The double Eulerian polynomial

Erik Aas
KTH

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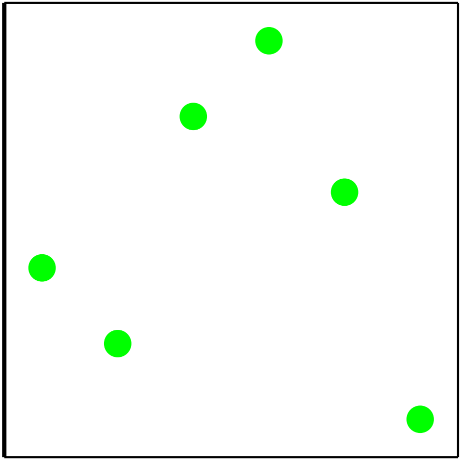
$\pi = 325641$



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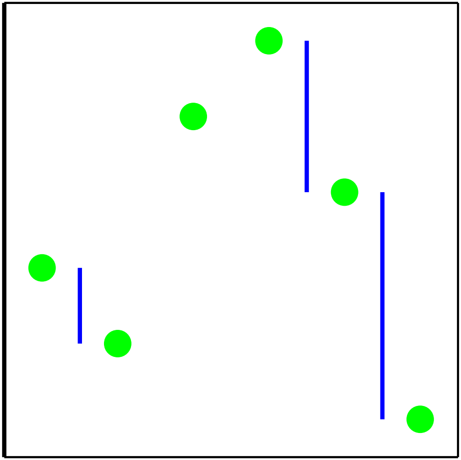

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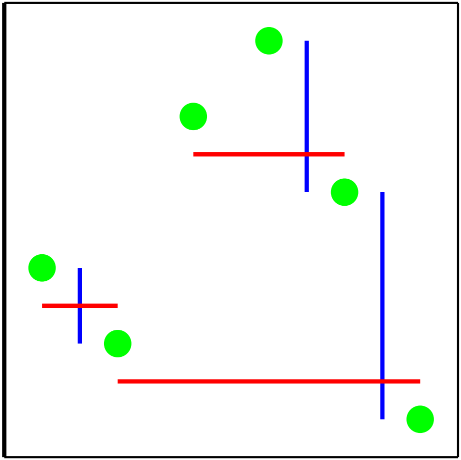

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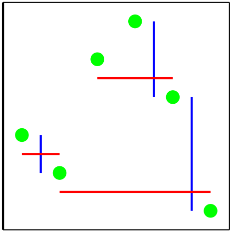

$\mathrm{des}(\pi) = 3$

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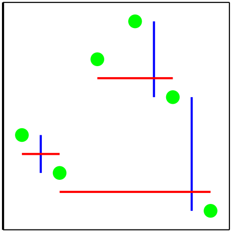

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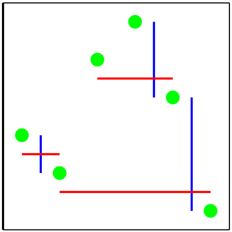

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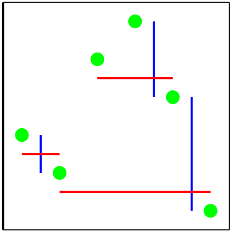

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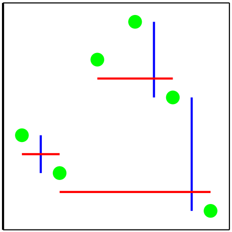

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$$A_4(x, y) = 1 + 10xy + xy^2 + x^2y + 10x^2y^2 + x^3y^3$$

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- $A_n(x,1) = \sum_{e\in\mathbb{I}_n} x^{\mathrm{asc}(e)} = \sum_{e\in\mathbb{I}_n} x^{\mathrm{row}(e)}$

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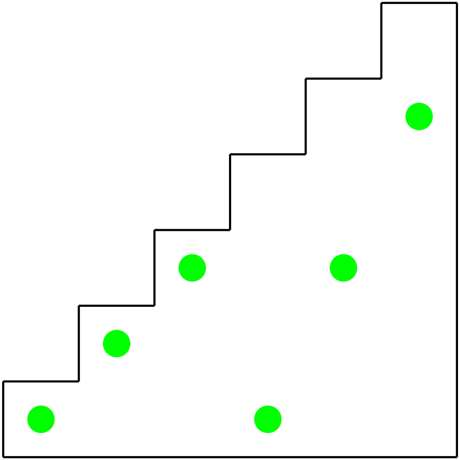
- Is $A_n(x,y)$ positive in the basis $(xy)^i(y+x)^j(1+yx)^{n-1-2i-j}$? [Gessel].
- $A_n(x,y) =^? \sum_{e\in\mathbb{I}_n} x^{\mathrm{asc}(e)}y^{\mathrm{row}(e)}$ [Visontai].

$e = 123135$



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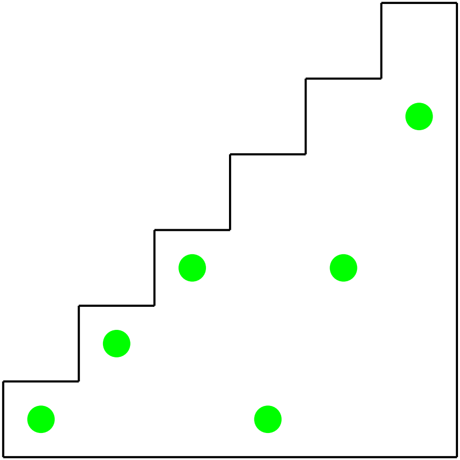

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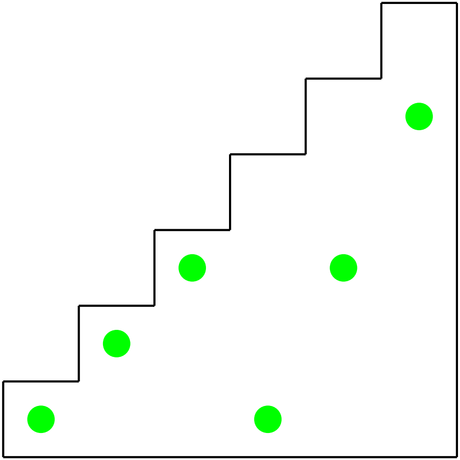

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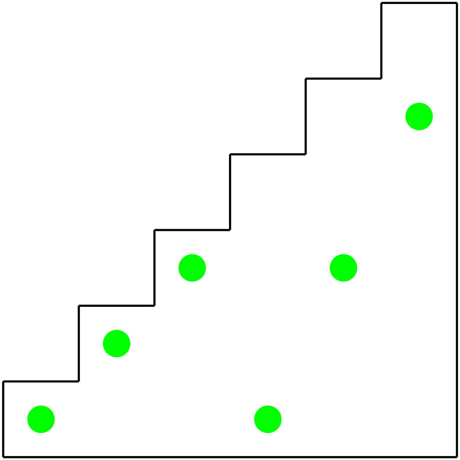

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$\mathrm{asc}(e) = 4$

$\mathrm{row}(e) = |\{e_i : e_i > 1\}| = 3$

Proof

Want to prove

$$\sum_{\pi \in \mathbb{S}_n} x^{\mathrm{des}(\pi)} y^{\mathrm{ides}(\pi)} = \sum_{e \in \mathbb{I}_n} x^{\mathrm{asc}(e)} y^{\mathrm{row}(e)}.$$

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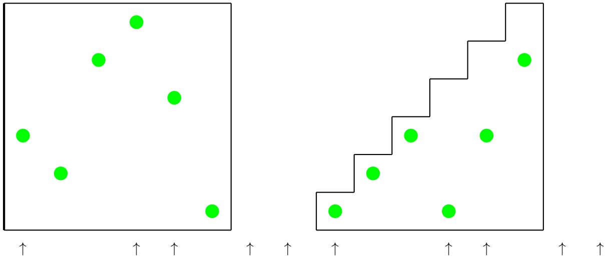
Fix S. Sufficient to prove:

$$\sum_{\pi \in \mathbb{S}_n, \mathsf{DES}(\pi) \supseteq S} y^{\mathsf{ides}(\pi)} = \sum_{e \in \mathbb{I}_n, \mathsf{ASC}(e) \supseteq S} y^{\mathsf{row}(e)}.$$

Choose $S = \{1, 4, 5, 7, 8\}$.



$\text{ides}(\pi) = \text{row}(e)$
$S \subseteq \text{DES}(\pi), \text{ASC}(e)$

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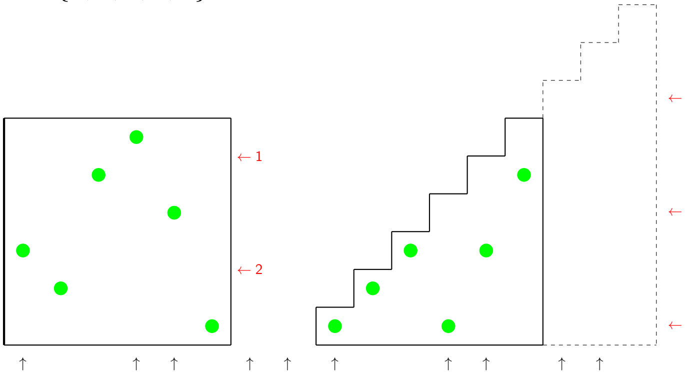

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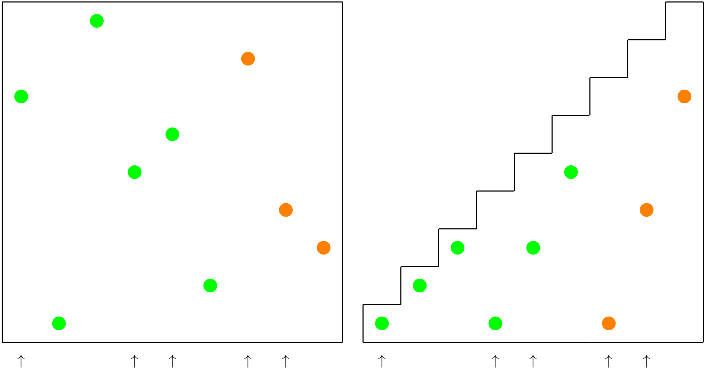

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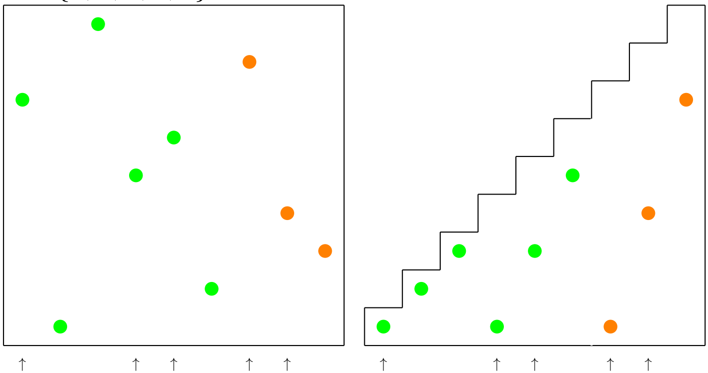

$1 + (2 - 1)$ new inverse descents.

2 new occupied rows.

"

$$\sum_{a=0}^{p} \binom{r-a-1}{a} \binom{a+s-t-1}{a} = \binom{r+s-t-1}{p}$$

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Thanks!