Pattern Avoidance in Poset Permutations

Sam Hopkins and Morgan Weiler

Massachusetts Institute of Technology and University of California, Berkeley

Permutation Patterns, Paris; July 5th, 2013

Section 1

Definitions and Introduction

Preliminary Definitions and Notation

Throughout, let P be a partially ordered set on n elements, under the relation $\prec$.

Definition

A *permutation* σ on P is a bijection

$$\sigma : \{1, \ldots, n\} \to P$$

- $\sigma_i := \sigma(i)$, the *entry* at the i^{th} position
- $\sigma = (\sigma_1, \cdots, \sigma_n)$
- S_P denotes the set of permutations on P
- B_n denotes the set of permutations on $\mathbb{B}_n$, the Boolean lattice on n elements

Pattern Containment

Throughout, for $a, b \in P$ let $a \sim b$ denote "a is incomparable to b."

Pattern Containment

Throughout, for $a, b \in P$ let $a \sim b$ denote "a is incomparable to b."

Definition

A *pattern* is a permutation considered only as a set of positions and a set of order relations (including $\sim$), one specified for each pair of positions. In that case we do not use parenthesis and commas, but simply let $\sigma = \sigma_1 \cdots \sigma_n$.

Pattern Containment

Throughout, for $a, b \in P$ let $a \sim b$ denote "a is incomparable to b."

Definition

A *pattern* is a permutation considered only as a set of positions and a set of order relations (including $\sim$), one specified for each pair of positions. In that case we do not use parenthesis and commas, but simply let $\sigma = \sigma_1 \cdots \sigma_n$.

Definition

Let P be a poset on n elements and Q a poset on k elements under the relation $\prec'$. For $\sigma \in S_P$ and $\pi \in S_Q$, we say σ *contains the pattern* π if there are k entries $\sigma_{i_1}, \ldots, \sigma_{i_k} \in \{\sigma_1, \ldots, \sigma_n\}$ with $i_1 < \cdots < i_k$ such that for all $1 \leq a < b \leq k$ we have

$$\sigma_{i_a} \prec, \succ, \sim \sigma_{i_b} \text{ if and only if } \pi_a \prec', \succ', \sim \pi_b \text{ respectively}$$

An Example and Further Definitions

Example

Let $\sigma \in B_3$ be given by $(\{2,3\},\{2\},\{1,3\},\{1,2,3\},\{1\},\emptyset,\{1,2\},\{3\})$. σ contains the pattern $\{1\}\{3\}\{1,2\}$ in the subsequence $(\{2\},\{1,3\},\{1,2\})$, but avoids $\emptyset\{1\}\{1,2\}$.

An Example and Further Definitions

Example

Let $\sigma \in B_3$ be given by $(\{2,3\},\{2\},\{1,3\},\{1,2,3\},\{1\},\emptyset,\{1,2\},\{3\})$. σ contains the pattern $\{1\}\{3\}\{1,2\}$ in the subsequence $(\{2\},\{1,3\},\{1,2\})$, but avoids $\emptyset\{1\}\{1,2\}$.

When considering patterns within chains of a poset we use the notation from permutations on sets $[k]=\{1,\ldots,k\}$; in this way the pattern $\emptyset\{1\}\{1,2\}$ can be represented 123. Else we use notation from the Boolean lattice on the smallest required number of elements.

An Example and Further Definitions

Example

Let $\sigma \in B_3$ be given by $(\{2,3\},\{2\},\{1,3\},\{1,2,3\},\{1\},\emptyset,\{1,2\},\{3\})$. σ contains the pattern $\{1\}\{3\}\{1,2\}$ in the subsequence $(\{2\},\{1,3\},\{1,2\})$, but avoids $\emptyset\{1\}\{1,2\}$.

When considering patterns within chains of a poset we use the notation from permutations on sets $[k] = \{1,\ldots,k\}$; in this way the pattern $\emptyset\{1\}\{1,2\}$ can be represented 123. Else we use notation from the Boolean lattice on the smallest required number of elements.

Definition

Let $Av_P(\sigma)$ denote the number of permutations in S_P which avoid σ.
Let $Av_n(\sigma)$ denote the number of permutations in B_n which avoid σ.

Motivation

We have two principle motivations, one from pattern avoidance theory and one from order theory:

Motivation

We have two principle motivations, one from pattern avoidance theory and one from order theory:

> To see how little order structure is necessary to recapture results from classical pattern avoidance (e.g. generalizes multiset permutations).

Motivation

We have two principle motivations, one from pattern avoidance theory and one from order theory:

> To see how little order structure is necessary to recapture results from classical pattern avoidance (e.g. generalizes multiset permutations).
>
> As a generalization of counting linear extensions of a poset. (What does stack "sorting" a partially ordered set look like?)

Easy Equivalences

Example

The *reverse* of $(\{1, 2\}, \emptyset, \{1\}, \{2\}) \in B_2$ is $(\{2\}, \{1\}, \emptyset, \{1, 2\})$.

Example

The *dual* of $(\{1, 2\}, \emptyset, \{1\}, \{2\}) \in B_2$ is $(\emptyset, \{1, 2\}, \{2\}, \{1\})$.

Easy Equivalences

Example

The *reverse* of $(\{1,2\}, \emptyset, \{1\}, \{2\}) \in B_2$ is $(\{2\}, \{1\}, \emptyset, \{1,2\})$.

Example

The *dual* of $(\{1,2\}, \emptyset, \{1\}, \{2\}) \in B_2$ is $(\emptyset, \{1,2\}, \{2\}, \{1\})$.

Fact

σ and its reverse are Wilf equivalent. If P is self-dual, σ and its dual are Wilf equivalent.

Wilf Classes of Length Two Patterns

There are three length two patterns, corresponding to the possible relations between distinct elements of any poset: 12, 21, and $\{1\}\{2\}$.

Wilf Classes of Length Two Patterns

There are three length two patterns, corresponding to the possible relations between distinct elements of any poset: 12, 21, and $\{1\}\{2\}$.

Definition

A *linear extension* is a bijection $\lambda : [n] \to P$ such that for all $1 \leq i < j \leq n$, we have $\lambda_i < \lambda_j \Rightarrow i < j$.

Example

The permutation

$$(\emptyset, \{1\}, \{3\}, \{1,3\}, \{2\}, \{1,2\}, \{2,3\}, \{1,2,3\})$$

is a linear extension of $\mathbb{B}_3$.

An Asymptotic Bound on Linear Extensions

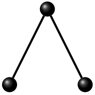
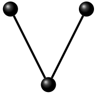
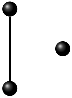
Linear extensions are total orderings of P which "respect" the partial order of P; in our language, these are exactly the 21-avoiding elements of S_P. We shall denote the number of linear extensions by $Av_n(21)$ so as to be consistent with our notation.

We have the asymptotic bound

$$\frac{\log Av_n(21)}{2^n} = \log \binom{n}{\lfloor n/2 \rfloor} - \frac{3}{2} \log e + O\left(\frac{\ln n}{n}\right)$$

with logarithms base 2 (Brightwell and Tetali).

Wilf Classes of Length Three Patterns for B_n

	123, 321
	132, 312, 213, 231
 or 	$\{1\}\{2\}\{1,2\}$, $\{1\}\{2\}\emptyset$, $\emptyset\{1\}\{2\}$, $\{1,2\}\{1\}\{2\}$
	$\{1\}\{1,2\}\{2\}$, $\{1\}\emptyset\{2\}$
	$\{1\}\{3\}\{1,2\}$, $\{1,2\}\{3\}\{1\}$
	$\{1\}\{1,2\}\{3\}$, $\{1,2\}\{1\}\{3\}$, $\{3\}\{1\}\{1,2\}$, $\{3\}\{1,2\}\{1\}$
	$\{1\}\{2\}\{3\}$

Section 2

The Patterns 132 and 123

Poset 132- vs. 123-avoidance

We want to mimic the classic bijection between 123- and 132-avoiders of Simion and Schmidt. However, fixing the *left-to-right minimal elements* (LRME) of σ, there may be many ways to fill in the remaining entries and avoid 123 or 132.

Poset 132- vs. 123-avoidance

We want to mimic the classic bijection between 123- and 132-avoiders of Simion and Schmidt. However, fixing the *left-to-right minimal elements* (LRME) of σ, there may be many ways to fill in the remaining entries and avoid 123 or 132.

Example

With $\sigma = (\{2,3\}, \{1\}, \{1,2\}, \{2\}, \emptyset, \{3\}, \{1,3\}, \{1,2,3\})$, the LRME are in red. Note that σ avoids 132 but the permutations

$(\{2,3\}, \{1\}, \{1,3\}, \{2\}, \emptyset, \{1,2\}, \{3\}, \{1,2,3\})$;

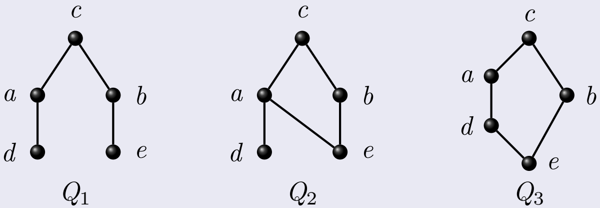
$(\{2,3\}, \{1\}, \{1,3\}, \{2\}, \emptyset, \{3\}, \{1,2\}, \{1,2,3\})$

have the same LRME in the same positions as σ and also avoid 132.

An injection from 132- to 123-avoiders

Theorem

We have $Av_P(132) \leq Av_P(123)$ *for any poset* P*, with strict inequality iff* P *contains one of* Q_1*,* Q_2*, or* Q_3 *below as an induced subposet:*



Proof sketch of injection

We construct an injection from the 132-avoiders to the 123-avoiders. Let σ be a 132-avoider.

Fix the positions of the LRME of σ. Let P' be the induced subposet of P on the non-LRME elements of σ. Label the non-LRME positions from left to right as $1, \ldots, k$. Each $x \in P'$ has a first position $\omega(x)$ it can occupy so that it is not an LRME. If, as we fill in non-LRME positions from left to right we always chose a maximal element among legal choices in P' we will avoid 123; if we always chose a minimal element we will avoid 132.

Proof sketch of injection

We construct an injection from the 132-avoiders to the 123-avoiders. Let σ be a 132-avoider.

Fix the positions of the LRME of σ. Let P' be the induced subposet of P on the non-LRME elements of σ. Label the non-LRME positions from left to right as $1, \ldots, k$. Each $x \in P'$ has a first position $\omega(x)$ it can occupy so that it is not an LRME. If, as we fill in non-LRME positions from left to right we always chose a maximal element among legal choices in P' we will avoid 123; if we always chose a minimal element we will avoid 132.

Example

Consider $\sigma = (\{2,3\}, \{1\}, \{1,2\}, \{2\}, \emptyset, \{3\}, \{1,3\}, \{1,2,3\})$. The LRME are in red. We have $\omega(\{1,3\}) = \omega(\{1,2\}) = \omega(\{1,2,3\}) = 1$, while $\omega(\{3\}) = 2$.

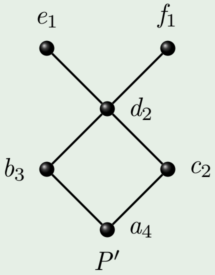
Proof sketch of injection cont'd

We say σ is ω-*legal* if $\omega(\sigma_i) \leq i$. Let $\Lambda^\omega \subseteq S_{P'}$ be the ω-legal perms.

$\Lambda^\omega_{\text{min}} \subseteq \Lambda^\omega$ which is left-to-right minimal;

$\Lambda^\omega_{\text{max}} \subseteq \Lambda^\omega$ which is left-to-right maximal.

Example



In the diagram on the left, each $x \in P'$ has as subscript $\omega(x)$.

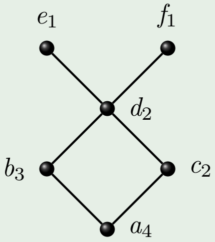
Proof sketch of injection cont'd

We say σ is ω-*legal* if $\omega(\sigma_i) \le i$. Let $\Lambda^\omega \subseteq S_{P'}$ be the ω-legal perms.

$\Lambda^\omega_{\text{min}} \subseteq \Lambda^\omega$ which is left-to-right minimal;

$\Lambda^\omega_{\text{max}} \subseteq \Lambda^\omega$ which is left-to-right maximal.

Example



P'

In the diagram on the left, each $x \in P'$ has as subscript $\omega(x)$.

$\sigma = f\ c\ b\ a\ d\ e$ is in $\Lambda^\omega_{\text{min}}$;

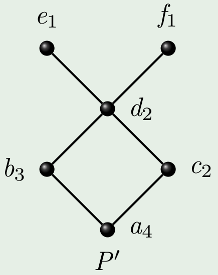
Proof sketch of injection cont'd

We say σ is ω-*legal* if $\omega(\sigma_i) \leq i$. Let $\Lambda^\omega \subseteq S_{P'}$ be the ω-legal perms.

$\Lambda^\omega_{\text{min}} \subseteq \Lambda^\omega$ which is left-to-right minimal;

$\Lambda^\omega_{\text{max}} \subseteq \Lambda^\omega$ which is left-to-right maximal.

Example



In the diagram on the left, each $x \in P'$ has as subscript $\omega(x)$.

$\sigma = f\ c\ b\ a\ d\ e$ is in $\Lambda^\omega_{\text{min}}$;
$\tau = f\ e\ d\ c\ b\ a$ is in $\Lambda^\omega_{\text{max}}$;

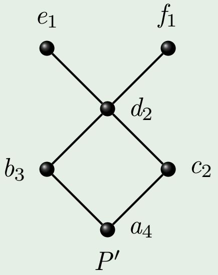
Proof sketch of injection cont'd

We say σ is ω-*legal* if $\omega(\sigma_i) \leq i$. Let $\Lambda^\omega \subseteq S_{P'}$ be the ω-legal perms.

$\Lambda^\omega_{\text{min}} \subseteq \Lambda^\omega$ which is left-to-right minimal;

$\Lambda^\omega_{\text{max}} \subseteq \Lambda^\omega$ which is left-to-right maximal.

Example



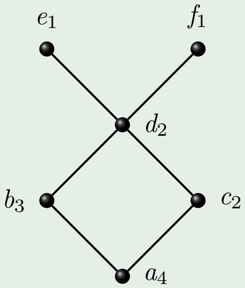
In the diagram on the left, each $x \in P'$ has as subscript $\omega(x)$.

$\sigma = f\ c\ b\ a\ d\ e$ is in $\Lambda^\omega_{\text{min}}$;
$\tau = f\ e\ d\ c\ b\ a$ is in $\Lambda^\omega_{\text{max}}$;
$\pi = f\ e\ d\ a\ b\ c$ is in neither.

Proof sketch of injection cont'd

Crucially, $x \geq y \Rightarrow \omega(x) \leq \omega(y)$. This allows an injective algorithm, call it $\phi\colon \Lambda^{\omega}_{\min} \to \Lambda^{\omega}_{\max}$. The algorithm considers each entry in turn, cycling through greater elements that could occupy that position.

Example



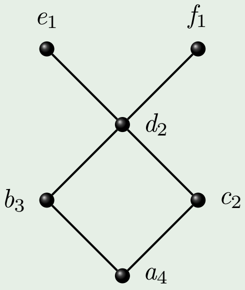
P'

$$\sigma =: \sigma^0 = \underline{\mathbf{f}}\ c\ b\ a\ d\ e$$

Proof sketch of injection cont'd

Crucially, $x \geq y \Rightarrow \omega(x) \leq \omega(y)$. This allows an injective algorithm, call it $\phi\colon \Lambda^{\omega}_{\text{min}} \to \Lambda^{\omega}_{\text{max}}$. The algorithm considers each entry in turn, cycling through greater elements that could occupy that position.

Example



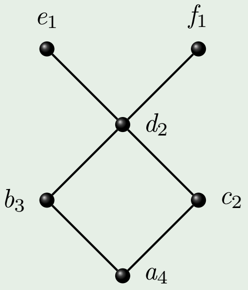
P'

$$\sigma =: \sigma^0 = \underline{\mathbf{f}}\ c\ b\ a\ d\ e$$
$$\sigma^1 = f\ \underline{\mathbf{c}}\ b\ a\ \underline{\mathbf{d}}\ \underline{\mathbf{e}}$$

Proof sketch of injection cont'd

Crucially, $x \geq y \Rightarrow \omega(x) \leq \omega(y)$. This allows an injective algorithm, call it $\phi \colon \Lambda^{\omega}_{\min} \to \Lambda^{\omega}_{\max}$. The algorithm considers each entry in turn, cycling through greater elements that could occupy that position.

Example



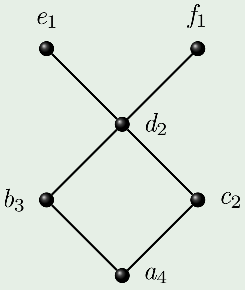
P'

$$\sigma =: \sigma^0 = \underline{\mathbf{f}}\ c\ b\ a\ d\ e$$
$$\sigma^1 = f\ \underline{\mathbf{c}}\ b\ a\ \underline{\mathbf{d}}\ \underline{\mathbf{e}}$$
$$\sigma^2 = f\ e\ \underline{\mathbf{b}}\ a\ c\ \underline{\mathbf{d}}$$

Proof sketch of injection cont'd

Crucially, $x \geq y \Rightarrow \omega(x) \leq \omega(y)$. This allows an injective algorithm, call it $\phi\colon \Lambda^{\omega}_{\min} \to \Lambda^{\omega}_{\max}$. The algorithm considers each entry in turn, cycling through greater elements that could occupy that position.

Example



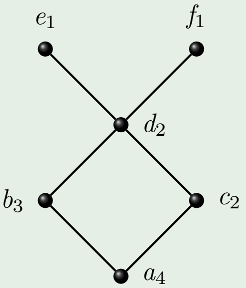
P'

$$\begin{aligned} \sigma =: \sigma^0 &= \underline{\mathbf{f}}\ c\ b\ a\ d\ e \\ \sigma^1 &= f\ \underline{\mathbf{c}}\ b\ a\ \underline{\mathbf{d}}\ \underline{\mathbf{e}} \\ \sigma^2 &= f\ e\ \underline{\mathbf{b}}\ a\ c\ \underline{\mathbf{d}} \\ \sigma^3 &= f\ e\ d\ \underline{\mathbf{a}}\ \underline{\mathbf{c}}\ b \end{aligned}$$

Proof sketch of injection cont'd

Crucially, $x \geq y \Rightarrow \omega(x) \leq \omega(y)$. This allows an injective algorithm, call it $\phi\colon \Lambda^{\omega}_{\min} \to \Lambda^{\omega}_{\max}$. The algorithm considers each entry in turn, cycling through greater elements that could occupy that position.

Example



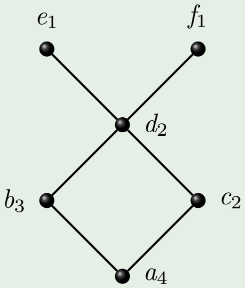
P'

$$
\begin{aligned}
\sigma =: \sigma^0 &= \underline{\mathbf{f}}\ c\ b\ a\ d\ e \\
\sigma^1 &= f\ \underline{\mathbf{c}}\ b\ a\ \underline{\mathbf{d}}\ \underline{\mathbf{e}} \\
\sigma^2 &= f\ e\ \underline{\mathbf{b}}\ a\ c\ \underline{\mathbf{d}} \\
\sigma^3 &= f\ e\ d\ \underline{\mathbf{a}}\ \underline{\mathbf{c}}\ b \\
\sigma^4 &= f\ e\ d\ c\ \underline{\mathbf{a}}\ \underline{\mathbf{b}}
\end{aligned}
$$

Proof sketch of injection cont'd

Crucially, $x \geq y \Rightarrow \omega(x) \leq \omega(y)$. This allows an injective algorithm, call it $\phi\colon \Lambda^{\omega}_{\min} \to \Lambda^{\omega}_{\max}$. The algorithm considers each entry in turn, cycling through greater elements that could occupy that position.

Example



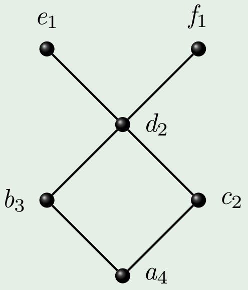
P'

$$
\begin{aligned}
\sigma =: \sigma^0 &= \underline{\mathbf{f}}\ c\ b\ a\ d\ e \\
\sigma^1 &= f\ \underline{\mathbf{c}}\ b\ a\ \underline{\mathbf{d}}\ \underline{\mathbf{e}} \\
\sigma^2 &= f\ e\ \underline{\mathbf{b}}\ a\ c\ \underline{\mathbf{d}} \\
\sigma^3 &= f\ e\ d\ \underline{\mathbf{a}}\ \underline{\mathbf{c}}\ b \\
\sigma^4 &= f\ e\ d\ c\ \underline{\mathbf{a}}\ \underline{\mathbf{b}} \\
\sigma^5 &= f\ e\ d\ c\ b\ \underline{\mathbf{a}}
\end{aligned}
$$

Proof sketch of injection cont'd

Crucially, $x \geq y \Rightarrow \omega(x) \leq \omega(y)$. This allows an injective algorithm, call it $\phi\colon \Lambda^{\omega}_{\min} \to \Lambda^{\omega}_{\max}$. The algorithm considers each entry in turn, cycling through greater elements that could occupy that position.

Example



P'

$$
\begin{aligned}
\sigma =: \sigma^0 &= \underline{\mathbf{f}}\ c\ b\ a\ d\ e \\
\sigma^1 &= f\ \underline{\mathbf{c}}\ b\ a\ \underline{\mathbf{d}}\ \underline{\mathbf{e}} \\
\sigma^2 &= f\ e\ \underline{\mathbf{b}}\ a\ c\ \underline{\mathbf{d}} \\
\sigma^3 &= f\ e\ d\ \underline{\mathbf{a}}\ \underline{\mathbf{c}}\ b \\
\sigma^4 &= f\ e\ d\ c\ \underline{\mathbf{a}}\ \underline{\mathbf{b}} \\
\sigma^5 &= f\ e\ d\ c\ b\ \underline{\mathbf{a}} \\
\sigma^6 &= f\ e\ d\ c\ b\ a = \phi(\sigma)
\end{aligned}
$$

Section 3

The Pattern {1}{1,2}{2} on the Boolean Lattice

What Kind of Permutations Avoid $\{1\}\{1,2\}\{2\}$?: V-shaped Permutations

Definition

$\sigma \in S_P$ is *V-shaped* if there is some $1 \leq i \leq n$ such that $(\sigma_1, \ldots, \sigma_i)$ is 12-avoiding and $(\sigma_{i+1}, \ldots, \sigma_n)$ is 21-avoiding.

What Kind of Permutations Avoid $\{1\}\{1,2\}\{2\}$?: V-shaped Permutations

Definition

$\sigma \in S_P$ is *V-shaped* if there is some $1 \leq i \leq n$ such that $(\sigma_1, \dots, \sigma_i)$ is 12-avoiding and $(\sigma_{i+1}, \dots, \sigma_n)$ is 21-avoiding.

V-shaped permutations avoid $\{1\}\{1,2\}\{2\}$, since there are no increases followed by decreases. The following diagram explains the name:

Example

The permutation $(\{1,3\}, \{3\}, \{1\}, \{2\}, \{2,3\}, \{1,2\}, \{1,2,3\})$ in B_3 avoids $\{1\}\{1,2\}\{2\}$.

{1,2,3}

{1,3} {2,3},{1,2}

{3},{1},{2}

The Asymptotic Approximation: Lower Bound

Definition

For $\sigma \in B_n$, let $\sigma - \emptyset$ denote $(\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_n)$, where $\sigma_i = \emptyset$.

We construct and count a subset of distinct V-shaped permutations on $\mathbb{B}_n - \emptyset$. (Since $\emptyset$ can never be part of a $\{1\}\{1,2\}\{2\}$ pattern, it can to anywhere.) This provides a basis for our lower bound on $Av_n(\{1\}\{1,2\}\{2\})$.

The Asymptotic Approximation: Lower Bound

Definition

For $\sigma \in B_n$, let $\sigma - \emptyset$ denote $(\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_n)$, where $\sigma_i = \emptyset$.

We construct and count a subset of distinct V-shaped permutations on $\mathbb{B}_n - \emptyset$. (Since $\emptyset$ can never be part of a $\{1\}\{1,2\}\{2\}$ pattern, it can to anywhere.) This provides a basis for our lower bound on $Av_n(\{1\}\{1,2\}\{2\})$.

Theorem

$$\frac{2^{n-1}}{n+1} \prod_{k=0}^{n} \left(\binom{n}{k} + 1 \right)! \leq Av_n(\{1\}\{1,2\}\{2\})$$

Towards an Upper Bound

Lemma

Let $\sigma \in B_n$. If σ has a subsequence abc such that $a \prec b$, $c \prec b$, and neither a nor c is empty, then σ contains a $\{1\}\{1,2\}\{2\}$ pattern.

Towards an Upper Bound

Lemma

Let $\sigma \in B_n$. If σ has a subsequence abc such that $a \prec b$, $c \prec b$, and neither a nor c is empty, then σ contains a $\{1\}\{1,2\}\{2\}$ pattern.

This provides a simpler description for $\{1\}\{1,2\}\{2\}$-avoidance. For all $x \in \mathbb{B}_n$, every element less than x in $\mathbb{B}_n \backslash \emptyset$ must be to the same side of x as all the others in any $\{1\}\{1,2\}\{2\}$-avoiding permutation.

δ Functions

Let $\delta : P \to \{L, R\}$. We are considering an arbitrary poset.

Definition

$\sigma \in S_P$ is δ-*legal* when the following hold

If $\delta(x) = L$ then x is to the left of all elements less than x.

If $\delta(x) = R$ then x is to the right of all elements less than x.

A Visualization of δ-legal Permutations

Example

The following is a δ-legal permutation:

$$(\{1,2,3\},\{1\},\emptyset,\{2,3\},\{3\},\{1,3\},\{2\},\{1,2\})$$

An appropriate function is given by $\delta : \{1\} \mapsto R; \{2\} \mapsto R; \{3\} \mapsto R; \{1,2\} \mapsto L; \{1,3\} \mapsto R; \{2,3\} \mapsto L; \{1,2,3\} \mapsto L$. Note that the images of the atoms (the singleton sets) are arbitrary.

Things Get More Complicated on B_4

Example

The following is a δ-legal permutation:

$$(\{1,2,3,4\},\{1,2\},\{1,3\},\{1\},\{2,3,4\},\{4\},\{1,4\},\emptyset,\{2,3\},\{3\},\{3,4\},$$
$$\{2\},\{2,4\},\{1,2,4\},\{1,2,3\},\{1,3,4\})$$

An appropriate function is given by sending everything in blue to L and everything in red to R.

Things Get More Complicated on B_4

Example

The following is a δ-legal permutation:

$$(\{1,2,3,4\},\{1,2\},\{1,3\},\{1\},\{2,3,4\},\{4\},\{1,4\},\emptyset,\{2,3\},\{3\},\{3,4\},\{2\},\{2,4\},\{1,2,4\},\{1,2,3\},\{1,3,4\})$$

An appropriate function is given by sending everything in blue to L and everything in red to R.

Thus it is possible to have a $\{1\}\{1,2\}\{2\}$-avoiding permutation which is not V-shaped.

Why are δ functions useful?

If we go back to our criterion for avoidance, it is now possible to restate it using our new language. σ avoids $\{1\}\{1,2\}\{2\}$ only if there is some $\delta\colon \mathbb{B}_n - \emptyset \to \{L, R\}$ such that $\sigma - \emptyset$ is δ-legal.

Since it is eas(ier) to count the number of δ-legal functions for a poset, the number of δ functions on $\mathbb{B}_n - \emptyset$ provides a basis for our upper bound on $Av_n(\{1\}\{1,2\}\{2\})$.

However, given a δ function, there could be many permutations for which that function displays δ-legality. So we also have to bound the number of possible ways to create a permutation which is δ-legal for a given δ-function.

The Asymptotic Approximation: Upper Bound

We give an upper bound for the number of δ-legal permutations given some δ function—the number of linear extensions of the initial poset, whence the $Av_n(21)$ term.

Here we relied on a result stating that a poset has fewer linear extensions than another it properly contains (Stachowiak).

Theorem

$$Av_n(\{1\}\{1,2\}\{2\}) \leq 2^{2^n-1+n} Av_n(21)$$

The Asymptotic Approximation: Upper Bound

We give an upper bound for the number of δ-legal permutations given some δ function—the number of linear extensions of the initial poset, whence the $Av_n(21)$ term.

Here we relied on a result stating that a poset has fewer linear extensions than another it properly contains (Stachowiak).

Theorem

$$Av_n(\{1\}\{1,2\}\{2\}) \leq 2^{2^n-1+n} Av_n(21)$$

Corollary

These two bounds imply

$$o(1) \leq \frac{\log(Av_n(\{1\}\{1,2\}\{2\})) - \log(Av_n(21))}{2^n} \leq 1 + o(1)$$

when considered alongside the Brightwell Tetali result.

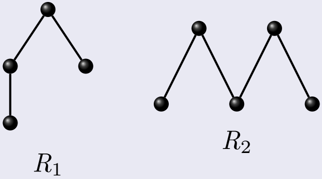
Section 4

Further Directions

An injection from $\{1\}\{1,2\}\{2\}$ to $\{1\}\{2\}\{1,2\}$?

Conjecture

We have $Av_P(\{1\}\{1,2\}\{2\}) \leq Av_P(\{1\}\{2\}\{1,2\})$ *for any poset* P*, with strict inequality iff* P *contains either of* R_1 *or* R_2 *below as an induced subposet:*



$\{1\}\{1,3\}\{2\}$ and $\{1\}\{3\}\{1,2\}$: Who knows?

The relationship between avoidance in the last pair of non-trivial length three patterns, $\{1\}\{1,2\}\{3\}$ and $\{1\}\{3\}\{1,2\}$, is more complicated.

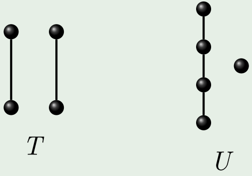
Example

With posets T and U as below, we have that

$$Av_T(\{1\}\{1,2\}\{3\}) < Av_T(\{1\}\{3\}\{1,2\}),$$

but

$$Av_U(\{1\}\{3\}\{1,2\}) < Av_U(\{1\}\{1,2\}\{3\}):$$



Connection to classical permutations: "gap" patterns

We are briefly in the world of classical permutations (so $Av_n(\pi)$ denotes the number of avoiders of π in S_n). Let $1 -_k 2 -_k 3$ mean the pattern 123, where there must be a gap of at least size k between the 1 and 2 and between the 2 and 3.

Example

12453 does not contain $1 -_1 2 -_1 3$.

Corollary

For $k \geq 1$, we have

$$Av_n(1 -_k 3 -_k 2) \leq Av_n(1 -_k 2 -_k 3),$$

with strict inequality iff $n \geq 3(k+1)+1$.

References

Graham R. Brightwell and Prasad Tetali. The number of linear extensions of the boolean lattice. Order, 20:333-345, 2003.

Sam Hopkins and Morgan Weiler. Pattern avoidance in permutations on the Boolean lattice. 2012. Eprint arxiv:1208.5718.

Rodica Simion and Frank W. Schmidt. Restricted permutations. European Journal of Combinatorics, 6:282-406, 1985.

Grzegorz Stachowiak. A relation between the comparability graph and the number of linear extensions. Order, 6:24-244, 1989.

Thanks

Thanks to NSF grant 1004624 and East Tennessee State University for funding, to our fellow REU participants, and our mentor, Anant Godbole.

And thanks to you for listening!