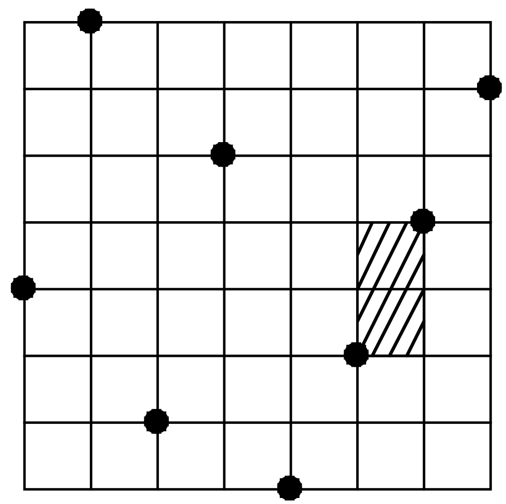

Figure 10: The graph of the permutation 48261357 with the occurrence $(4,6)$ highlighted

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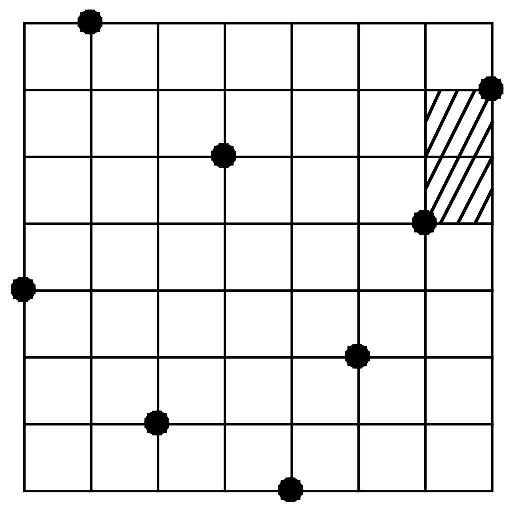


Figure 11: The graph of the permutation 48261357 with the occurrence $(4,6)$ highlighted

Cycle occurrence of frame pattern.

The frame pattern μ can occur in a cycle in the same way it can occur in a permutation except now you may have occurrences "wrap around."

For example the pair $(4,6)$ is a cycle-occurrence of μ in $(1,3,6,5,7,2,8,4)$.

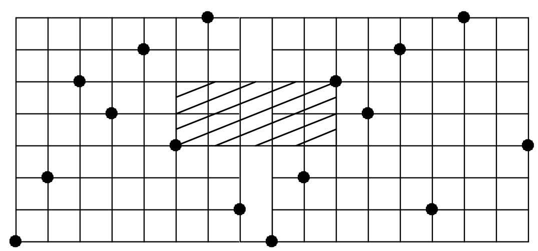


Figure 12: The graph of the permutation 48261357 with the occurrence $(4,6)$ highlighted

Notice that any consecutive integers $(a, a+1)$ form a cycle-occurrence of μ because every other integer b has the property $b < a$ or $a+1 < b$. Therefore every cycle $C \in S_n$ has at least $n-1$ cycle-occurrences of μ from each consecutive pair

$$(1,2),(2,3),\ldots,(n-1,n).$$

In fact, notice that the only cycle-occurrences of μ in the cycle $(1,2,3,\ldots,n-1,n)$ are these consecutive pairs.

Non-trivial cycle occurrences

Consider the polynomials

$$NT_{n,\mu}(q) = \sum_{C \text{ is an } n\text{-cycle}} q^{NT_\mu(C)}$$

where $NT_\mu(C)$ is the number of non-trivial cycle-occurrences of μ in C.

Clearly the constant term is 1 because of the cycle $(1, 2, \ldots, n)$ and the $q^{\binom{n-1}{2}}$ coefficient is 1 because of the cycle $(1, n, n-1, \ldots, 3, 2)$.

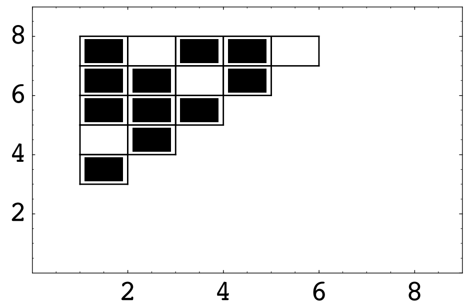
Table 1: Polynomials $NT_{n,\mu}(q)$

$NT_{1,\mu}(q)$	1
$NT_{2,\mu}(q)$	1
$NT_{3,\mu}(q)$	$1+q$
$NT_{4,\mu}(q)$	$1+3\,q+q^2+q^3$
$NT_{5,\mu}(q)$	$1+6\,q+6\,q^2+7\,q^3+2\,q^4+q^5+q^6$
$NT_{6,\mu}(q)$	$1+10\,q+20\,q^2+31\,q^3+23\,q^4+15\,q^5+13\,q^6+3\,q^7+2\,q^8+q^9+q^{10}$
$NT_{7,\mu}(q)$	$1+15\,q+50\,q^2+106\,q^3+135\,q^4+126\,q^5+119\,q^6+66\,q^7+46\,q^8+25\,q^9+19\,q^{10}+5\,q^{11}+3\,q^{12}+2\,q^{13}+q^{14}+q^{15}$
$NT_{8,\mu}(q)$	$1+21\,q+105\,q^2+301\,q^3+561\,q^4+736\,q^5+850\,q^6+726\,q^7+603\,q^8+418\,q^9+299\,q^{10}+174\,q^{11}+101\,q^{12}+65\,q^{13}+33\,q^{14}+27\,q^{15}+7\,q^{16}+5\,q^{17}+3\,q^{18}+2\,q^{19}+q^{20}+q^{21}$
$NT_{9,\mu}(q)$	$1+28\,q+196\,q^2+742\,q^3+1870\,q^4+3311\,q^5+4820\,q^6+5541\,q^7+5675\,q^8+5007\,q^9+4055\,q^{10}+3093\,q^{11}+2116\,q^{12}+1461\,q^{13}+888\,q^{14}+646\,q^{15}+338\,q^{16}+217\,q^{17}+126\,q^{18}+80\,q^{19}+44\,q^{20}+35\,q^{21}+11\,q^{22}+7\,q^{23}+5\,q^{24}+3\,q^{25}+2\,q^{26}+q^{27}+q^{28}$
$NT_{10,\mu}(q)$	$1+36\,q+336\,q^2+1638\,q^3+5328\,q^4+12253\,q^5+22392\,q^6+32864\,q^7+41488\,q^8+45433\,q^9+44119\,q^{10}+40008\,q^{11}+32781\,q^{12}+25689\,q^{13}+18551\,q^{14}+13710\,q^{15}+9137\,q^{16}+6179\,q^{17}+3971\,q^{18}+2568\,q^{19}+1640\,q^{20}+1098\,q^{21}+640\,q^{22}+374\,q^{23}+251\,q^{24}+148\,q^{25}+100\,q^{26}+56\,q^{27}+46\,q^{28}+15\,q^{29}+11\,q^{30}+7\,q^{31}+5\,q^{32}+3\,q^{33}+2\,q^{34}+q^{35}+q^{36}$

let $\mathcal{NM}_\mu(C)$ be the set of *non-matches* of C and let $NM_\mu(C) = |\mathcal{NM}_\mu(C)|$. Notice that $NM_\mu(C) + NT_\mu(C) = \binom{n-1}{2}$. Consider the polynomials

$$NM_{n,\mu}(q) = \sum_{C \text{ is an } n\text{-cycle}} q^{NM_\mu(C)}.$$

Theorem 0.1. *For $k < n-2$, $NM_{n,\mu}(q)|_{q^k} = a(k)$ where $a(k)$ is the number of integer partitions of k.*



{{1, 3}, {1, 5}, {1, 6}, {1, 7}, {2, 4}, {2, 5}, {2, 6}, {3, 5}, {3, 7}, {4, 6}, {4, 7}}

{4, 11, {1, 4, 5, 2, 7, 3, 6}}

Figure 13: The non-matches of $(1, 4, 5, 2, 7, 3, 6)$

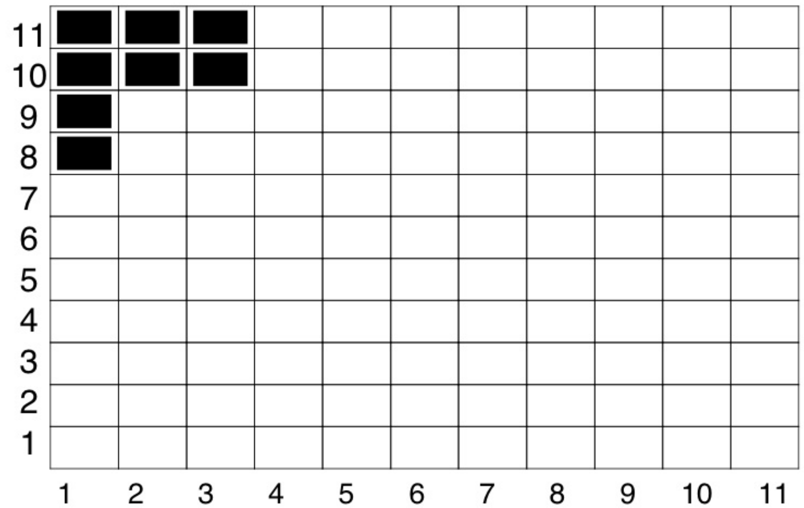


Figure 14: The non-matches of $(1, 7, 6, 5, 4, 11, 10, 3, 2, 9, 8)$

Bond

Define a *bond* of a cycle C to be a pair of consecutive integers $(a, a+1)$ such that a is followed by $a+1$ in the cycle C.

For example the bonds of the cycle $(1, 4, 5, 6, 2, 7, 8, 3)$ are the pairs $(4, 5)$, $(5, 6)$ and $(7, 8)$.

Contraction

Define the *contraction* of a cycle C, $\mathrm{cont}(C)$ to be the result when the bonds are collected together and the reduction is taken.

For example, the contraction of $(1, 4, 5, 6, 2, 7, 8, 3)$ is $(1, 4, 2, 5, 3)$.

Incontractible

A cycle is called *incontractible* if it has no bonds. For example, $(1, 4, 2, 5, 3)$ is incontractible.

Counting non-trivial matches.

Theorem 0.2. *For a cycle $C \in S_n$,*

$$NT_\mu(C) = NT_\mu(cont(C)) \tag{1}$$

For example the matches of the cycle $(1, 4, 5, 6, 3, 2, 7, 8)$ are the pairs

$$(1, 4), (1, 3), (3, 7), (2, 7), (2, 4).$$

Notice that $\text{cont}(1, 4, 5, 6, 3, 2, 7, 8) = (1, 4, 3, 2, 5)$ and the matches are the pairs

$$(1, 4), (1, 3), (3, 5), (2, 5), (2, 4).$$

So the number of non-trivial matches is determined by the contraction of the cycle.

Theorem 0.3. *Let A be an* incontractible *cycle of length ℓ. The number of n-cycles C such that $cont(C) = A$ is $\binom{n-1}{\ell-1}$.*

For example, if $A = (1, 3, 2, 4)$ and $n = 6$ then there are $\binom{5}{3} = 10$ cycles of length 6 that contract to $(1, 3, 2, 4)$.

1—3—2—4,5,6	123
1—3,4—2—5,6	124
1—3,4,5—2—6	125
1—4—2,3—5,6	134
1—4,5—2,3—6	135
1—5—2,3,4—6	145
1,2—4—3—5,6	234
1,2—4,5—3—6	235
1,2—5—3,4—6	245
1,2,3—5—4—6	345

Corollary 0.4.

$$NT_{n,\mu}(q)|_{q^k} = \sum_{\substack{A \text{ is incontractible} \\ NT_\mu(A)=k}} \binom{n-1}{|A|-1} \tag{2}$$

Table 2: Incontractible cycles with k non-trivial cycle-occurrences

$k=0$	(1)
1	$(1,3,2)$
2	$(1,3,2,4)$, $(1,3,5,4,2), (1,4,2,5,3)$

$$NT_{n,\mu}(q)|_{q^0} = 1. \tag{3}$$

$$NT_{n,\mu}(q)|_{q^1} = \binom{n-1}{2}. \tag{4}$$

$$NT_{n,\mu}(q)|_{q^2} = \binom{n-1}{3} + \binom{n-1}{4} + \binom{n-1}{4}. \tag{5}$$

Incontractible cycles

Let $\mathcal{IC}_n$ be the set of *incontractible* n-cycles. It is clear now that understanding *incontractible* cycles is the key to explaining the coefficients of $NT_{n,\mu}(q)$. Consider the polynomial

$$NTI_{n,\mu}(q) = \sum_{A \in \mathcal{IC}_n} q^{NT_\mu(A)}.$$

Setting $q = 1$, the number of incontractible n-cycles is the same as the number of derangements of length $n - 1$.

Table 3: The number of incontractible cycles

$NTI_{1,\mu}(1)$	1
$NTI_{2,\mu}(1)$	0
$NTI_{3,\mu}(1)$	1
$NTI_{4,\mu}(1)$	2
$NTI_{5,\mu}(1)$	9
$NTI_{6,\mu}(1)$	44
$NTI_{7,\mu}(1)$	265
$NTI_{8,\mu}(1)$	1854
$NTI_{9,\mu}(1)$	14833
$NTI_{10,\mu}(1)$	133496
$NTI_{11,\mu}(1)$	1334961

Let IC_n be the number of incontractible cycles of length n then we have the recursion

$$IC_n = (n-2)IC_{n-1} + (n-2)IC_{n-2}.$$

Which leads to the result about derangements.

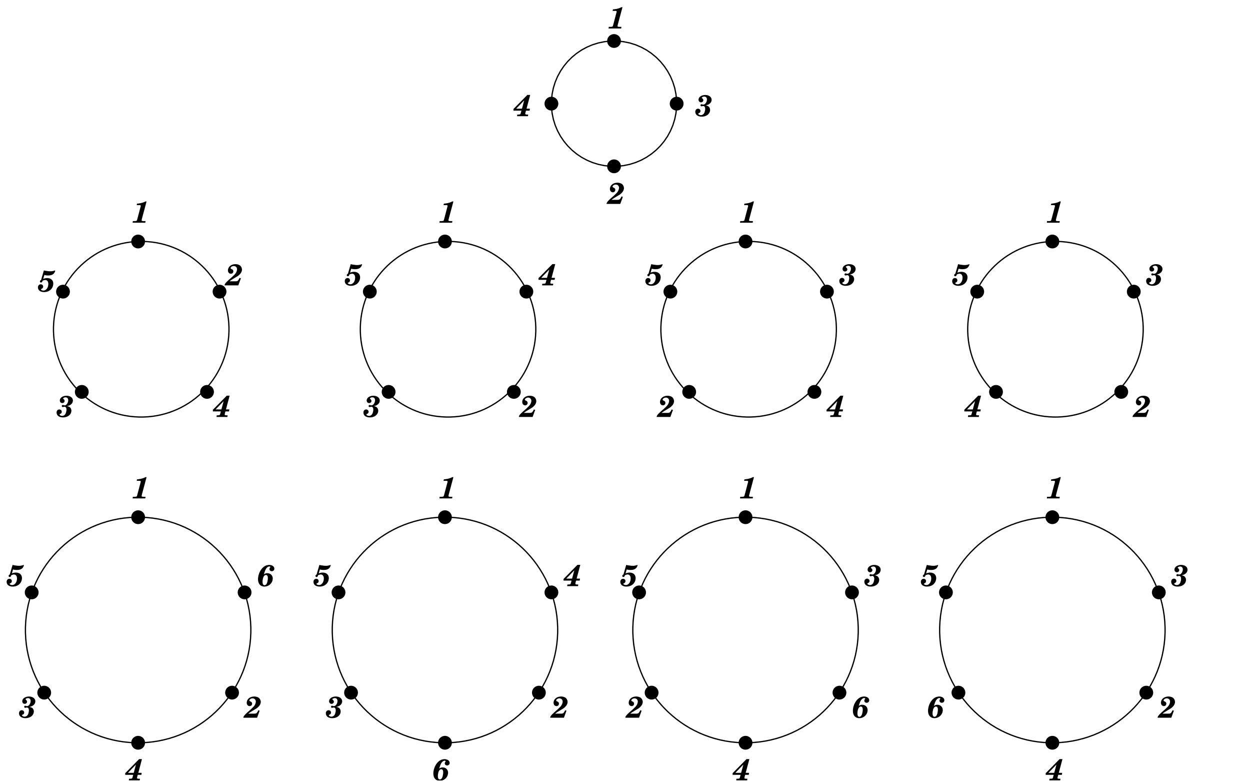


Figure 15: expanding (1,4,3,2)

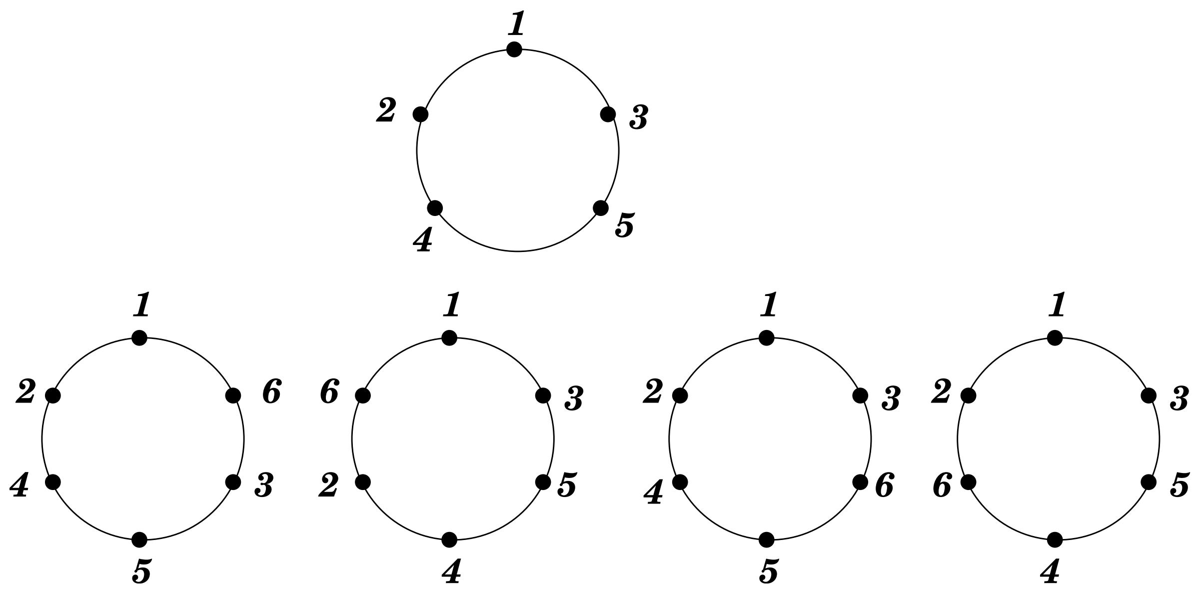


Figure 16: inserting 6 into (1,3,5,4,2)

Corollary 0.5. *For all* $n \geq 2$*, the lowest power of* q *appearing in* $NTI_{2n,\mu}(q)$ *and* $NTI_{2n+1,\mu}(q)$ *is* q^n*.*

Table 4: Polynomials $NTI_{n,\mu}(q)$

$NTI_{1,\mu}(q)$	1
$NTI_{2,\mu}(q)$	0
$NTI_{3,\mu}(q)$	q
$NTI_{4,\mu}(q)$	$q^2 + q^3$
$NTI_{5,\mu}(q)$	$2\,q^2 + 3\,q^3 + 2\,q^4 + q^5 + q^6$
$NTI_{6,\mu}(q)$	$6\,q^3 + 13\,q^4 + 10\,q^5 + 8\,q^6 + 3\,q^7 + 2\,q^8 + q^9 + q^{10}$
$NTI_{7,\mu}(q)$	$5\,q^3 + 27\,q^4 + 51\,q^5 + 56\,q^6 + 48\,q^7 + 34\,q^8 + 19\,q^9 + 13\,q^{10}$ $+5\,q^{11} + 3\,q^{12} + 2\,q^{13} + q^{14} + q^{15}$
$NTI_{8,\mu}(q)$	$29\,q^4 + 134\,q^5 + 255\,q^6 + 327\,q^7 + 323\,q^8 + 264\,q^9 + 187\,q^{10} + 139\,q^{11} +$ $80\,q^{12} + 51\,q^{13} + 26\,q^{14} + 20\,q^{15} + 7\,q^{16} + 5\,q^{17} + 3\,q^{18} + 2\,q^{19} + q^{20} + q^{21}$
$NTI_{9,\mu}(q)$	$14\,q^4 + 181\,q^5 + 694\,q^6 + 1413\,q^7 + 2027\,q^8 + 2307\,q^9 + 2139\,q^{10} + 1841\,q^{11} +$ $1392\,q^{12} + 997\,q^{13} + 652\,q^{14} + 458\,q^{15} + 282\,q^{16} + 177\,q^{17} + 102\,q^{18} + 64\,q^{19} +$ $36\,q^{20} + 27\,q^{21} + 11\,q^{22} + 7\,q^{23} + 5\,q^{24} + 3\,q^{25} + 2\,q^{26} + q^{27} + q^{28}$
$NTI_{10,\mu}(q)$	$130\,q^5 + 1128\,q^6 + 3965\,q^7 + 8509\,q^8 + 13444\,q^9 + 16918\,q^{10} +$ $18015\,q^{11} + 17121\,q^{12} + 14712\,q^{13} + 11663\,q^{14} + 8784\,q^{15} + 6347\,q^{16} +$ $4406\,q^{17} + 2945\,q^{18} + 1920\,q^{19} + 1280\,q^{20} + 819\,q^{21} + 541\,q^{22} +$ $311\,q^{23} + 206\,q^{24} + 121\,q^{25} + 82\,q^{26} + 47\,q^{27} + 37\,q^{28} + 15\,q^{29} +$ $11\,q^{30} + 7\,q^{31} + 5\,q^{32} + 3\,q^{33} + 2\,q^{34} + q^{35} + q^{36}$
$NTI_{11,\mu}(q)$	$42\,q^5 + 1071\,q^6 + 7174\,q^7 + 24345\,q^8 + 54877\,q^9 + 93836\,q^{10} +$ $129891\,q^{11} + 153883\,q^{12} + 161486\,q^{13} + 153805\,q^{14} + 134845\,q^{15} + 111673\,q^{16} +$ $87617\,q^{17} + 66415\,q^{18} + 47877\,q^{19} + 34288\,q^{20} + 23538\,q^{21} + 16386\,q^{22} +$ $10865\,q^{23} + 7272\,q^{24} + 4704\,q^{25} + 3198\,q^{26} + 2038\,q^{27} + 1388\,q^{28} + 870\,q^{29} +$ $563\,q^{30} + 347\,q^{31} + 237\,q^{32} + 151\,q^{33} + 102\,q^{34} + 62\,q^{35} + 48\,q^{36} +$ $22\,q^{37} + 15\,q^{38} + 11\,q^{39} + 7\,q^{40} + 5\,q^{41} + 3\,q^{42} + 2\,q^{43} + q^{44} + q^{45}$

Charge Graph

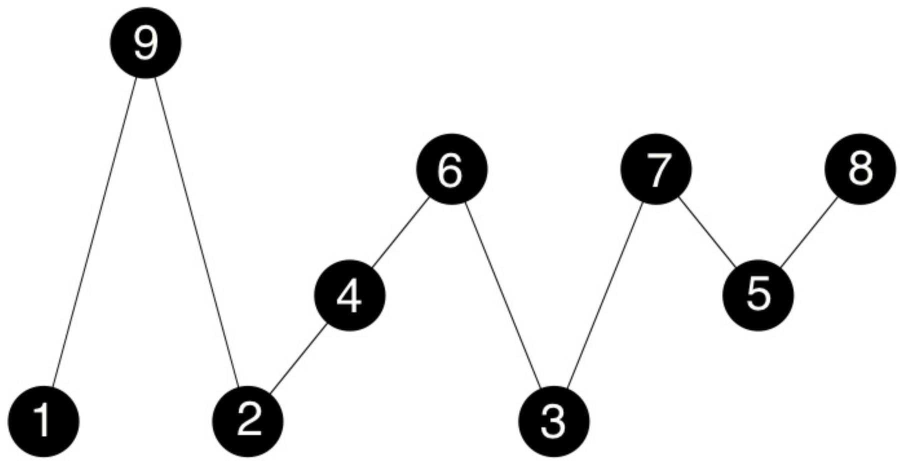


Figure 17: The charge graph of $(1, 9, 2, 4, 6, 3, 7, 5, 8)$

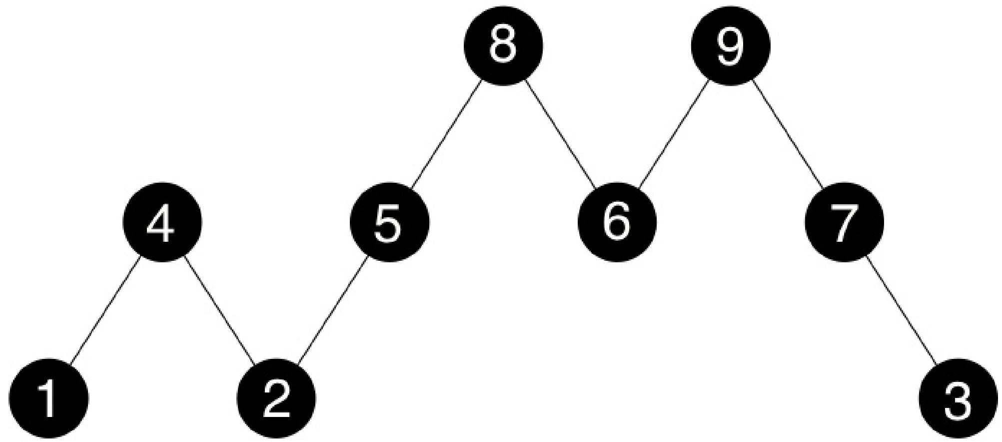


Figure 18: The charge graph of $(1, 4, 2, 5, 8, 6, 9, 7, 3)$

For C a $2n+1$-cycle,

Theorem 0.6. *The charge graph of C is a Dyck path of semi-length n $\iff$ C is incontractible with exactly n μ-matches(minimum).*

Thank you for coming
see you later