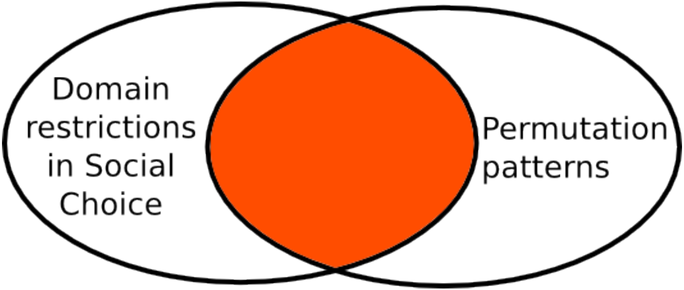
Single-peaked preference profiles and permutation patterns: A unified perspective

Martin Lackner

Vienna University of Technology

July 4, 2013
Permutation Patterns 2013

Outline


Domain restrictions in Social Choice
Permutation patterns

Social choice

Social choice deals with combining the preferences of individuals to reach a collective decision, e.g., voting.

Social choice

Social choice deals with combining the preferences of individuals to reach a collective decision, e.g., voting.

- Preferences of each individual (votes) . . . total orders

Social choice

Social choice deals with combining the preferences of individuals to reach a collective decision, e.g., voting.

- Preferences of each individual (votes) ... total orders
- Preference profile (election) ... multiset of total orders

Social choice

Social choice deals with combining the preferences of individuals to reach a collective decision, e.g., voting.

- Preferences of each individual (votes) . . . total orders
- Preference profile (election) . . . multiset of total orders
- Social choice function . . . mapping from profiles to sets of candidates (winners)

Social choice

Social choice deals with combining the preferences of individuals to reach a collective decision, e.g., voting.

- Preferences of each individual (votes) . . . total orders
- Preference profile (election) . . . multiset of total orders
- Social choice function . . . mapping from profiles to sets of candidates (winners)
- Social welfare function . . . mapping from profiles to total orders (ranking)

Social choice

Social choice deals with combining the preferences of individuals to reach a collective decision, e.g., voting.

- Preferences of each individual (votes) . . . total orders
- Preference profile (election) . . . multiset of total orders
- Social choice function . . . mapping from profiles to sets of candidates (winners)
- Social welfare function . . . mapping from profiles to total orders (ranking)
- Example: Plurality voting.

Arrow's impossibility theorem

Theorem (Arrow, 1951)

There is no social welfare function that satisfies the following criteria:

- More than two options
- (Pareto efficiency) If every individual prefers a over b, then a is prefered to b in the outcome.
- (Independence of irrelevant alternatives) The relative ranking of two options in the outcome is not influenced by a third candidate.
- (Non-dictatorship) There is no dictator.

Arrow's impossibility theorem

Theorem (Arrow, 1951)

There is no social welfare function that satisfies the following criteria:

- More than two options
- (Pareto efficiency) If every individual prefers *a* over *b*, then *a* is prefered to *b* in the outcome.
- (Independence of irrelevant alternatives) The relative ranking of two options in the outcome is not influenced by a third candidate.
- (Non-dictatorship) There is no dictator.

For social choice functions: Gibbard-Satterthwaite

Arrow's impossibility theorem

Theorem (Arrow, 1951)

There is no social welfare function that satisfies the following criteria:

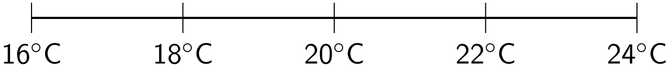
- More than two options
- (Pareto efficiency) If every individual prefers a over b, then a is prefered to b in the outcome.
- (Independence of irrelevant alternatives) The relative ranking of two options in the outcome is not influenced by a third candidate.
- (Non-dictatorship) There is no dictator.

For social choice functions: Gibbard-Satterthwaite

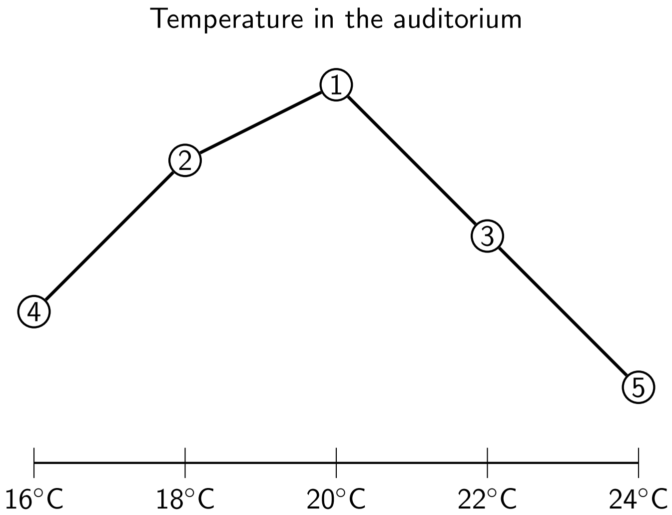
One way to deal with these limitations: Domain restrictions

Single-peaked profiles

Temperature in the auditorium

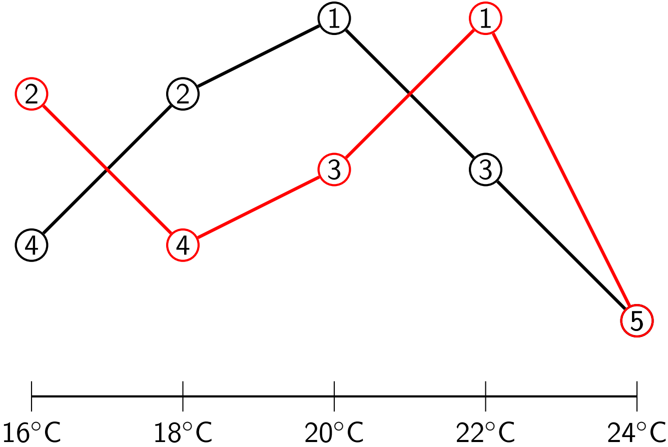


Single-peaked profiles



Single-peaked profiles

Temperature in the auditorium



A characterization of single-peakedness

Theorem (Ballester, Haeringer 2011)

A preference profile is single-peaked if and only if

1. there do not exist candidates a, b, c, d and votes V_1, V_2 such that
   - $V_1 : a > b > c, d > b$ holds and
   - $V_2 : c > b > a, d > b$ holds

   AND
2. there do not exist candidates a, b, c and votes V_1, V_2, V_3 such that
   - $V_1 : b > a, c > a$ holds and
   - $V_2 : a > b, c > b$ holds and
   - $V_3 : a > c, b > c$ holds.

A characterization of single-peakedness

Theorem (Ballester, Haeringer 2011)

A preference profile is single-peaked if and only if

1. there do not exist candidates a, b, c, d and votes V_1, V_2 such that
   - $V_1 : a > b > c, d > b$ holds and
   - $V_2 : c > b > a, d > b$ holds

   AND
2. there do not exist candidates a, b, c and votes V_1, V_2, V_3 such that
   - $V_1 : b > a, c > a$ holds and
   - $V_2 : a > b, c > b$ holds and
   - $V_3 : a > c, b > c$ holds.

Similar characterizations exist for many other domain restrictions: single-crossing, single-caved, group-separable, etc.

Generalization: Configuration Containment

Definition

Let k, m be positive integers. Furthermore, let $\mathcal{C}$ be a multiset of partial orders over $[k]$ and let $\mathcal{P}$ be a multiset of total orders over $[m]$. We refer to $\mathcal{C}$ as a *configuration* and to $\mathcal{P}$ as a *profile*.

Generalization: Configuration Containment

Definition

Let k, m be positive integers. Furthermore, let $\mathcal{C}$ be a multiset of partial orders over $[k]$ and let $\mathcal{P}$ be a multiset of total orders over $[m]$. We refer to $\mathcal{C}$ as a *configuration* and to $\mathcal{P}$ as a *profile*.

The profile $\mathcal{P}$ *contains* configuration $\mathcal{C}$ if there exist an injective function f from $\mathcal{C}$ into $\mathcal{P}$ and an injective function g from $[k]$ into $[m]$ such that, for any $a, b \in [k]$ and $O \in \mathcal{C}$, it holds that if $a\,O\,b$ then $g(a)\,f(O)\,g(b)$.

Generalization: Configuration Containment

Definition

Let k, m be positive integers. Furthermore, let $\mathcal{C}$ be a multiset of partial orders over $[k]$ and let $\mathcal{P}$ be a multiset of total orders over $[m]$. We refer to $\mathcal{C}$ as a *configuration* and to $\mathcal{P}$ as a *profile*.

The profile $\mathcal{P}$ *contains* configuration $\mathcal{C}$ if there exist an injective function f from $\mathcal{C}$ into $\mathcal{P}$ and an injective function g from $[k]$ into $[m]$ such that, for any $a, b \in [k]$ and $O \in \mathcal{C}$, it holds that if $a\,O\,b$ then $g(a)\,f(O)\,g(b)$.

Remark: g is not a matching (not increasing)

Generalization: Configuration Containment

Definition

Let k, m be positive integers. Furthermore, let $\mathcal{C}$ be a multiset of partial orders over $[k]$ and let $\mathcal{P}$ be a multiset of total orders over $[m]$. We refer to $\mathcal{C}$ as a *configuration* and to $\mathcal{P}$ as a *profile*.

The profile $\mathcal{P}$ *contains* configuration $\mathcal{C}$ if there exist an injective function f from $\mathcal{C}$ into $\mathcal{P}$ and an injective function g from $[k]$ into $[m]$ such that, for any $a, b \in [k]$ and $O \in \mathcal{C}$, it holds that if $a\,O\,b$ then $g(a)\,f(O)\,g(b)$.

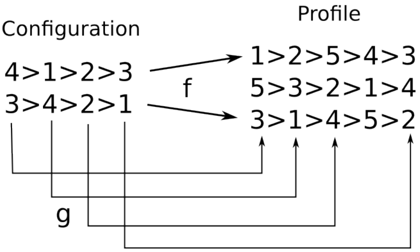
Configuration	Profile
4>1>2>3	1>2>5>4>3
3>4>2>1	5>3>2>1>4
	3>1>4>5>2

Generalization: Configuration Containment

Definition

Let k, m be positive integers. Furthermore, let $\mathcal{C}$ be a multiset of partial orders over $[k]$ and let $\mathcal{P}$ be a multiset of total orders over $[m]$. We refer to $\mathcal{C}$ as a *configuration* and to $\mathcal{P}$ as a *profile*.

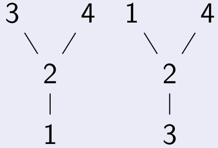
The profile $\mathcal{P}$ *contains* configuration $\mathcal{C}$ if there exist an injective function f from $\mathcal{C}$ into $\mathcal{P}$ and an injective function g from $[k]$ into $[m]$ such that, for any $a, b \in [k]$ and $O \in \mathcal{C}$, it holds that if $a\, O\, b$ then $g(a)\, f(O)\, g(b)$.



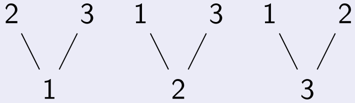
$$g : 1 \mapsto 2, 2 \mapsto 4, 3 \mapsto 3, 4 \mapsto 1$$

A characterization of single-peakedness

$\alpha-$configuration



worst-diverse configuration



Theorem (Ballester, Haeringer 2011)

A preference profile is single-peaked if and only if it does contain neither $\alpha-$ nor worst-diverse configurations.

Relation to permutation patterns

Every permutation pattern matching problem can be translated into a configuration containment problem:

Theorem

Let $\pi = (\pi_1 \ldots \pi_k)$ and $\sigma = (\sigma_1 \ldots \sigma_m)$ be permutations. The profile

$$\mathcal{P} = \{1 < 2 < \cdots < m,\ 1 < 2 < \cdots < m,\ \sigma_1 < \sigma_2 < \cdots < \sigma_m\}$$

contains the configuration

$$\mathcal{C} = \{1 < 2 < \cdots < k,\ 1 < 2 < \cdots < k,\ \pi_1 < \pi_2 < \cdots < \pi_k\}$$

if and only if σ contains π.

Relation to permutation patterns (ctd.)

Theorem

Let $\pi = (\pi_1 \ldots \pi_k)$ and $\sigma = (\sigma_1 \ldots \sigma_m)$ be permutations. The profile

$$\mathcal{P} = \{1 < 2 < \cdots < m,\ \sigma_1 < \sigma_2 < \cdots < \sigma_m\}$$

contains the configuration

$$\mathcal{C} = \{1 < 2 < \cdots < k, \pi_1 < \pi_2 < \cdots < \pi_k\}$$

if and only if σ contains either π or π^{-1}.

Computational formulation

Configuration Containment

Instance: A profile $\mathcal{P}$ and a set of configurations Γ

Question: Is there a $\mathcal{C} \in \Gamma$ that is contained in $\mathcal{P}$?

Hardness results (1)

Theorem

The CONFIGURATION CONTAINMENT is NP-complete, even if $|\mathcal{P}| = 2$, $\Gamma = \{\mathcal{C}\}$ and $|\mathcal{C}| = 2$.

Hardness results (1)

Theorem

The Configuration Containment is NP-complete, even if $|\mathcal{P}| = 2$, $\Gamma = \{\mathcal{C}\}$ and $|\mathcal{C}| = 2$.

Proof idea: Reduction from Permutation Pattern Matching.
For each pattern T, text T find P', T' such that

- the inverse of P' is not contained in T' and
- P' is contained in T' iff P is contained in T.

Hardness results (2)

Theorem

The Configuration Containment parameterized by the length of the longest configuration is W[1]-complete, even if $|\mathcal{P}| = 3$, $\Gamma = \{\mathcal{C}\}$ and $|\mathcal{C}| = 3$.

Hardness results (2)

Theorem

The CONFIGURATION CONTAINMENT parameterized by the length of the longest configuration is W[1]-complete, even if $|\mathcal{P}| = 3$, $\Gamma = \{\mathcal{C}\}$ and $|\mathcal{C}| = 3$.

Proof idea: Parameterized reduction from SEGREGATED PERMUTATION PATTERN MATCHING [Bruner, L. 2013]

Research directions

Counting/probability

- How many single-peaked profiles are there (for fixed m, n)?

Research directions

Counting/probability

- How many single-peaked profiles are there (for fixed m, n)?
- Single-crossing?

Research directions

Counting/probability

- How many single-peaked profiles are there (for fixed m, n)?
- Single-crossing?
- Other configurations/domain restrictions.

Research directions

Counting/probability

- How many single-peaked profiles are there (for fixed m, n)?
- Single-crossing?
- Other configurations/domain restrictions.
- Nearly single-peaked: Voter deletion, candidate deletion, etc.

Research directions

Counting/probability

- How many single-peaked profiles are there (for fixed m, n)?
- Single-crossing?
- Other configurations/domain restrictions.
- Nearly single-peaked: Voter deletion, candidate deletion, etc.

Algorithms

- Single-peaked ... $\mathcal{O}(m \cdot n)$ (longest configuration $k = 4$)

Research directions

Counting/probability

- How many single-peaked profiles are there (for fixed m, n)?
- Single-crossing?
- Other configurations/domain restrictions.
- Nearly single-peaked: Voter deletion, candidate deletion, etc.

Algorithms

- Single-peaked ... $\mathcal{O}(m \cdot n)$ (longest configuration $k = 4$)
- Single-crossing ... $\mathcal{O}(m^2 \cdot n)$ (longest configuration $k = 6$)

Research directions

Counting/probability

- How many single-peaked profiles are there (for fixed m, n)?
- Single-crossing?
- Other configurations/domain restrictions.
- Nearly single-peaked: Voter deletion, candidate deletion, etc.

Algorithms

- Single-peaked ... $\mathcal{O}(m \cdot n)$ (longest configuration $k = 4$)
- Single-crossing ... $\mathcal{O}(m^2 \cdot n)$ (longest configuration $k = 6$)
- Universal configuration containment algorithm faster than $\mathcal{O}(m^k \cdot n)$?

Summary

- Configuration containment:
  captures the most important domain restrictions
- Permutation patterns occur as a special case
- This work connects the two main topics of my (unfinished) PhD thesis: domain restrictions and permutation patterns. I am very interested in feedback.