Isomorphisms between pattern classes

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Outline of talk

Isomorphisms and automorphisms

A bijection Γ from one pattern class $\mathcal{A}$ to another pattern class $\mathcal{B}$ is an *isomorphism* if, for all $\alpha_1, \alpha_2 \in \mathcal{A}$, we have

$$\alpha_1 \preceq \alpha_2 \text{ if and only if } \Gamma(\alpha_1) \preceq \Gamma(\alpha_2)$$

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Find all isomorphisms

i.e. find all triples $(\mathcal{A}, \mathcal{B}, \Gamma)$

Symmetries

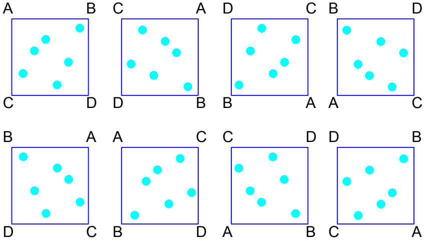

Figure: The symmetries of the pattern containment order

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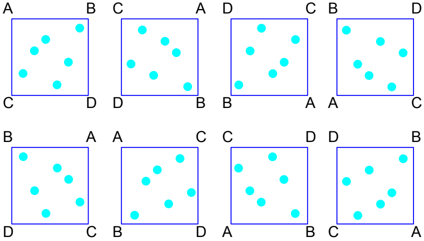

Figure: The symmetries of the pattern containment order

Theorem (Rebecca Smith)

The symmetries of the square are the only automorphisms of the pattern containment order.

Not all isomorphisms are symmetries

Question

Can we have an isomorphism Γ *between pattern classes* $\mathcal{A}$ *and* $\mathcal{B}$ *(i.e. a triple* $(\mathcal{A}, \mathcal{B}, \Gamma)$*) which is not realized by a symmetry?*

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YES – in fact. . .

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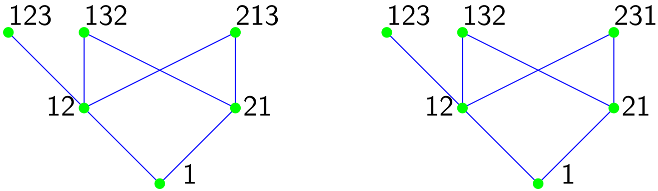
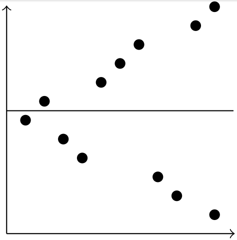
YES – in fact. . .



Two isomorphic pattern classes not equivalent under symmetry

The 'exotic' automorphism of $\mathrm{Av}(132, 312)$



6 7 5 4 8 9 10 3 2 11 12 1

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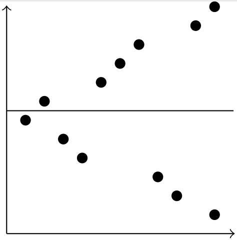

6 7 5 4 8 9 10 3 2 11 12 1

$\downarrow \eta$

abbaaabbaab

Encode a permutation π by a word in $\{a, b\}^*$ with every symbol $\pi(i)$ except the first being represented by a letter (a if $\pi(i) > \pi(1)$, and b if $\pi(i) < \pi(1)$).

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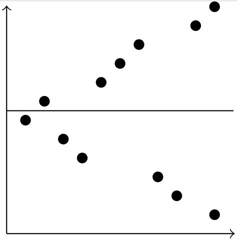

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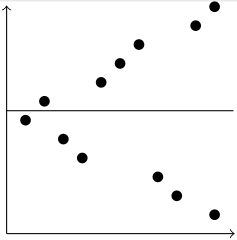

$$\begin{array}{c} 6\ 7\ 5\ 4\ 8\ 9\ 10\ 3\ 2\ 11\ 12\ 1 \\ \downarrow \eta \\ abbaaabbaab \end{array}$$

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The exotic automorphism ξ is induced by reversal of words.

$$\begin{array}{ccc} 6\ 7\ 5\ 4\ 8\ 9\ 10\ 3\ 2\ 11\ 12\ 1 & \xrightarrow{\eta} & abbaaabbaab \\ \downarrow \xi & & \downarrow \textit{reversal} \\ 6\ 5\ 7\ 8\ 4\ 3\ 9\ 10\ 11\ 2\ 1\ 12 & \xleftarrow{\eta^{-1}} & baabbaaabba \end{array}$$

Weeding out triples

- We don't have to find *all* triples.
- Given a triple $(\mathcal{A}, \mathcal{B}, \Gamma)$ and symmetries α, β then $(\mathcal{A}\alpha, \mathcal{B}\beta, \alpha^{-1}\Gamma\beta)$ is also a triple. We classify only to within this equivalence.
- Given a triple $(\mathcal{A}, \mathcal{B}, \Gamma)$ we may be able to extend Γ to larger pattern classes $\mathcal{A} \cup \{\alpha\}, \mathcal{B} \cup \{\beta\}$ by defining $\Gamma(\alpha) = \beta$. We are interested in "maximal" isomorphisms that cannot be extended in this way.
- Main result: a list of all maximal, inequivalent isomorphisms.

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Covering and reconstruction

Theorem (Rebecca Smith)

Let β_1, β_2 be two permutations with the same set of lower covers. Then $\beta_1 = \beta_2$ unless

1. $\{\beta_1, \beta_2\} = \{12, 21\}$,
2. $\{\beta_1, \beta_2\} \subseteq \{132, 213, 231, 312\}$, *or*
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$\mathcal{T} = \{1, 12, 21, 132, 213, 231, 312, 2413, 3142\}$

Corollary

Let $\mathcal{A}$ be a pattern class containing $\mathcal{T}$. Let $(\mathcal{A}, \mathcal{B}_1, \Gamma_1)$ and $(\mathcal{A}, \mathcal{B}_2, \Gamma_2)$ be two triples. Suppose $\Gamma_1|_{\mathcal{T}} = \Gamma_2|_{\mathcal{T}}$. Then $\Gamma_1 = \Gamma_2$ (and $\mathcal{B}_1 = \mathcal{B}_2$).

A process for constructing maximal triples $(\mathcal{A}, \mathcal{B}, \Gamma$

- Begin with any triple $(\mathcal{T}, \mathcal{T}, \Gamma)$. Put $\mathcal{A} = \mathcal{T}$ and $\mathcal{B} = \mathcal{T}$.
- (This step repeated until very tired).
  - Select a permutation α not in $\mathcal{A}$ whose lower covers do belong to $\mathcal{A}$.
  - Apply Γ to these lower covers to produce a set C of permutations in $\mathcal{B}$.
  - If there is a β whose set of lower covers is exactly C extend Γ by setting $\Gamma(\alpha) = \beta$, add α to $\mathcal{A}$ and β to $\mathcal{B}$.
  - If there is no such β add α to a list L of known basis elements of $\mathcal{A}$.
- After not too many repeats we seem to stop discovering new basis elements of $\mathcal{A}$. At this point we merely have to demonstrate that there are no more to be found; i.e. that Γ extends to $\mathrm{Av}(L)$.

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Typical example

- Initially Γ is defined on $\mathcal{T}$ by fixing all elements of $\mathcal{T}$ except for 2413 and 3142 which it exchanges.
- $\Gamma(123)$ determined as 123 because only 12 is a lower cover of 123, $\Gamma(12) = 12$, and 123 is the only permutation whose lower covering set is $\{12\}$. Similarly $\Gamma(321) = 321$.
- $\Gamma(1324) = 1324$ because the lower covers of 1324 are $\{123, 132, 213\}$ which map to $\{123, 132, 213\}$ and 1324 is the unique permutation with these lower covers.
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Typical example continued

- What about $\Gamma(13524)$? The lower covers of 13524 are $\{2413, 1423, 1324, 1243, 1342\}$ and these map to $\{3142, 1423, 1324, 1243, 1342\}$. There is a unique permutation with these lower covers – 14253 – and so this is the image of 13524. But. . .
- What about the image of 41253? The lower covers are $\{1243, 3142, 4123, 3124\}$ which ought to map to $\{1243, 2413, 4123, 3124\}$. But there is no permutation with this lower covering set. So Γ cannot be defined on 41253 and we put it into the basis of $\mathcal{A}$.
- Similar calculations produce 18 basis elements of length 5 and 4 of length 6. *We don't seem to find any of lengths 7, 8, 9.*
- To prove that there aren't any more basis elements we analyze the class $\mathcal{X}$ defined by the existing basis elements and show there is an isomorphism on $\mathcal{X}$ that extends the original Γ.

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Typical example continued: the structure of $\mathcal{X}$

- $\mathcal{X}$ has simple permutations $\{2413, 3142, 24153, 31524, 35142, 42513\}$
- None of these has any non-trivial inflation
- $\mathcal{X}$ is both $\oplus$ and $\ominus$ closed. Hence
- $\mathcal{X}$ is the $\oplus, \ominus$ closure of the set $\{1, 2413, 3142, 24153, 31524, 35142, 42513\}$

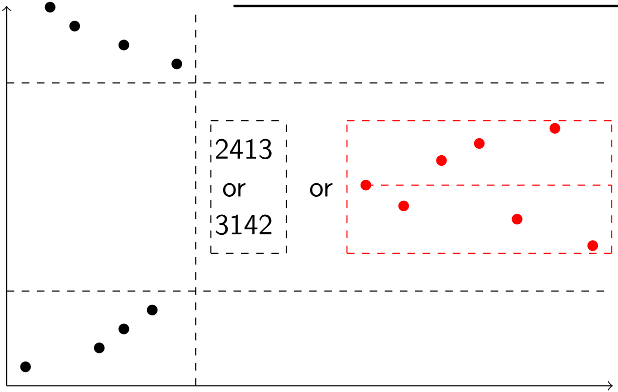
It is now easy to prove that the class really does have an isomorphism (indeed an automorphism) that extends the original Γ. Γ is defined on a permutation α by fixing all the points of α except for those that lie in intervals 2413 and 3142, which it exchanges.

The handle turns

- Carry out the above process with all 6 inequivalent triples $(\mathcal{T}, \mathcal{T}, \Gamma)$
- In all cases no basis elements of lengths more than 6 seem to arise
- Analyse the structure of the classes $\mathcal{X}$ defined by the basis elements that have been found
- Display an isomorphism defined on $\mathcal{X}$ that extends the original Γ
- Conclude that $\mathcal{X}$ is the maximal class to which the original Γ can be extended

The other maximal isomorphisms

$\mathcal{T}$	12	21	132	213	231	312	2413	3142
$\Gamma(\mathcal{T})$	12	21	132	231	213	312	2413	3142



Black points fixed,
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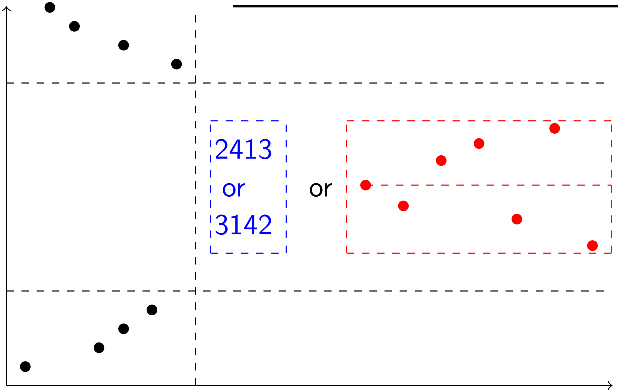

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$$\bigwedge \cup \bigvee \cup < \cup > \cup \boxed{\begin{matrix} 2143 & 3412 \\ 2413 & 3142 \\ 25314 & 41352 \end{matrix}}$$

The isomorphism maps Λ and V to themselves each with an exotic twist, exchanges $<$ and $>$ but with an exotic twist, swaps 2143 and 3412, and fixes 2413, 3142, 25314 and 41352.

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The isomorphism maps $\mathsf{V} \longrightarrow \, > \, \longrightarrow \Lambda \longrightarrow \, < \, \longrightarrow \mathsf{V}$ in a 4-cycle, with an exotic twist at each stage, swaps 2143 and 3412, and fixes 2413 and 3142.

Conclusions and consequences

1. Every isomorphism between pattern classes is, to within symmetry, the restriction of one of 6 maximal isomorphisms $(\mathcal{A}, \mathcal{B}, \Gamma)$
2. In each maximal case there is also a symmetry mapping $\mathcal{A}$ to $\mathcal{B}$
3. The structure and enumeration of each maximal $\mathcal{A}$ is known
4. If $f : \mathcal{A} \longrightarrow \mathcal{B}$ is any isomorphism that is not a symmetry then
   1. $\mathcal{A}$ is partially well-ordered
   2. $\mathcal{A}$ has at least one basis element of length at most 5
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