Block Patterns in Stirling Permutations

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Permutation Patterns 2013

Outline

A new notion of patterns that

- generalizes patterns in permutations/words/trees, and
- has applications to operad theory.

Outline:

- Stirling permutations
- Blocks and block patterns
- Block patterns of height 1
- Applications
  - Strong Wilf-type equivalence
  - Generating functions

Stirling Permutations

Definition

The *Stirling permutations of order n* ($\mathcal{Q}_n$) are the rearrangements of

$$\{1^2, 2^2, \ldots, n^2\}$$

such that, $\forall i$, every element between two i's is greater than i. (Equivalently, they avoid the classical pattern 212.)

- Example: 4415778852213663.
- Non-example: 4366431577885221

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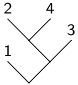
Basic Facts

- Introduced in [Gessel and Stanley, 1978].
- Patterns studied by Janson, Kuba, Panholzer, others.
- Bijection between Stirling permutations and (a class of) labeled binary trees.



$\longleftrightarrow 133221$

- Observe recursively that $|\mathcal{Q}_n| = (2n-1)!!$.
- This implies

$$\sum_{n\geq 0} |\mathcal{Q}_n| \frac{t^n}{n!} = \frac{1}{\sqrt{1-2t}}.$$

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The *i*th *block* of $\sigma \in \mathcal{Q}_n$, written $[i, i]_\sigma$, is the subsequence of σ beginning and ending with i.

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Two blocks are *comparable* if they are contained in all of the same blocks except themselves.

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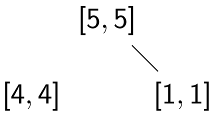

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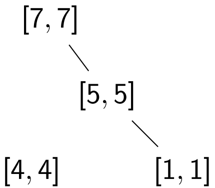

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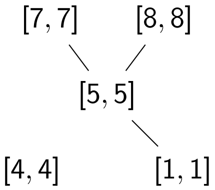

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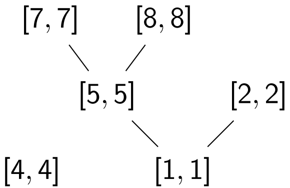

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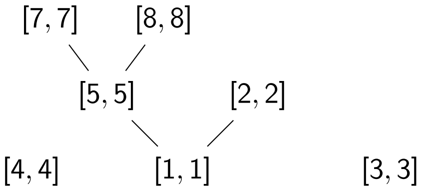

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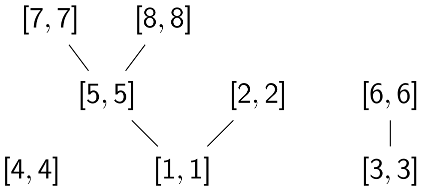

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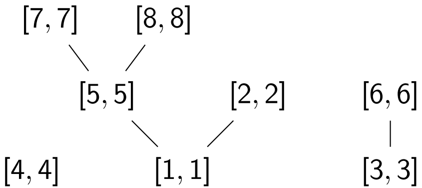

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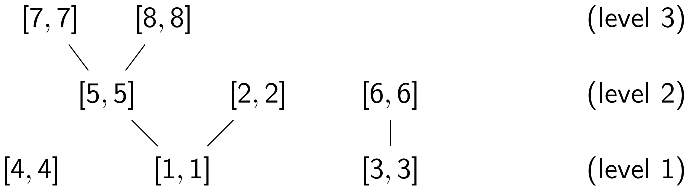

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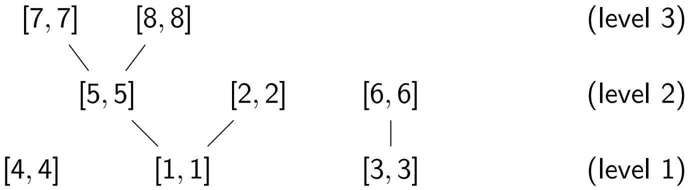

- The *level* of a block is the number of blocks containing it.
- The *height* of a Stirling permutation is its maximum level.

Restricting Height

- Restricting height leads to well-known sets of objects.
- Stirling perms with height $= 1 \leftrightarrow$ permutations.

$$441155223366 \longleftrightarrow 415236$$

- Stirling perms with height $\leq 2 \leftrightarrow$ ordered cycle decomps.

$$455413322661 \longleftrightarrow (4,5)(1,3,2,6)$$

  - Counted by ordered Stirling numbers of first kind.

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- 0 times at level 3

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- Form a vincular pattern $v(\tau)$ by underlining everywhere except between elements of tau that are consecutive and equal.
- An occurrence of τ as a *consecutive block pattern* ($\text{cons}(\tau)$) is just an occurrence of $v(\tau)$.

$$331221 \to \underline{3}\,\underline{312}\,\underline{21}$$

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Why Block Patterns?

- *Naturally correspond to patterns in labeled trees!*
- Inherently account for trivial symmetries.
- More specific motivation:
  - Consecutive block patterns = tree patterns in [Dotsenko, 2012].
  - Block patterns = labeled versions of patterns in [Rowland, 2010], [Dairyko et al., 2012].

What's Been Done With Block Patterns?

- Some (consecutive) Wilf equivalence [Dotsenko, 2012]
- Some (consecutive) asymptotic results
- Patterns of height 1 (which correspond to combs in trees)

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$66 \mid 99377 \mid 55 \mid$

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29937732|88|4554|1661

Notation

For patterns $a, c \in \mathfrak{S}_\infty$, let

$$P_c^a(t,z) = \sum_{n\geq 0} \frac{t^n}{n!} \sum_{\substack{\pi\in\mathfrak{S}_n \\ \pi \text{ avoids } a}} z^{\#\ c \text{ in } \pi}.$$

For sequences of block patterns $A, C : \{1, 2, \ldots\} \to \mathcal{Q}_\infty$,

$$\mathcal{Q}_n^A = \{\sigma \in \mathcal{Q}_n \text{ avoiding } A_i \text{ at level } i\}$$

$$Q_C^A(t;\vec{x};\vec{y}) =$$

$$\sum_{n\geq 0} \frac{t^n}{n!} \sum_{\sigma\in\mathcal{Q}_n^A} \prod_{i\geq 1} x_i^{\#\ C_i \text{ at level } i \text{ in } \sigma}\, y_i^{\#\text{ blocks at level } i}$$

Level 1 Patterns

Theorem

Assume that A_1, C_1 have height 1, i.e. they are just permutations.

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- $Q_{(C_2, C_3, \ldots)}^{(A_2, A_3, \ldots)}(t; \vec{x}; \vec{y})$

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- $P_{C_1}^{A_1}(t, z)$
- $Q_{(C_2, C_3, \ldots)}^{(A_2, A_3, \ldots)}(t; \vec{x}; \vec{y})$

then $Q_C^A(t; \vec{x}; \vec{y})$ is equal to

$$P_{C_1}^{A_1}\left(y_1 \int_0^t Q_{(C_2, C_3, \ldots)}^{(A_2, A_3, \ldots)}(u; x_2, x_3, \ldots; y_2, y_3, \ldots)\mathrm{d}u, x_1\right)$$

Proof Sketch

$$P_{C_1}^{A_1}\left(y_1\int_0^t Q_{(C_2,C_3,\ldots)}^{(A_2,A_3,\ldots)}(u;x_2,x_3,\ldots;y_2,y_3,\ldots)\mathrm{d}u,x_1\right)$$

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3. Permute the level 1 blocks in some way.

Application to Strong Wilf-Type Equivalence

Corollary

If $P^{A_i}_{C_i}(t,z) = P^{A'_i}_{C'_i}(t,z)$ *for all* i*, then*

$$Q^{A}_{C}(t;\vec{x};\vec{y}) = Q^{A'}_{C'}(t;\vec{x};\vec{y}).$$

Proof.

The previous theorem provides a functional equation both Q's must satisfy. □

E.g. $A_i = \text{class}(112233), A'_i = \text{class}(113322)$.

Applications to Generating Functions

- Restrict to Stirling permutations of height ≤ 2.
  - Stirling numbers (of both kinds)
  - Alteration of zig-zag numbers
- Ignore blocks beyond level 1.
  - Bessel polynomials

Height ≤ 2, Increasing at Level 1

$A = (2211, \emptyset, 11, \emptyset, \ldots),\ C = (\emptyset, \ldots).$

$$
\begin{aligned}
P^{21}_{\emptyset}(t, z) &= \exp(t) \\
Q^{A}_{C}(t; \vec{x}; \vec{y}) &= \exp\left(y_1 \int_0^t \frac{\mathrm{d}u}{1 - u y_2} \right) \\
&= \exp\left(\frac{-y_1}{y_2} \log(1 - t y_2) \right) \\
&= (1 - t y_2)^{-y_1/y_2}
\end{aligned}
$$

Stirling numbers of the first kind!
(OEIS A008275)

Height ≤ 2, Increasing at Level 2

$A = (\emptyset, 2211, 11, \emptyset, \ldots),\ C = (\emptyset, \ldots).$

$$P_{\emptyset}^{\emptyset}(t, z) = \frac{1}{1 - t}$$

$$Q_C^A(t; \vec{x}; \vec{y}) = \frac{1}{1 - y_1 \int_0^t \exp(u y_2) \mathrm{d}u} = \frac{y_2}{y_2 - y_1(\exp(t y_2) - 1)}$$

Ordered Stirling numbers of second kind! (OEIS A019538)

Height ≤ 2, Counting Level 1 "Descents"

$A = (\emptyset, \emptyset, 11, \emptyset, \ldots),\ C = (\text{cons}(2211), \emptyset, \ldots).$

$$P^{\emptyset}_{\underline{21}}(t, z) = \frac{z-1}{z - \exp\left(t(z-1)\right)}$$

$$Q^{A}_{C}(t; \vec{x}; \vec{y}) = \frac{x_1 - 1}{x_1 - \exp\left((x_1-1)y_1 \int_0^t \frac{1}{1-uy_2}\mathrm{d}u\right)}$$

$$= \frac{x_1 - 1}{x_1 - (1-ty_2)^{y_1(1-x_1)/y_2}}$$

Refinement of ordered Stirling numbers (first kind).
Not in OEIS.

$$1,\ x+2,\ x^2+7x+6,\ x^3+17x^2+46x+24,\ \ldots.$$

Height ≤ 2, Zig-Zag at Level 1

$$A = (\{\mathrm{cons}(123), \mathrm{cons}(321)\}, \emptyset, 1, \emptyset, \dots), C = (\emptyset, \dots).$$

$$P_{\emptyset}^{\{\underline{123},\underline{321}\}}(t,z) = \sec t + \tan t$$

$$Q_C^A(t; \vec{x}; \vec{y}) = \sec\left(y_1 \int_0^t \frac{\mathrm{d}u}{1-uy_2}\right) + \tan\left(y_1 \int_0^t \frac{\mathrm{d}u}{1-uy_2}\right)$$

$$= \sec\left(\frac{-y_1}{y_2}\log(1-ty_2)\right) + \tan\left(\frac{-y_1}{y_2}\log(1-ty_2)\right)$$

Not in OEIS (even with y's set to 1).
1, 1, 2, 7, 34, 210, ...

Ignoring Higher Blocks, Increasing at Level 1

$A = (2211, \emptyset, \ldots),\ C = (\emptyset, \ldots).$

$$P^{21}_{\emptyset}(t, z) = \exp(t)$$
$$Q^{A}_{C}(t; \vec{x}; y, 1, \ldots) = \exp\left(y(1 - \sqrt{1 - 2t})\right)$$

Bessel polynomials! (OEIS A001497)
Definition of Bessel polynomials gives coefficient of $\frac{t^n}{n!} y^k$:

$$\frac{(2n-k-1)!}{2^{n-k}(n-k)!(k-1)!}$$

Ignoring Higher Blocks, Counting Lvl. 1 "Descents"

$A = (\emptyset, \ldots), \; C = (\text{cons}(2211), \emptyset, \ldots).$

$$P^{\emptyset}_{\underline{21}}(t, z) = \frac{z - 1}{z - \exp\left(t(z-1)\right)}$$

$$Q^{A}_{C}(t; \vec{x}; y, 1, \ldots) = \frac{x_1 - 1}{x_1 - \exp\left(y(x_1 - 1)\int_0^t \frac{\mathrm{d}u}{\sqrt{1-2u}}\right)} = \frac{x_1 - 1}{x_1 - \exp\left(y(x_1 - 1)(1 - \sqrt{1 - 2t})\right)}$$

Not in OEIS.

Ignoring Higher Blocks, Avoiding 123 at Level 1

$A = (\text{cons}(112233), \emptyset, \ldots)$, $C = (\emptyset, \ldots)$. As proven in [Elizalde and Noy, 2003]

$$P_{\emptyset}^{123}(t, z) = \frac{\sqrt{3}e^{t/2}}{2\cos\left(\frac{\sqrt{3}}{2}t + \frac{\pi}{6}\right)}$$

$$Q_C^A(t; \vec{x}; y, 1, \ldots) = P_{\emptyset}^{123}(y(1 - \sqrt{1 - 2t}), x)$$

Recap

- Stirling permutations
- Blocks and block patterns
- Block patterns of height 1
- Applications
  - Strong Wilf-type equivalence
  - Generating functions

Extensions

- Dotsenko's trees generally correspond to *colored k-Stirling permutations*.
- *Colored* means each $i \in \{1, \dots, n\}$ is labeled with an integer in $\{0, 1, \dots, c-1\}$.
- *k-Stirling permutations* are just 212-avoiding rearrangements of

$$\{1^k, 2^k, \dots, n^k\}.$$

- Our theorem still applies, but there are fewer known generating functions.

Future Work?

- Patterns of height > 1?
- Interplay between block and normal patterns?
- Stanley-Wilf limits?

Citations I

Dairyko, M., Pudwell, L., Tyner, S., and Wynn, C. (2012).
Non-Contiguous Pattern Avoidance in Binary Trees.
ArXiv e-prints.

Dotsenko, V. (2012).
Pattern avoidance in labelled trees.
Séminaire Lotharingien de Combinatoire, B67b.

Elizalde, S. and Noy, M. (2003).
Consecutive patterns in permutations.
Advanced in Applied Math, 30:110–125.

Gessel, I. and Stanley, R. P. (1978).
Stirling polynomials.
Journal of Combinatorial Theory, Series A, 24(1):24–33.

Citations II

Janson, S., Kuba, M., and Panholzer, A. (2011).
Generalized stirling permutations, families of increasing trees and urn models.
Journal of Combinatorial Theory, Series A, 118(1):94–114.

Rowland, E. S. (2010).
Pattern avoidance in binary trees.
Journal of Combinatorial Theory, Series A, 117:741–758.

Thank you!