# Rationality for subclasses of Av(321)

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# The goal

**Theorem**

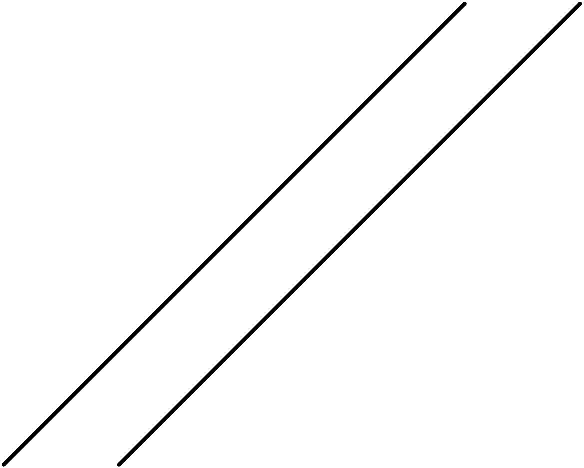
*Every proper finitely based subclass of* Av(321) *has a rational generating function.*

# But first

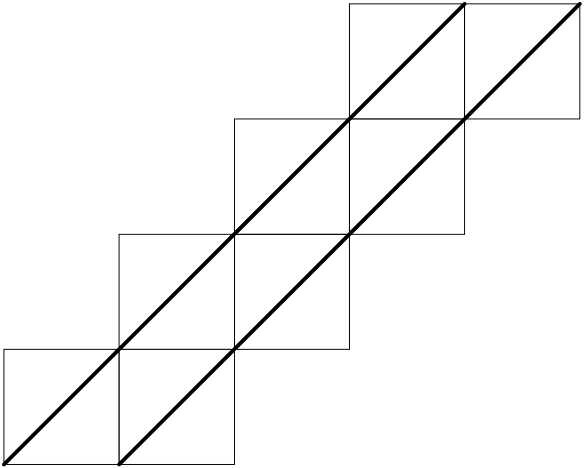
**Theorem**

*Every proper subclass of* Av(312) *has a rational generating function.*

# Av(321)



# Av(321)



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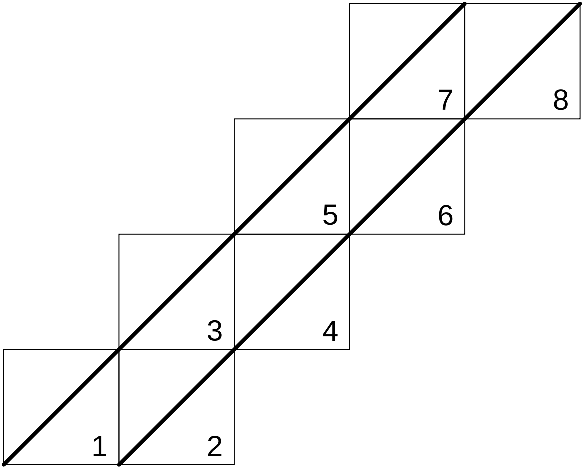

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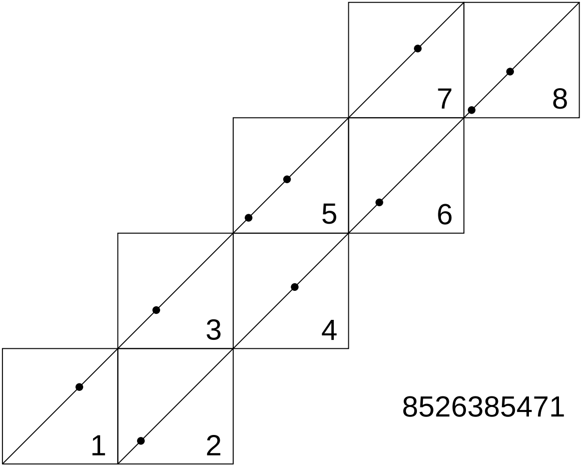


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# Av(321)

- We have a map from sequences of positive integers to elements of Av(321)
- The map is surjective, but not injective
- We can make the map bijective by only considering sequences that correspond to *greedy* griddings, and by disallowing factors of commuting elements in decreasing order
- For fixed $c, s > 0$ the word $(c(c-1)\dots 1)^s$ represents a special permutation – a family of $s$ parallel increasing oscillations of length $c$ – denoted $\sigma_{c,s}$

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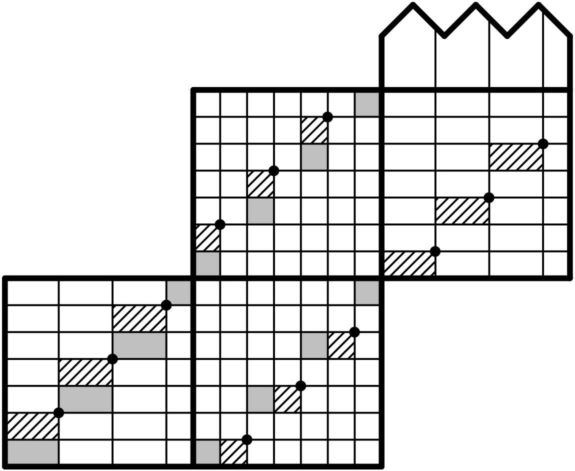
- And distant boxes can be linked by increasing oscillations between them

But, the family $\sigma_{c,s}$ is universal for Av(321), so any proper subclass avoids one of them – i.e. parallel oscillations in a subclass can be very thick, or very long, but not both.

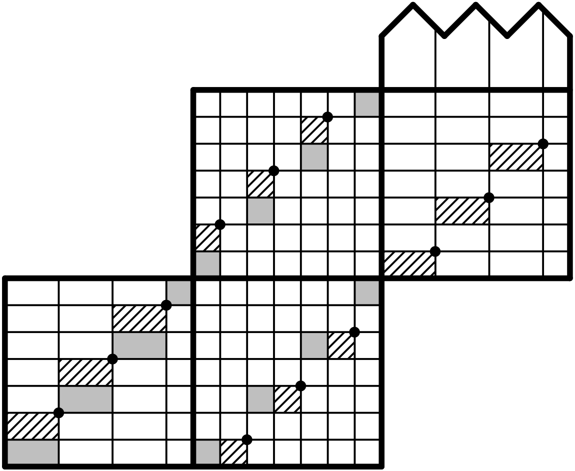
# Supposing we avoid $\sigma_{4,4}$

Consider an abstract permutation avoiding $\sigma_{4,4}$, and specifically its first four boxes.



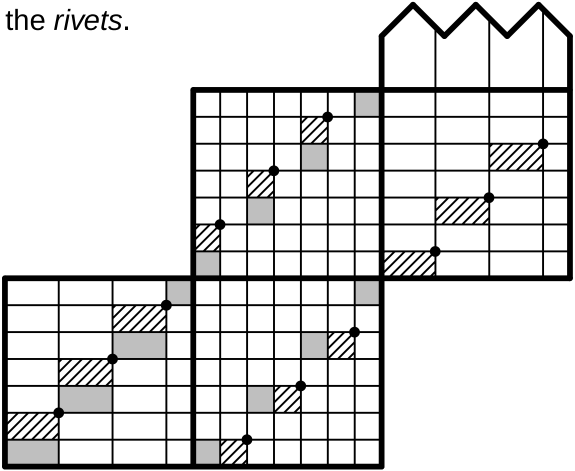
# Supposing we avoid $\sigma_{4,4}$

There is some maximal parallel oscillation from the first to the fourth box (in the picture it has size 3 – the biggest it can be).



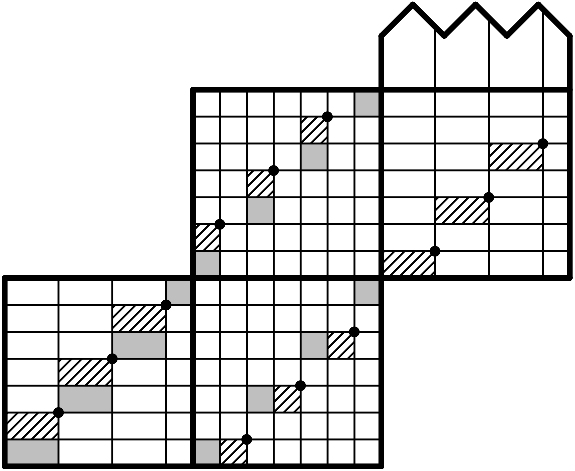
# Supposing we avoid $\sigma_{4,4}$

Take the *minimal* such oscillation – this literally corresponds to the “first” $(4321)^3$ subword, and means that certain cells (hatched) are known to be empty. Call the points of this oscillation the *rivets*.



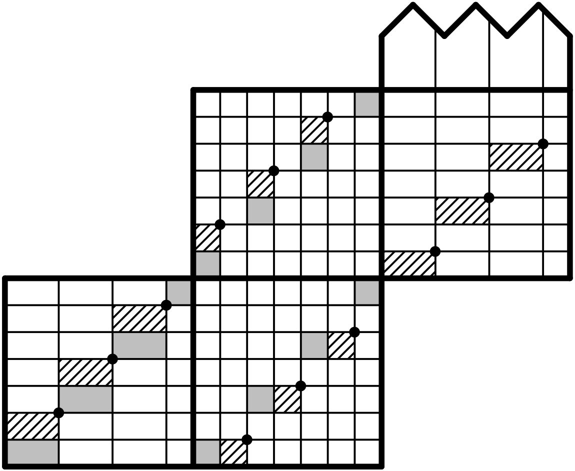
# Supposing we avoid $\sigma_{4,4}$

Elsewhere (grey cells) there may be some *entangled* elements – these are defined by being part of an oscillation parallel to the rivets that terminates in the first box.



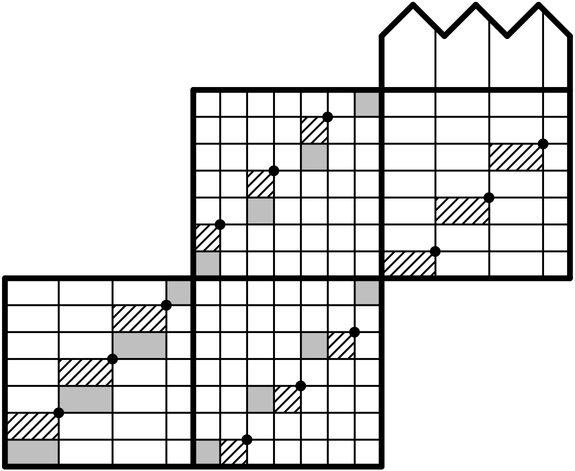
# Supposing we avoid $\sigma_{4,4}$

Entangled elements for instance are the elements of 321 (or 21, or 1) subwords occurring completely between two consecutive rivets.



# Supposing we avoid $\sigma_{4,4}$

The entangled elements, together with the rivets form the first *panel*.



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**Observation**

*The relationship between a non-rivet, $x$, in a given panel, and any non-rivet, $y$, in a preceding panel is determined by their relationships to the rivets (otherwise there would be an entanglement and $x$ would have been in the preceding panel).*

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# And so ...

- The panels are really just decorated (rivets) elements of a geometric grid class so there is a regular language describing them
- Each permutation can be described by its sequence of panels, with the inter-relationships between rivets being further indicated by the state of some automaton for processing the panel words
- So the class $\mathrm{Av}(321, \sigma_{s,c})$ can be described by a regular language and hence has a rational generating function
- To add further basis elements, e.g. $\pi$ just forbid “any subword that looks like it could be a copy of $\pi$ embedded in a sequence of panels” (easy if $\pi$ is $\oplus$-indecomposable, a bit of work for the general case)

# Speculations

- We know that growth rates of subclasses of Av(321) don't change when extra articulation points are included – we may be able to investigate this in more detail at the level of generating functions once the proof is implemented.
- Is there a general Catalan dichotomy? What extra conditions are required?

# A word from our sponsors . . .

A new version of *PermLab* will be released shortly. It includes support in the GUI for classes of involutions, more flexibility in producing animations, and hopefully fixes the bugs that arose when parallel computations were interrupted.

CS at Otago are hiring a confirmation-path lecturer (tenure track assistant professor). Closing date for applications is July 19 – see me for more information if interested (or on behalf of someone who might be!)

Thank you