Mahonian-Stirling statistics on matchings and restricted permutations

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Permutation statistics

- Inversions: $\mathrm{inv}(\sigma) = \#$ pairs $i < j$ such that $\sigma(i) > \sigma(j)$

$$\sigma = 23514, \ \mathrm{inv}(\sigma) = 1 + 1 + 2 + 0 + 0 = 4$$

$$\sum_{\sigma \in S_n} q^{\mathrm{inv}(\sigma)} = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1})$$

  - st is a Mahonian statistic if $\sum_{\sigma \in S_n} q^{\mathrm{st}(\sigma)} = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1})$

- Cycles: $\sigma = 23514 = (12354)$, $\mathrm{cyc}(\sigma) = 1$

$$\sum_{\sigma \in S_n} t^{\mathrm{cyc}(\sigma)} = t(t+1)(t+2)\cdots(t+n-1)$$

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$$\sum_{\sigma \in S_n} t^{\mathrm{cyc}(\sigma)} = t(t+1)(t+2)\cdots(t+n-1)$$

But ...

$$\sum_{\sigma \in S_n} q^{\mathrm{inv}(\sigma)} t^{\mathrm{cyc}(\sigma)} \neq t(t+q)(t+q+q^2)\cdots(t+q+\cdots+q^{n-1})$$

Pairs of permutation statistics

$$\sum_{\sigma \in S_n} q^{\mathrm{inv}(\sigma)} t^{?(\sigma)} = t(t+q)(t+q+q^2)\cdots(t+q+\cdots+q^{n-1})$$

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Theorem (Björner-Wachs, 1991)

$$\sum_{\sigma \in S_n} q^{\mathrm{inv}(\sigma)} t^{\mathrm{rlmin}(\sigma)} = t(t+q)(t+q+q^2)\cdots(t+q+\cdots+q^{n-1})$$

In fact,

$$\sum_{\sigma \in S_n} q^{\mathrm{inv}(\sigma)} \prod_{i \in \mathrm{Rlminl}(\sigma)} t_i = t_1(t_2+q)\cdots(t_n+q+\cdots+q^{n-1})$$

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Theorem (Foata-Han, 2009; Petersen, 2011)

$$\sum_{\sigma \in S_n} q^{\mathrm{sor}(\sigma)} t^{\mathrm{cyc}(\sigma)} = t(t+q)(t+q+q^2)\cdots(t+q+\cdots+q^{n-1})$$

The partner statistics

- Right-to-left minimum letters:

$$\mathrm{Rlminl}(\sigma) = \#\{\sigma(i) : \sigma(i) < \sigma(j) \text{ for all } j > i\}$$

$$\mathrm{rlmin}(\sigma) = |\mathrm{Rlminl}(\sigma)|$$

$$\mathrm{Rlminl}(23514) = \{1, 4\} \qquad \mathrm{rlmin}(23514) = 2$$

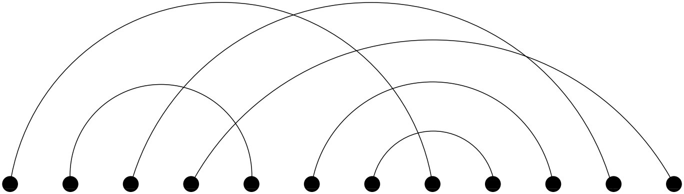
- Sorting index: $\mathrm{sor}(\sigma)$ = the total distance the elements of σ move in the *Straight selection sort* algorithm [Knuth]

$$23514 \to 23415 \to 23145 \to 21345 \to 12345$$

$$\mathrm{sor}(23514) = 2 + 1 + 1 + 1 + 0 = 5$$

Matchings

A (perfect) matching is a set partition with blocks of size exactly 2.



From permutations to matchings

Permutations $\leftrightarrow$ Matchings of type $\underbrace{\text{open}, \ldots, \text{open}}_{n}, \underbrace{\text{close}, \ldots, \text{close}}_{n}$

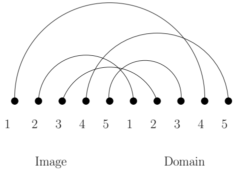


Figure: The permutation $\sigma = 23514$.

- inv $\leftrightarrow$ nestings
- rlmin $\leftrightarrow$ arcs that are not nested below anything

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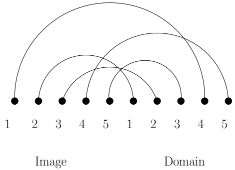


Figure: The permutation $\sigma = 23514$.

- inv $\leftrightarrow$ nestings $= \mathrm{ne}(M)$
- rlmin $\leftrightarrow$ arcs that are not nested below anything $= \mathrm{rlmin}(M)$

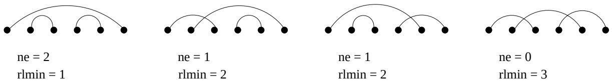
These statistics can be be defined for matchings of any type.

Distribution of $(\mathrm{ne}, \mathrm{rlmin})$

- $D =$ a fixed possible type of a matching
  Example: $\mathrm{open}, \mathrm{open}, \mathrm{close}, \mathrm{open}, \mathrm{close}, \mathrm{close}$

Distribution of (ne, rlmin)

- $D =$ a fixed possible type of a matching
  Example: open, open, close, open, close, close



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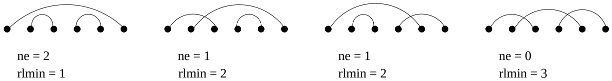

$$\sum_{M \in \mathcal{M}(D)} q^{\mathrm{ne}(M)} \prod_{i \in \mathrm{Rlminl}(M)} t_i = q^2 t_1 + q t_1 t_2 + q t_1 t_3 + t_1 t_2 t_3$$

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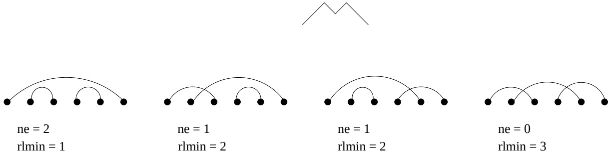
  Example: open, open, close, open, close, close



$$\sum_{M\in\mathcal{M}(D)} q^{\mathrm{ne}(M)} \prod_{i\in \mathrm{Rlminl}(M)} t_i = t_1(t_2+q)(t_3+q)$$

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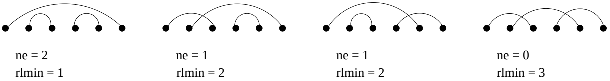

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Theorem

$$\sum_{M \in \mathcal{M}_{2n}(D)} q^{\mathrm{ne}(M)} \prod_{i \in \mathrm{Rlminl}(M)} t_i = \prod_{k=1}^{n} (t_k + q + \cdots + q^{h_k})$$

- Follows from a bijection due to de Sainte-Catherine (1993).

Distribution of (ne, rlmin)

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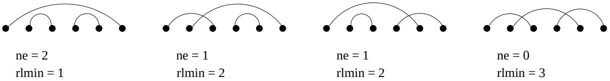
  Example: open, open, close, open, close, close



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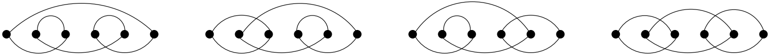
- h_k = the starting level of the k-th up step in D

$$h_1 = 0, h_2 = 1, h_3 = 1$$

- cyc(M) = # cycles in the graph of M when the non-nonesting matching of same type is drawn in the bottom half

Example:



sor = 2
cyc = 1

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cyc = 3

cyc and sor for matchings

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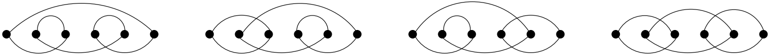

sor = 2	sor = 1	sor = 1	sor = 0
cyc = 1	cyc = 2	cyc =2	cyc = 3

- sor(M) = total distance the arcs "move" when the matching M is sorted to be non-nesting

$$\sum_{M \in \mathcal{M}_{2n}(D)} q^{\mathrm{sor}(M)} \prod_{i \in \mathrm{Cyc}(M)} t_i = q^2 t_1 + q t_1 t_2 + q t_1 t_3 + t_1 t_2 t_3$$

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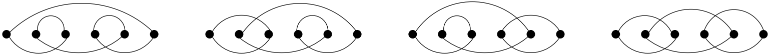

sor = 2, cyc = 1 | sor = 1, cyc = 2 | sor = 1, cyc =2 | sor = 0, cyc = 3

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$$\sum_{M \in \mathcal{M}_{2n}(D)} q^{\mathrm{sor}(M)} \prod_{i \in \mathrm{Cyc}(M)} t_i = t_1(t_2+q)(t_3+q)$$

Main result

Theorem (P.)

$$\sum_{M \in \mathcal{M}_{2n}(D)} q^{\mathrm{sor}(M)} \prod_{i \in \mathrm{Cyc}(M)} t_i = \prod_{k=1}^{n} (t_k + q + \cdots + q^{h_k})$$

where

$h_k = \#$ *openers* $- \#$ *closers to the left of the* k*-th opener in* D.

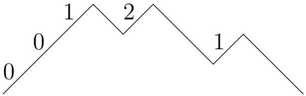
Corollary

$$\sum_{M \in \mathcal{M}_{2n}(D)} q^{\mathrm{sor}(M)} \prod_{i \in \mathrm{Cyc}(M)} t_i = \sum_{M \in \mathcal{M}_{2n}(D)} q^{\mathrm{ne}(M)} \prod_{i \in \mathrm{Rlminl}(M)} t_i$$

Proof outline

- Bijection: weighted Dyck paths $\longrightarrow$ Matchings
- Dyck path
  - starts at the origin
  - steps $(1,1)$ and $(1,-1)$
  - ends at the x-axis
  - never goes below the x-axis
- Weighted Dyck paths (Histoire d'Hermite)
  - weight w_k is assigned to the k-th $(1,1)$ step, $0 \leq k \leq n$
  - restriction: $0 \leq w_k \leq h_k$,
    where h_k is the height of the k-th $(1,1)$ step

The map: Weighted Dyck paths → Matchings (P.)



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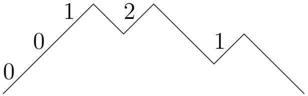
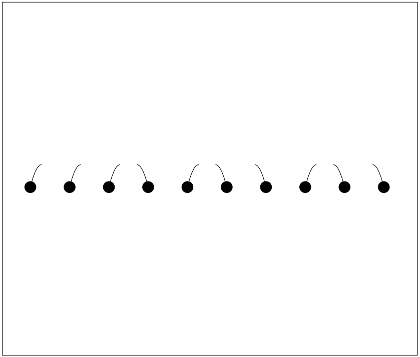


$\longrightarrow$



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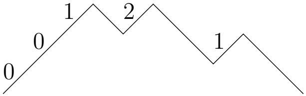

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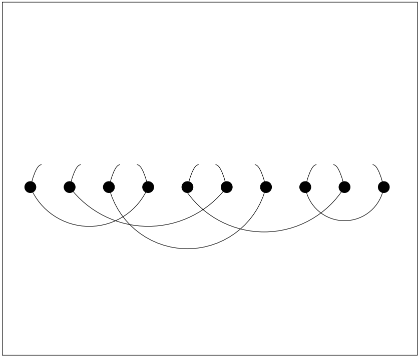

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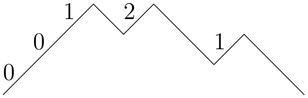

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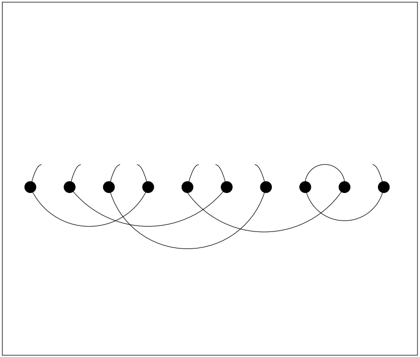

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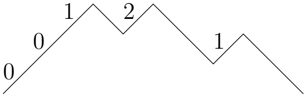

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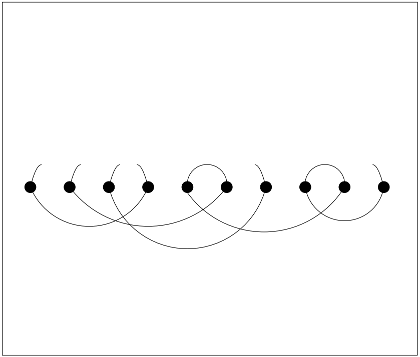

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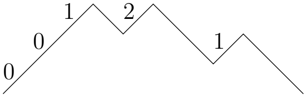

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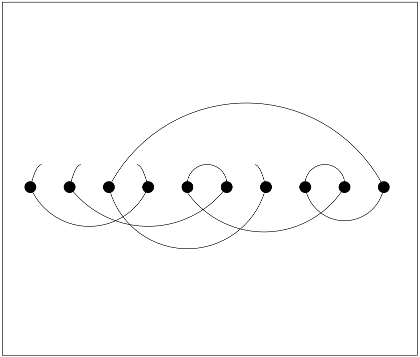

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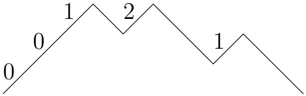

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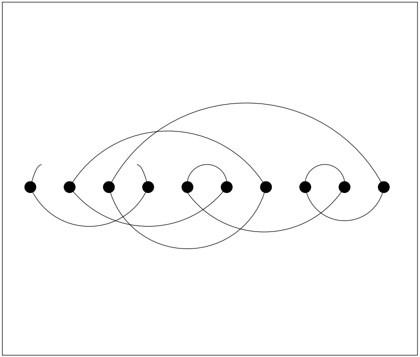

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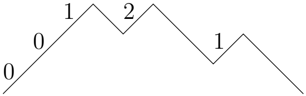

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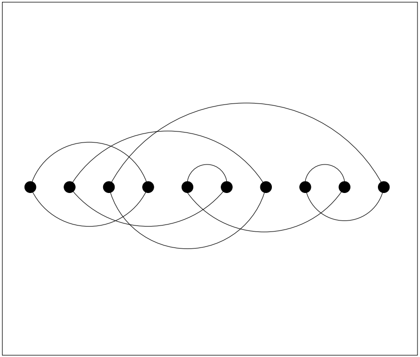

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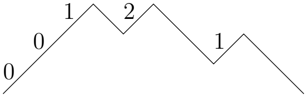

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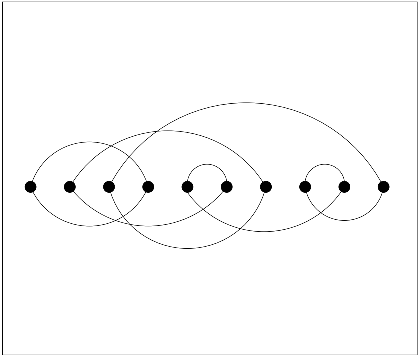

- zero weights $\leftrightarrow$ Cycles of M
- $0 + 0 + 1 + 2 + 1 = \mathrm{sor}(M)$

Generating function

Corollary

$$\sum_{M \in \mathcal{M}_{2n}(D)} q^{\mathrm{sor}(M)} \prod_{i \in \mathrm{Cyc}(M)} t_i = \prod_{k=1}^{n} (t_k + q + \cdots + q^{h_k}),$$

where $\mathrm{Cyc}(M)$ *is the set of minimal vertices in the cycles of* M*.*

Generating function

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where $\mathrm{Cyc}(M)$ *is the set of minimal vertices in the cycles of* M*. Therefore,*

$$\sum_{M\in\mathcal{M}_{2n}(D)} q^{\mathrm{sor}(M)} \prod_{i\in\mathrm{Cyc}(M)} t_i = \sum_{M\in\mathcal{M}_{2n}(D)} q^{\mathrm{ne}(M)} \prod_{i\in\mathrm{Rlminl}(M)} t_i .$$

Back to permutations

- There is a one-to-one correspondence between $\mathcal{M}_{2n}(D)$ and permutations that "fit" in a Young diagram with boundary D.

Corollary (P.)

Let $\mathbf{r} = 1 \leq r_1 \leq r_2 \leq \cdots \leq r_n$ *and* $S_{\mathbf{r}} = \{\sigma \in S_n : \sigma(i) \leq r_i\}$. *Then*

$$\sum_{\sigma \in S_{\mathbf{r}}} q^{\mathrm{sor}(\sigma)} \prod_{i \in \mathrm{Cyc}(\sigma)} t_i = \sum_{\sigma \in S_{\mathbf{r}}} q^{\mathrm{inv}(\sigma)} \prod_{i \in \mathrm{Rlminl}(\sigma)} t_i.$$

Additional properties of the bijections

Moreover, define

- Left-to-right maxima positions

$$\mathrm{Lrmaxp}(\sigma) = \{i : \sigma(j) < \sigma(i), \forall j < i\}$$

Refined result of Foata and Han (2009):

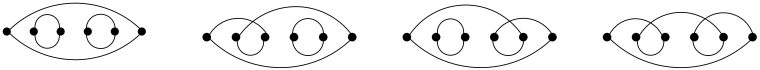
Corollary (P.)

The multisets $\{(\mathrm{sor}(\sigma), \mathrm{Cyc}(\sigma), \mathrm{Lrmaxp}(\sigma)) : \sigma \in S_{\mathbf{r}}\}$ *and* $\{(\mathrm{inv}(\sigma), \mathrm{Rlminl}(\sigma), \mathrm{Lrmaxp}(\sigma)) : \sigma \in S_{\mathbf{r}}\}$ *are equal.*

Changing the bottom matching

Example: $\sigma_0 = 321$ in the bottom half

$\sigma = 231$ $\quad$ $\sigma = 132$ $\quad$ $\sigma = 213$ $\quad$ $\sigma = 123$



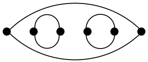
$\sigma_0^{-1}\sigma = 123$	$\sigma_0^{-1}\sigma = 321$	$\sigma_0^{-1}\sigma = 132$	$\sigma_0^{-1}\sigma = 312$
$\mathrm{cyc} = 3$	$\mathrm{cyc} = 2$	$\mathrm{cyc} = 2$	$\mathrm{cyc} = 1$

- Fix the bottom matching $M_0 \in \mathcal{M}_{2n}(D)$. We can define $\mathrm{cyc}(M, M_0)$ and $\mathrm{sor}(M, M_0)$.

Changing the bottom matching

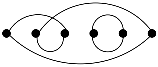
Example: $\sigma_0 = 321$ in the bottom half

$\sigma = 231$



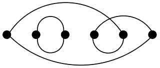
$\sigma_0^{-1}\sigma = 123$
$\text{cyc} = 3$

$\sigma = 132$



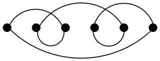
$\sigma_0^{-1}\sigma = 321$
$\text{cyc} = 2$

$\sigma = 213$



$\sigma_0^{-1}\sigma = 132$
$\text{cyc} = 2$

$\sigma = 123$



$\sigma_0^{-1}\sigma = 312$
$\text{cyc} = 1$

- Fix the bottom matching $M_0 \in \mathcal{M}_{2n}(D)$. We can define $\mathrm{cyc}(M, M_0)$ and $\mathrm{sor}(M, M_0)$.

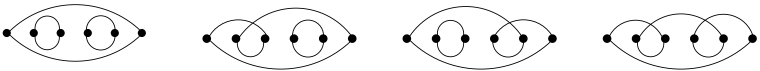
Theorem (P.)

$$\sum_{M \in \mathcal{M}_{2n}(D)} q^{\mathrm{sor}(M,M_0)} \prod_{i \in \mathrm{Cyc}(M,M_0)} t_i = \prod_{k=1}^{n} (t_k + q + \cdots + q^{h_k}),$$

Changing the bottom matching

Example: $\sigma_0 = 321$ in the bottom half

$\sigma = 231$ | $\sigma = 132$ | $\sigma = 213$ | $\sigma = 123$



$\sigma_0^{-1}\sigma = 123$, cyc $= 3$ | $\sigma_0^{-1}\sigma = 321$, cyc $= 2$ | $\sigma_0^{-1}\sigma = 132$, cyc $= 2$ | $\sigma_0^{-1}\sigma = 312$, cyc $= 1$

- Fix the bottom matching $M_0 \in \mathcal{M}_{2n}(D)$. We can define $\mathrm{cyc}(M, M_0)$ and $\mathrm{sor}(M, M_0)$.

Corollary

Let $\mathbf{r} = 1 \leq r_1 \leq r_2 \leq \cdots \leq r_n$ *and* $S_{\mathbf{r}} = \{\sigma \in S_n : \sigma(i) \leq r_i\}$. *Fix* $\sigma_0 \in S_{\mathbf{r}}$. *Then*

$$\sum_{\sigma \in S_{\mathbf{r}}} t^{\mathrm{cyc}(\sigma_0^{-1}\sigma)} = \sum_{\sigma \in S_{\mathbf{r}}} t^{\mathrm{cyc}(\sigma)} = \prod_{k=1}^{n} (t + r_k - k).$$

Signed permutations

Theorem (Petersen, 2011)

$$\sum_{\sigma\in B_n} q^{\mathrm{inv}_B(\sigma)} t^{\mathrm{nmin}(\sigma)} = \sum_{\sigma\in B_n} q^{\mathrm{sor}_B(\sigma)} t^{\ell'_B(\sigma)} = \prod_{i=1}^{n}(1 + t[2i]_q - t)$$

- $\mathrm{inv}_B(\sigma) = \#\{1 \leq i < j \leq n : \sigma(i) > \sigma(j)\}$
  $+\#\{1 \leq i < j \leq n : -\sigma(i) > \sigma(j)\} + N(\sigma)$
- $\mathrm{nmin}(\sigma) = \#\{i : \sigma(i) > |\sigma(j)| \text{ for some } j > i\} + N(\sigma)$
- $\mathrm{sor}_B(134\bar{2}2\bar{4}\bar{3}\bar{1}) = 5 + 5 + 1 = 11$

$$134\bar{2}2\bar{4}3\bar{1} \to \bar{4}3\bar{1}\bar{2}21\bar{3}4 \to \bar{4}\bar{3}\bar{1}\bar{2}2134 \to \bar{4}\bar{3}\bar{2}\bar{1}1234$$

- $\ell'_B(\sigma)$ – reflection length

Bicolored matchings

Theorem (P.)

Let $\mathbf{r} = 1 \leq r_1 \leq r_2 \leq \cdots \leq r_n$ *and* $B_{\mathbf{r}} = \{\sigma \in B_n : |\sigma(i)| \leq r_i\}$ *(restricted signed permutations). Then*

$$\sum_{\sigma \in B_{\mathbf{r}}} q^{\mathrm{sor}_B(\sigma)} t^{\mathrm{nmin}_B(\sigma)} = \sum_{\sigma \in B_{\mathbf{r}}} q^{\mathrm{inv}_B(\sigma)} t^{\ell'_B(\sigma)}.$$

Signed permutations $\leftrightarrow$ Bicolored matchings

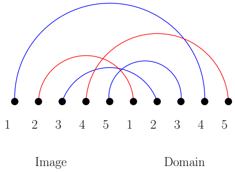


Figure: The permutation $\sigma = 2\bar{3}\bar{5}\bar{1}4$.

Thank you.