Andrei Asinowski (Technion / FU Berlin)

# The “even part” of Baxter permutations

A joint work with
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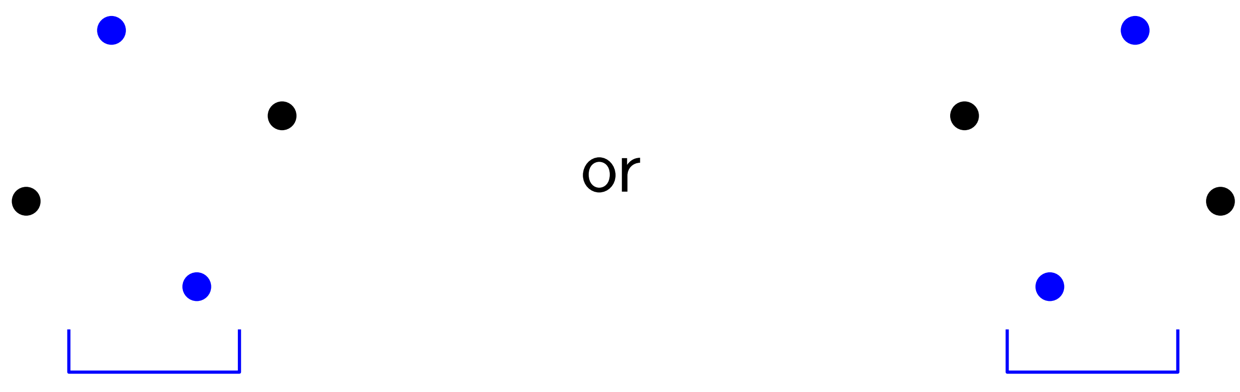
Mireille Bousquet-Mélou (LaBRI),
Toufik Mansour (U. of Haifa),
Ron Pinter (Technion)

A *Baxter permutation* is a $(2-41-3,\ 3-14-2)$-avoiding permutation.

That is: the diagram of $\pi$ contains no quadruple of points such that their relative position is




where the blue points are in adjacent columns of $\pi$.

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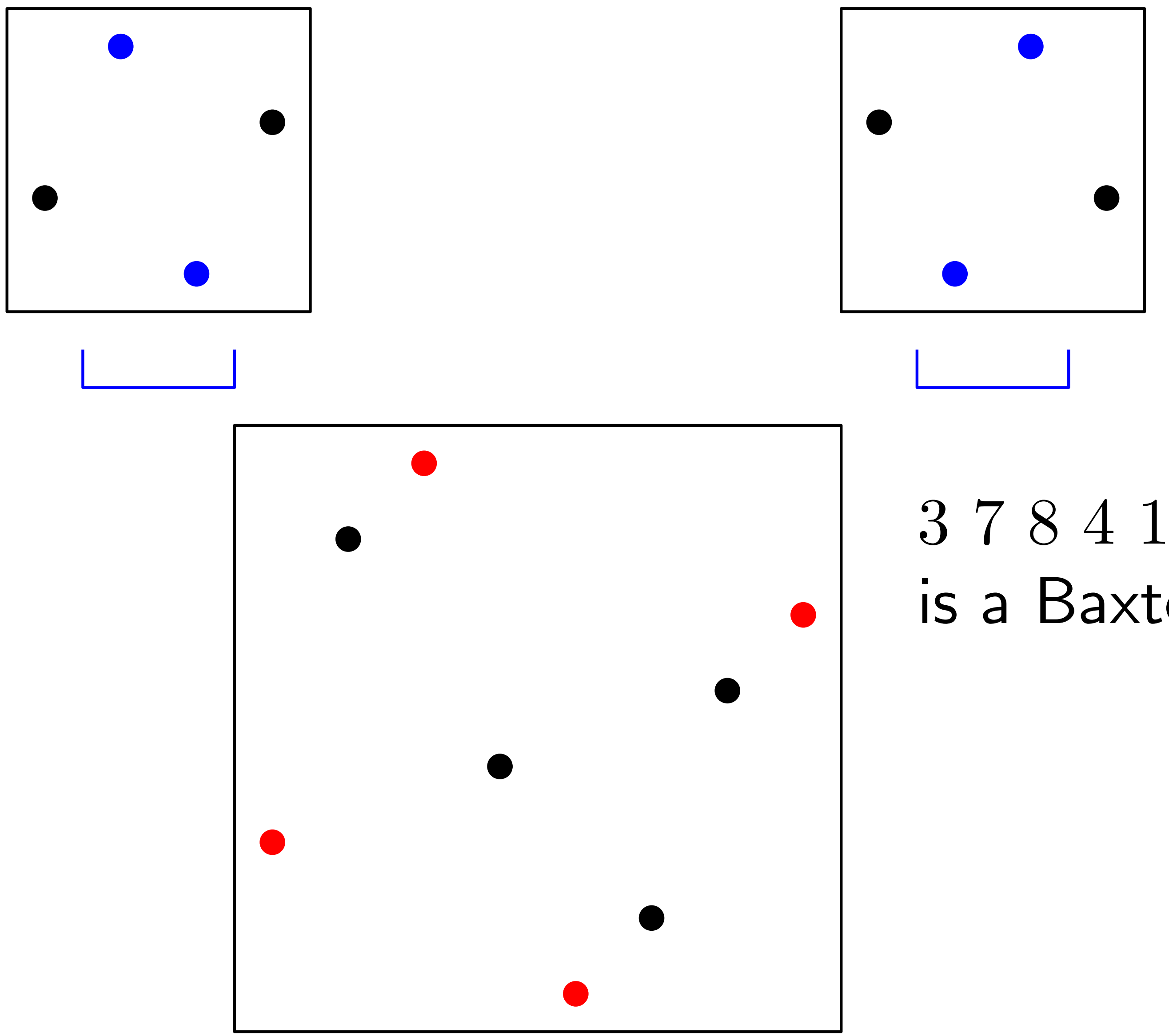

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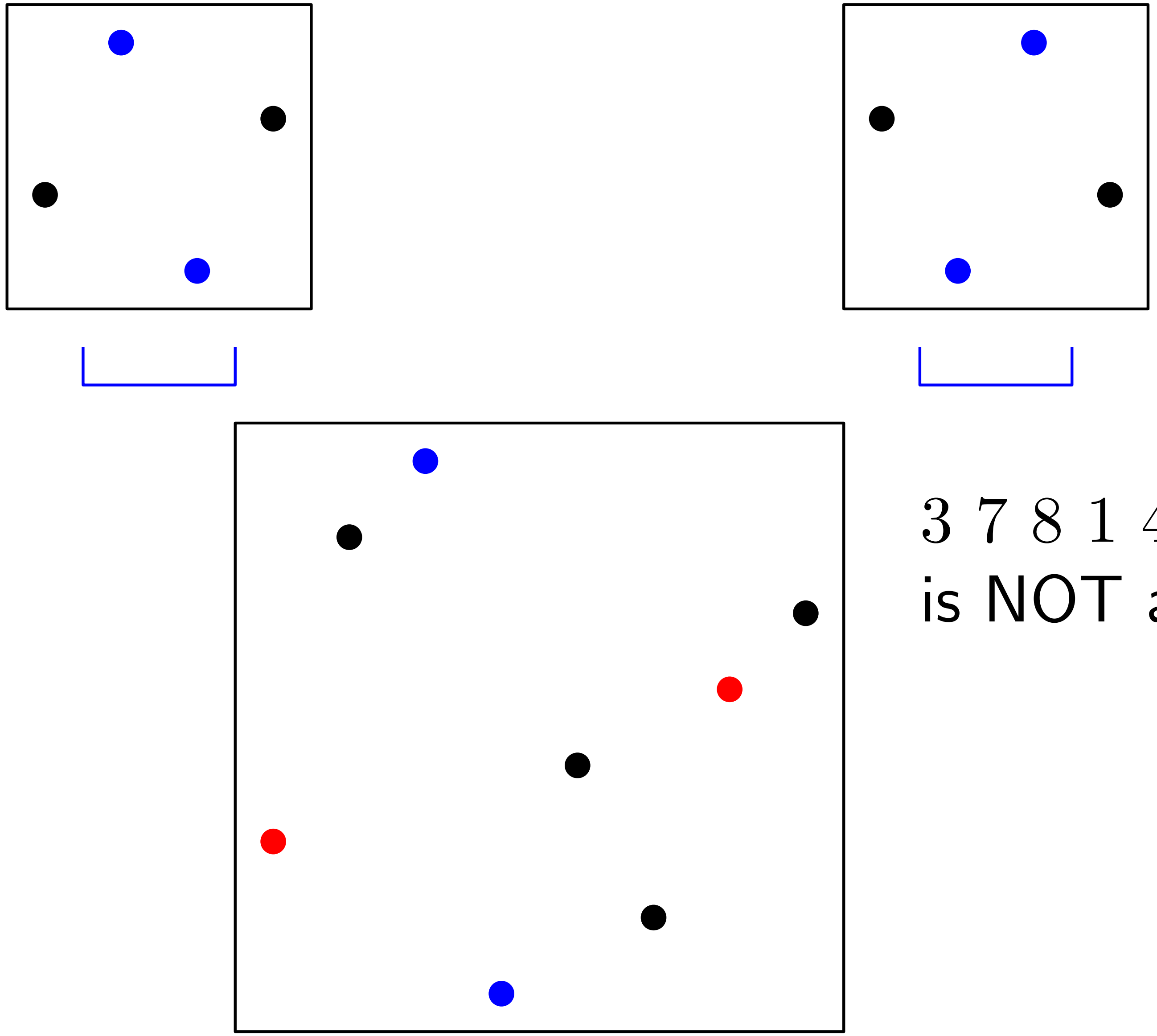

$3\ 7\ 8\ 1\ 4\ 2\ 5\ 6$
is NOT a Baxter perm.

The original definition (Baxter and Joichi, 63):
Let $f, g : [0,1] \to [0,1]$ two continuous functions that commute under composition: $h := g \circ f = f \circ g$.
Let $S$ be the set of fixed points of $h$.
Then the actions of $f$ and of $g$ on $S$ are permutations of $S$ (inverses of each other).

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Let $x \in S$. (That is: $h(x) = x$.) Denote $y = f(x)$.

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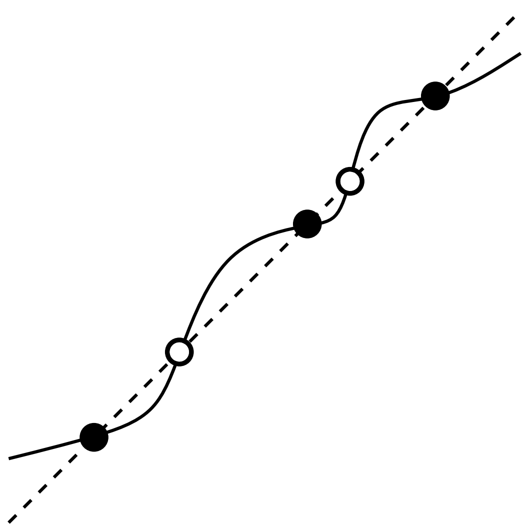
Assume, further, that $S$ is a finite set of odd size: $S = \{x_1, x_2, \ldots, x_{2n+1}\}$, where $x_1 < x_2 < \cdots < x_{2n+1}$; and that in the odd points $h(x) - x$ turns from $+$ to $-$, and in the even points $h(x) - x$ turns from $-$ to $+$.



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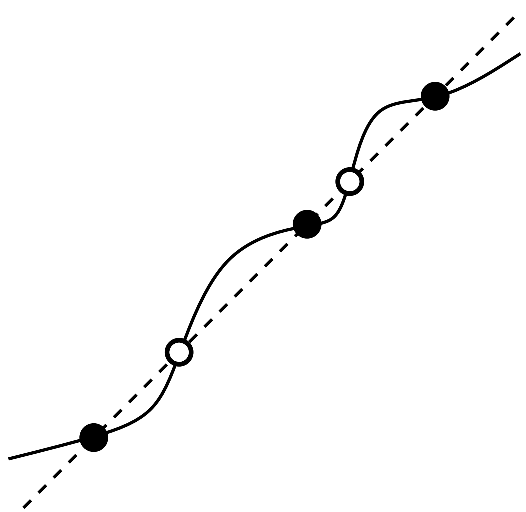

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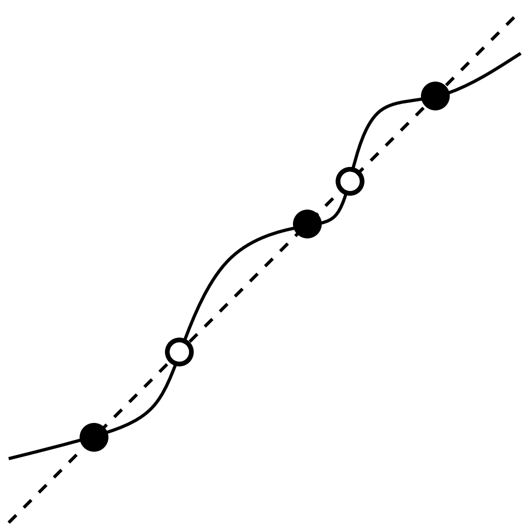

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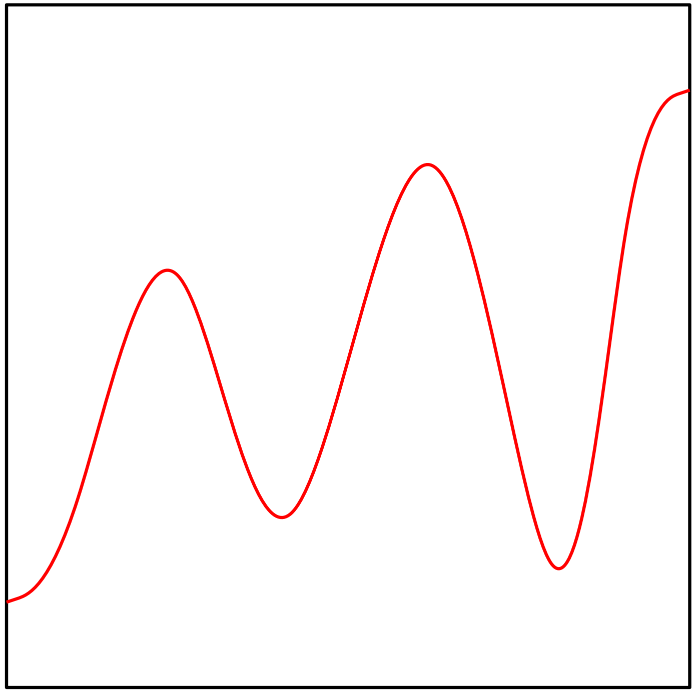
A *complete Baxter permutation* is a permutation of $[2n+1]$ that can be obtained in this way.

Among the properties of complete Baxter permutation:

1. It is a permutation of $[2n+1]$.
2. Odd numbers are mapped to odd numbers, even to even.
3. Even values are uniquely determined by odd values.

# A monotone meandre interpretation (Boyce, 67)



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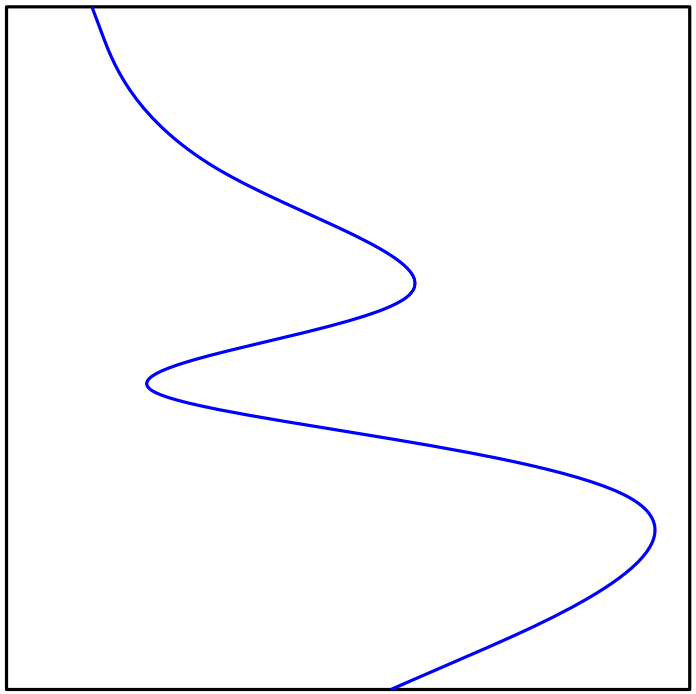

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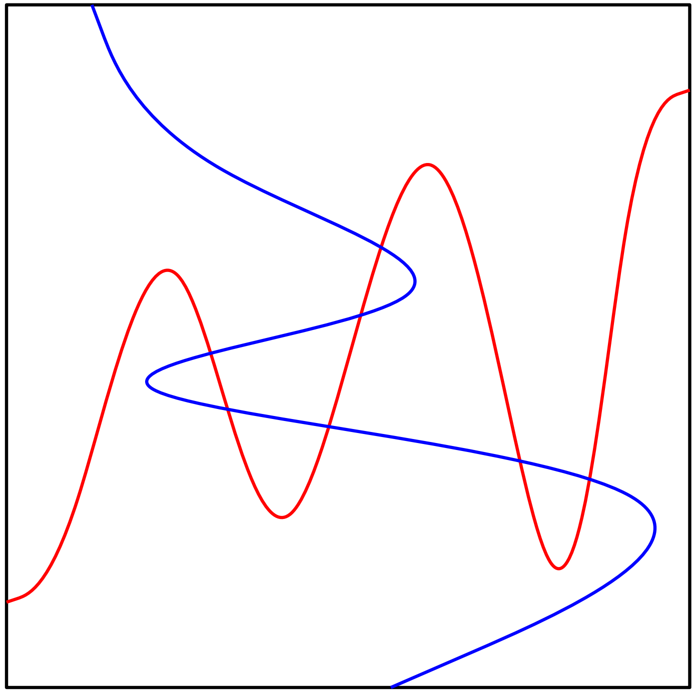

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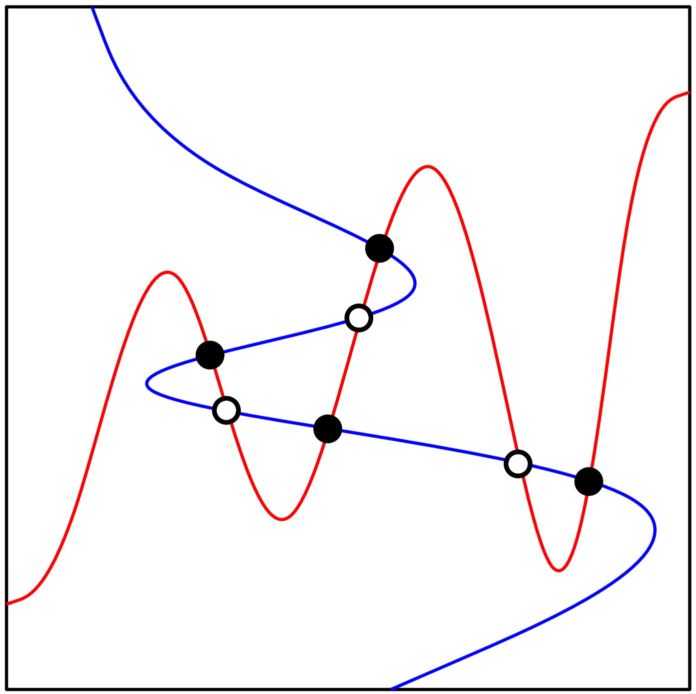

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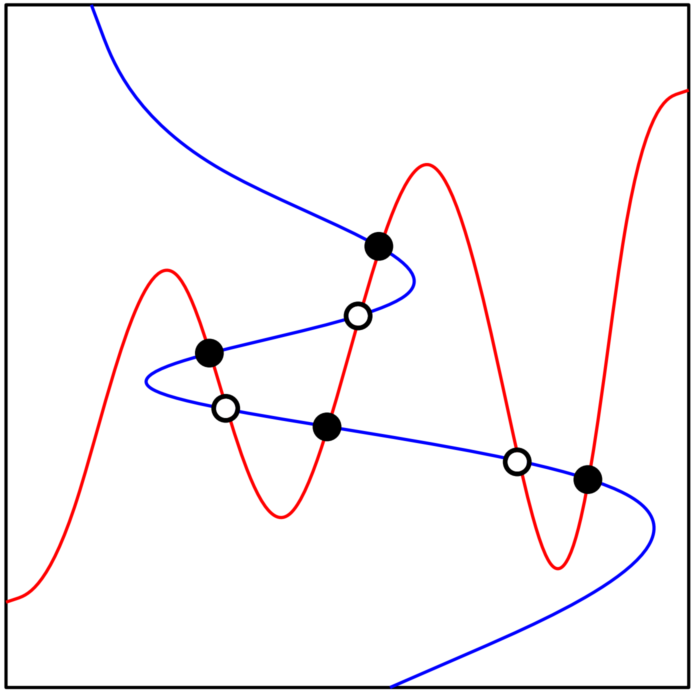

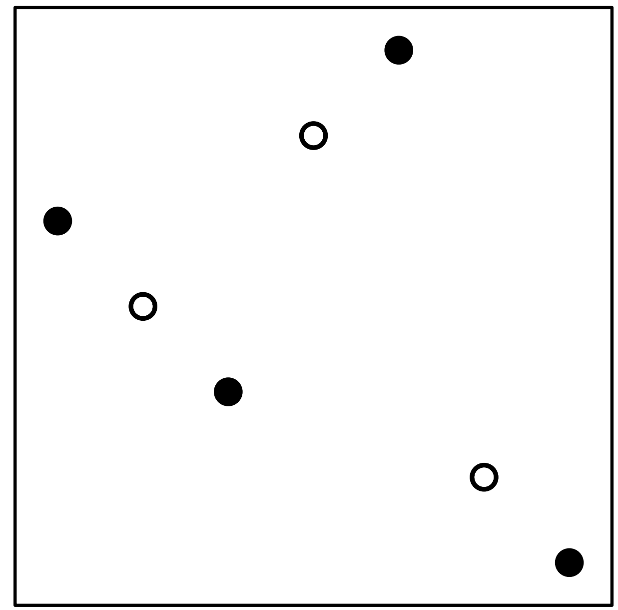

$$5\ 4\ 3\ 6\ 7\ 2\ 1$$

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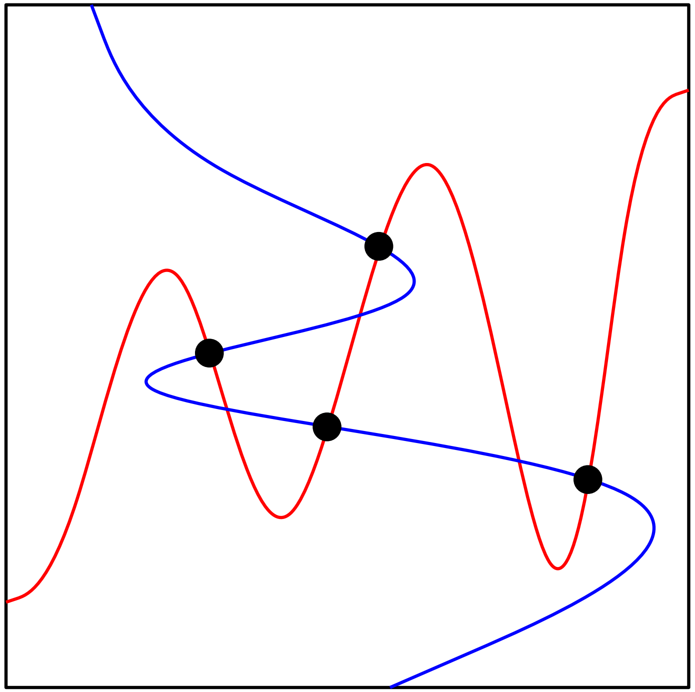

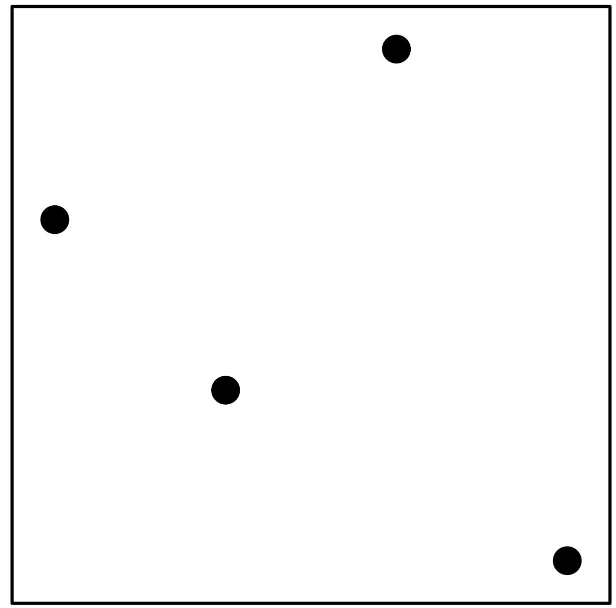

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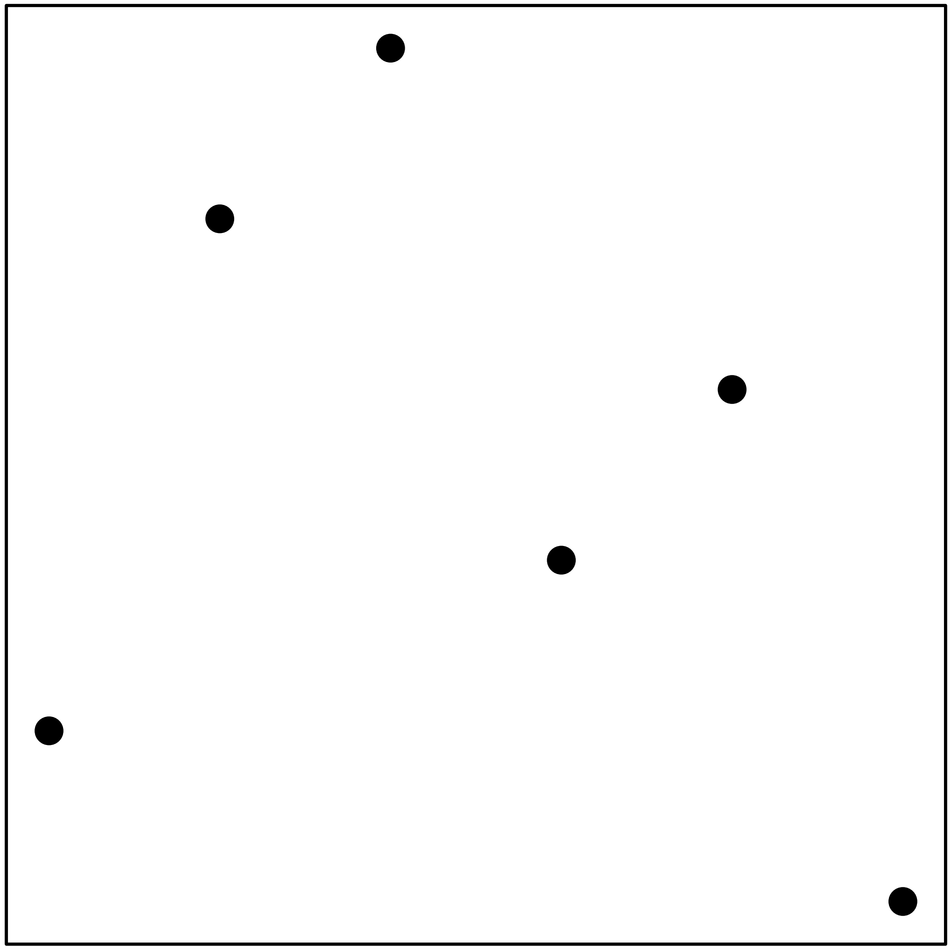
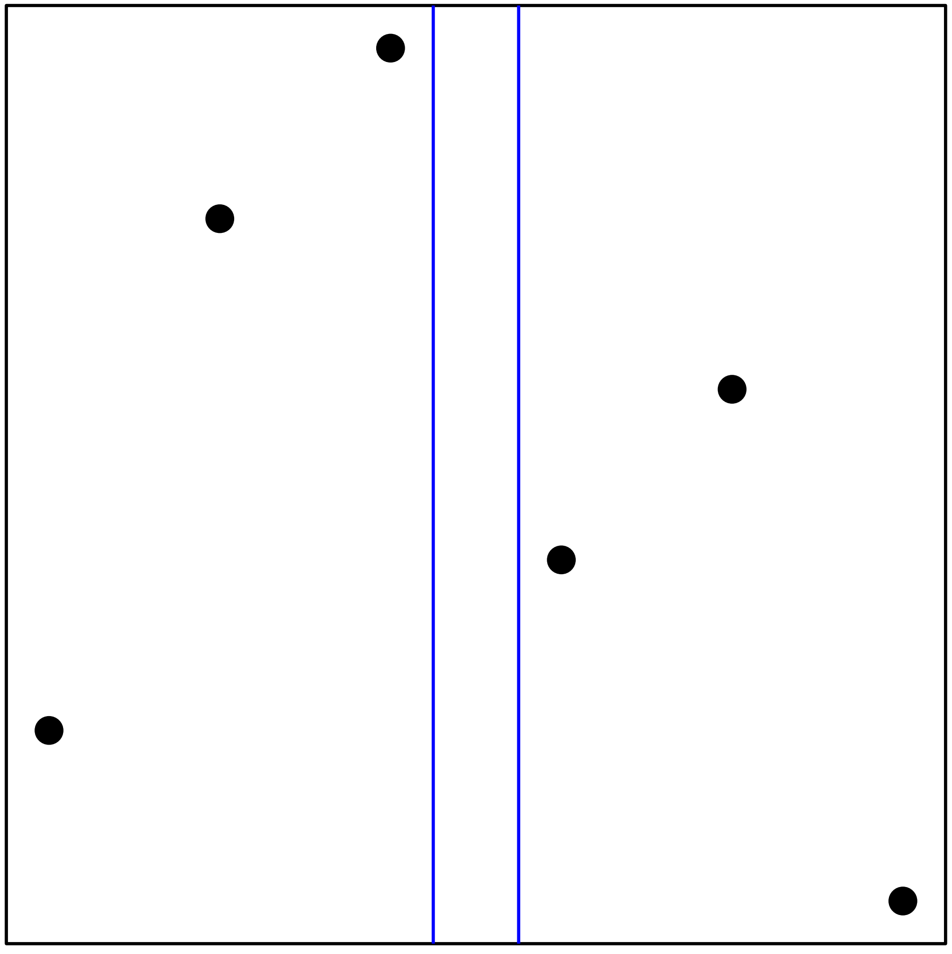
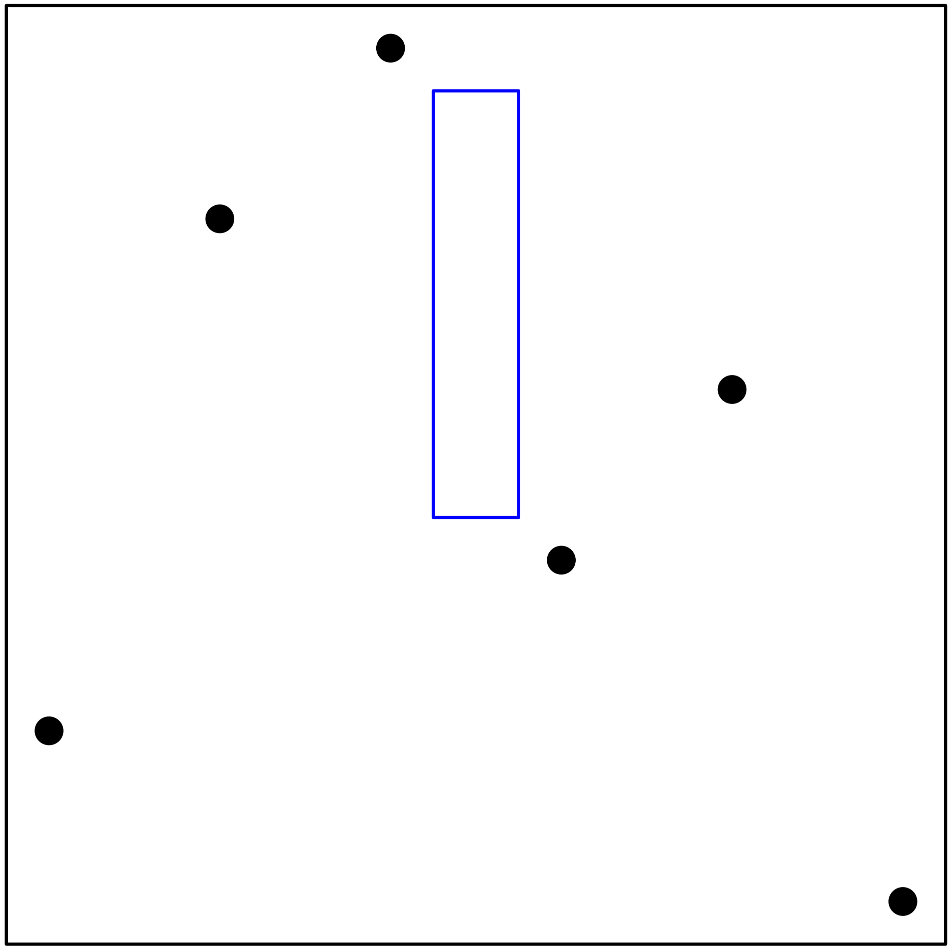
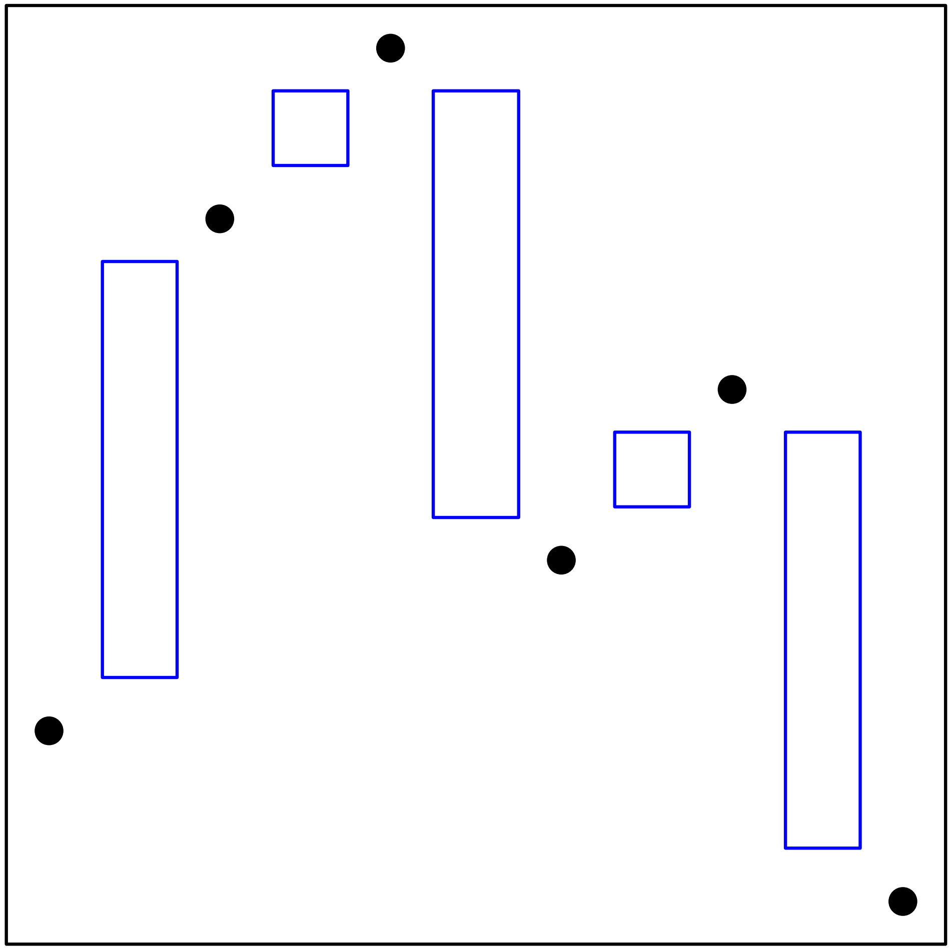
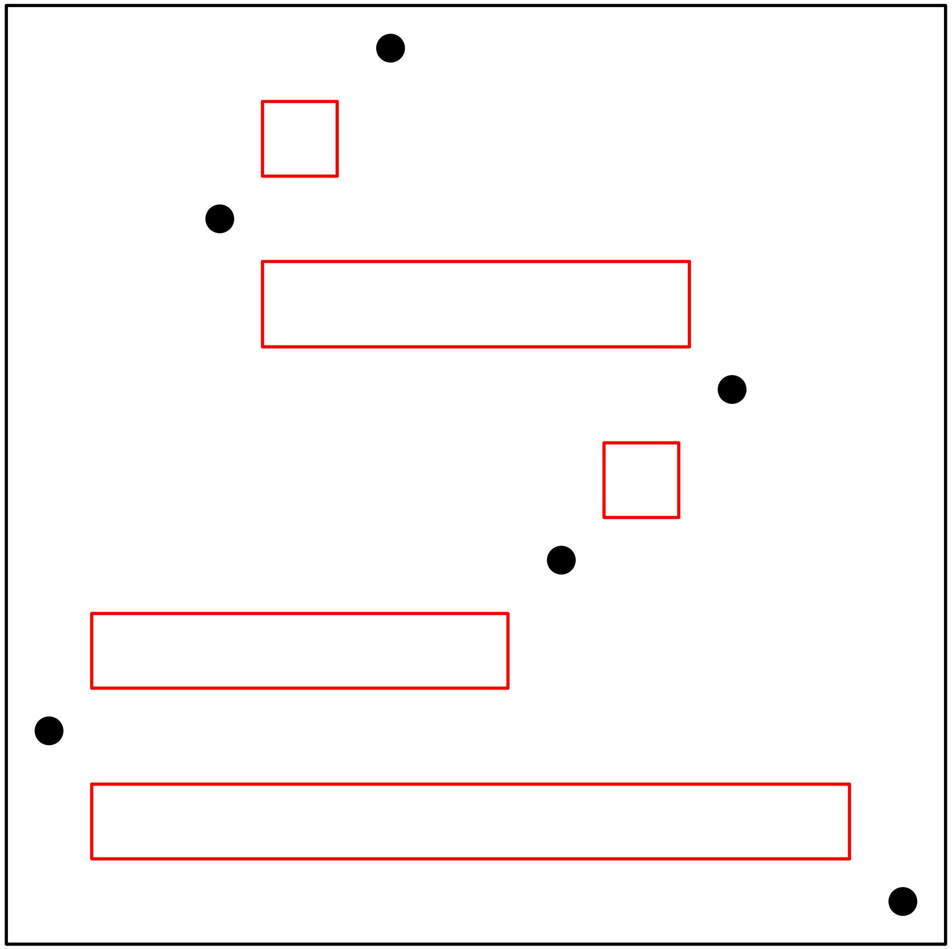
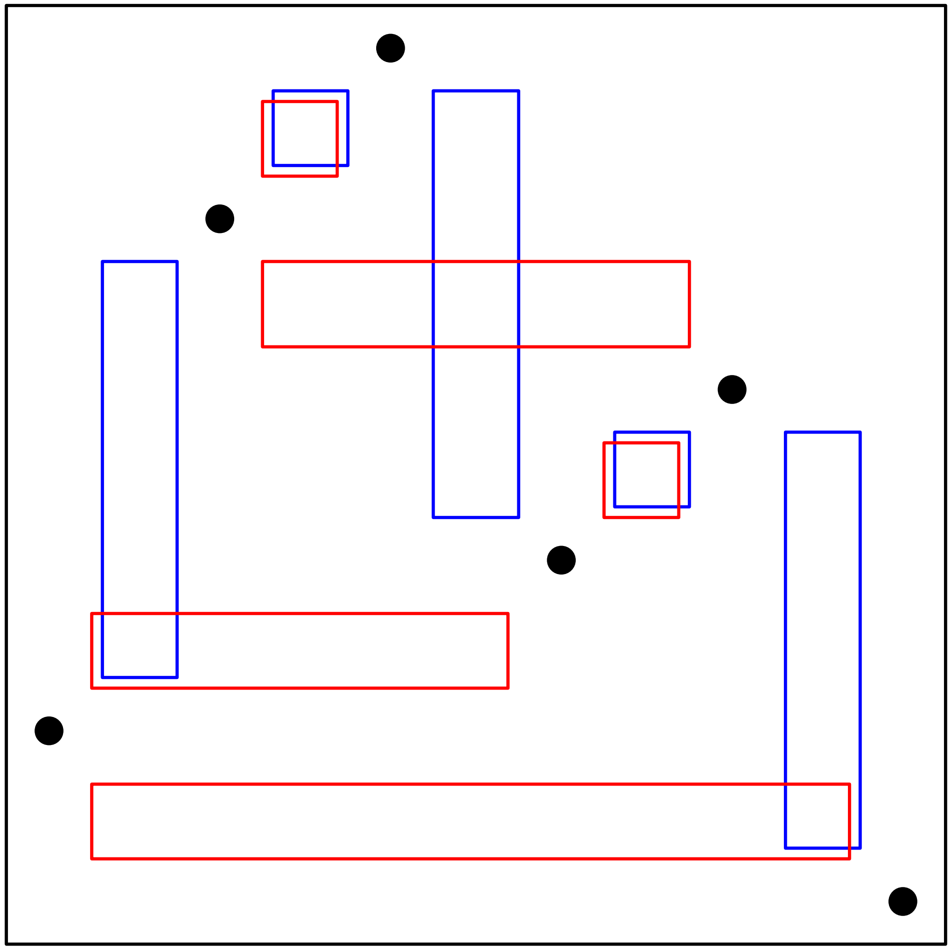


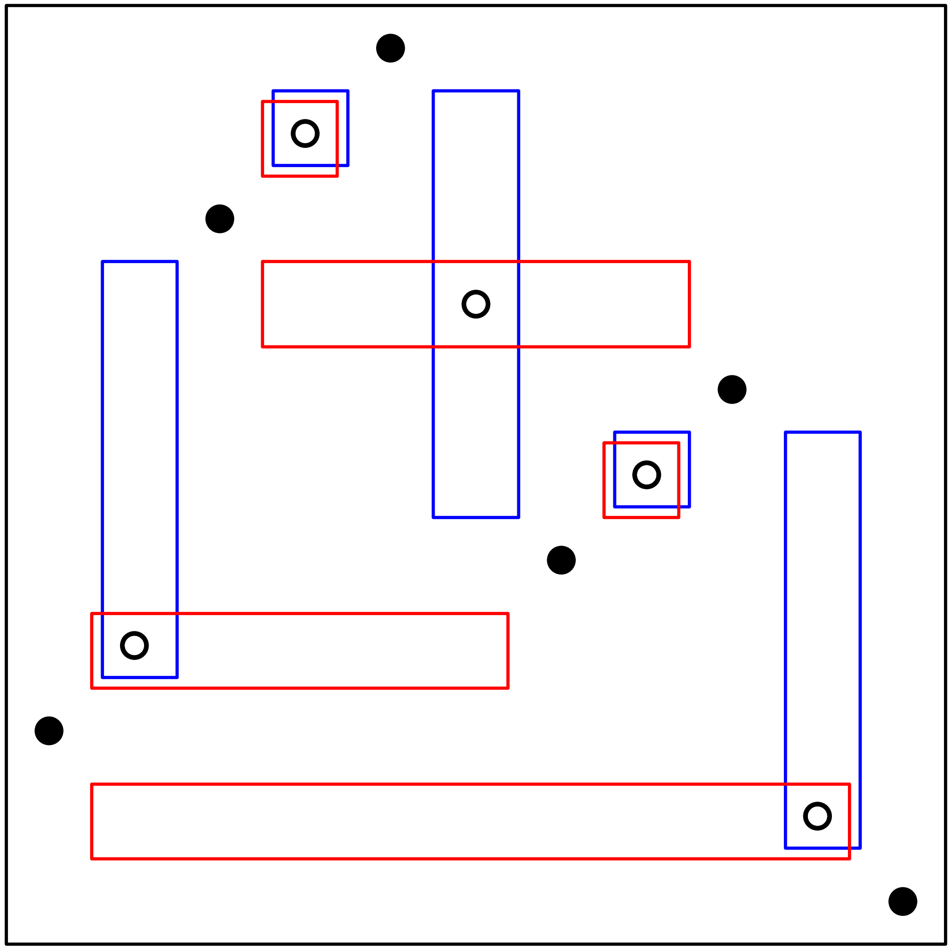


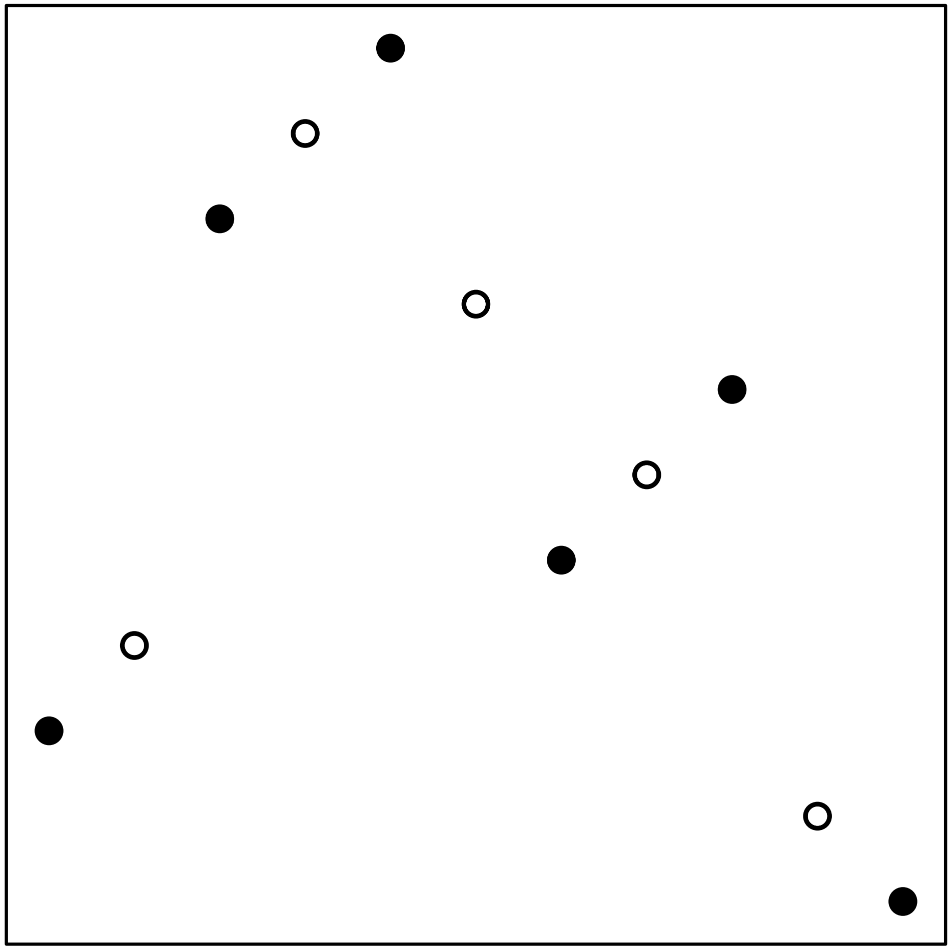


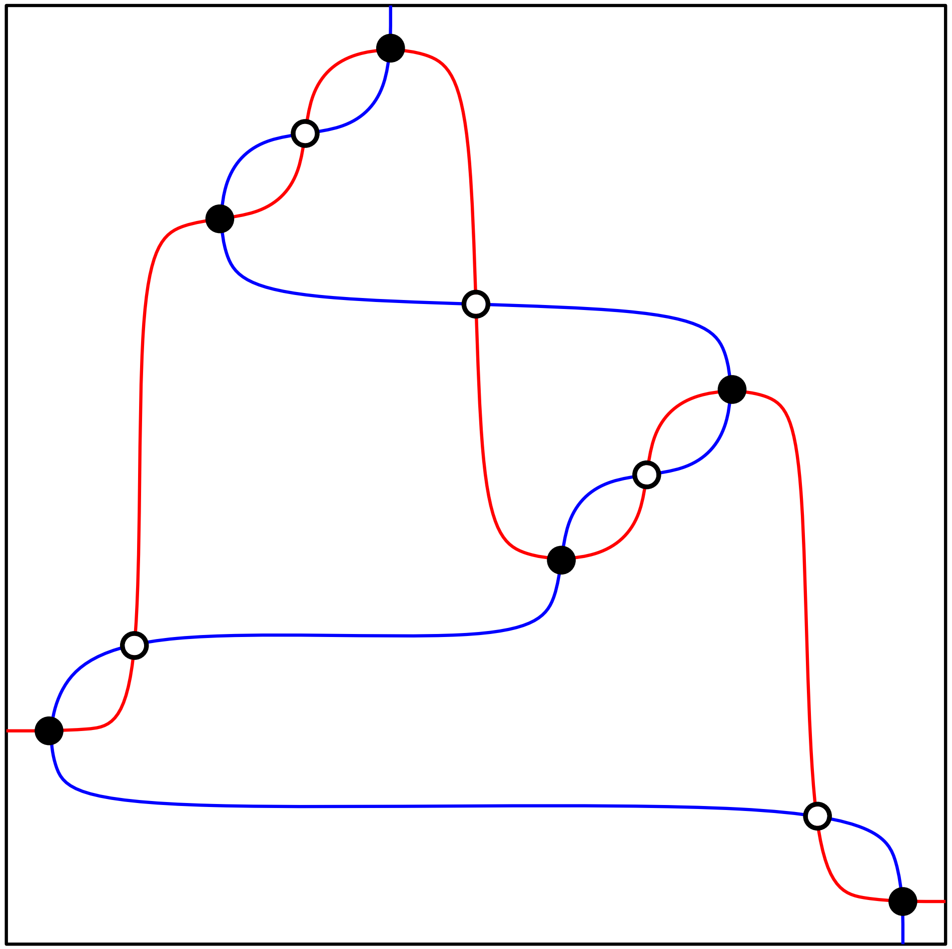


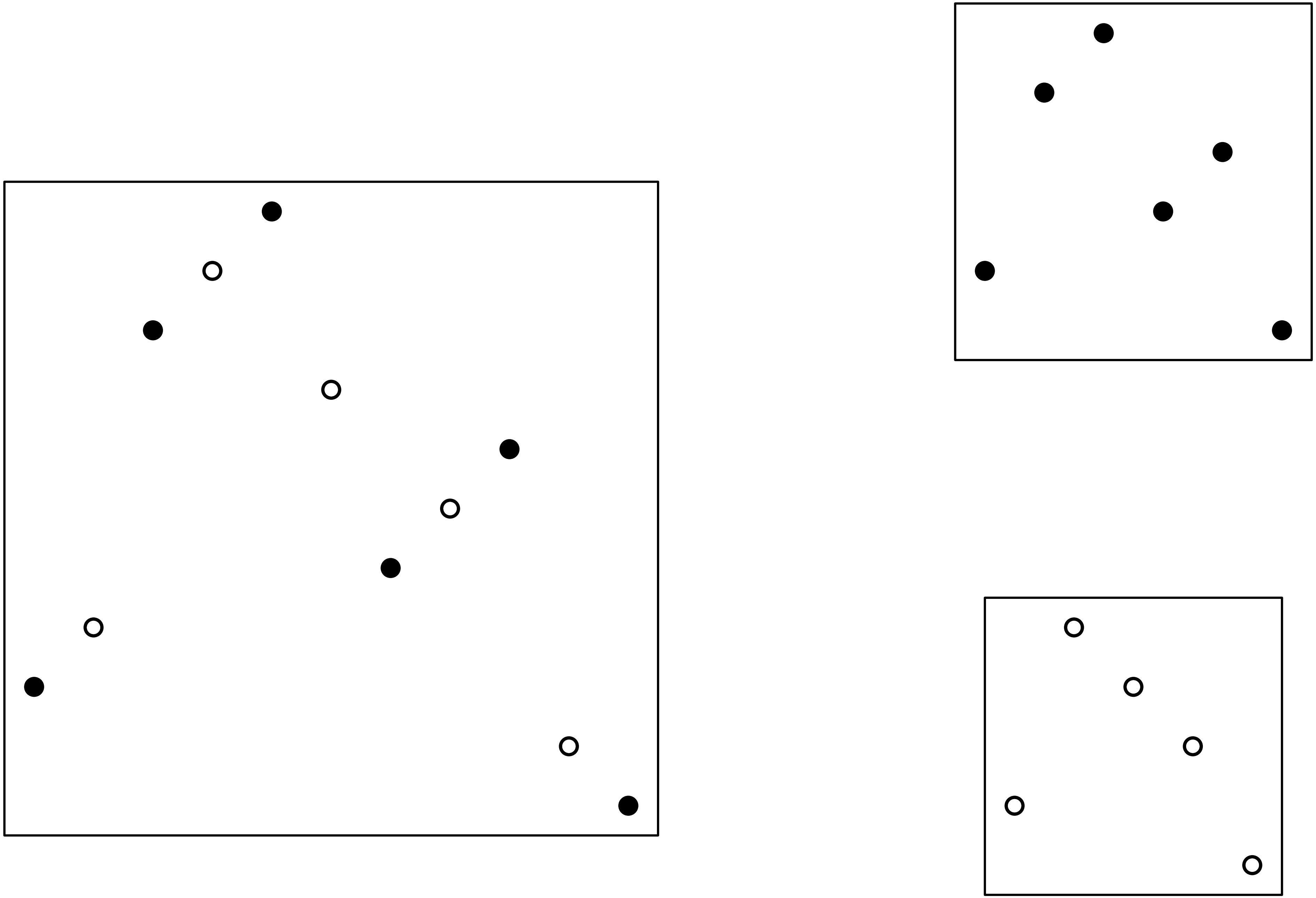


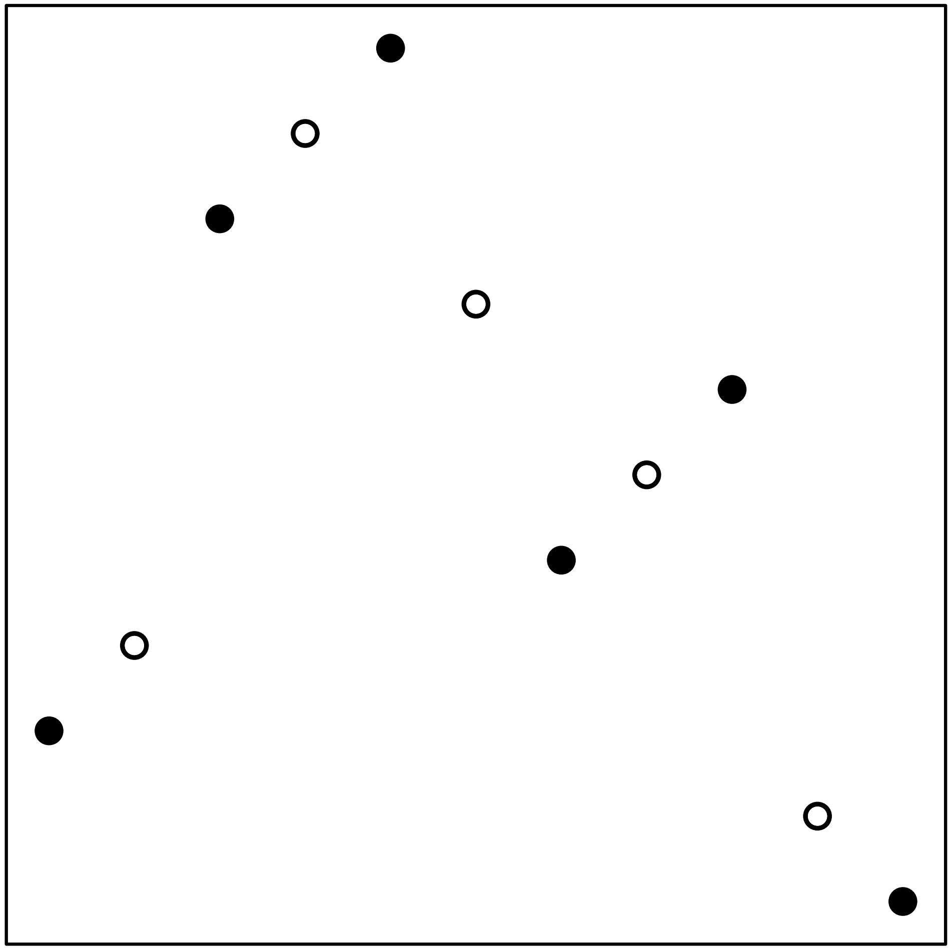




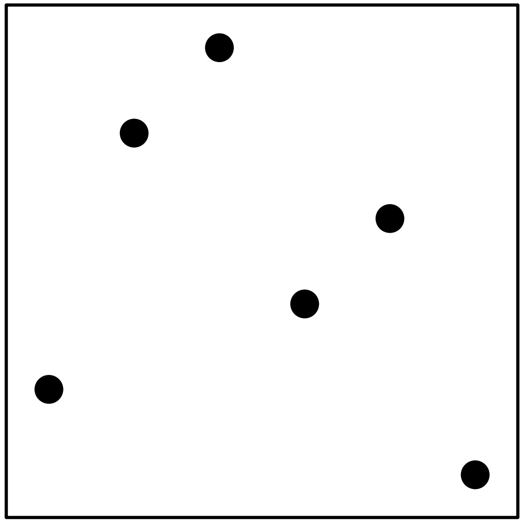




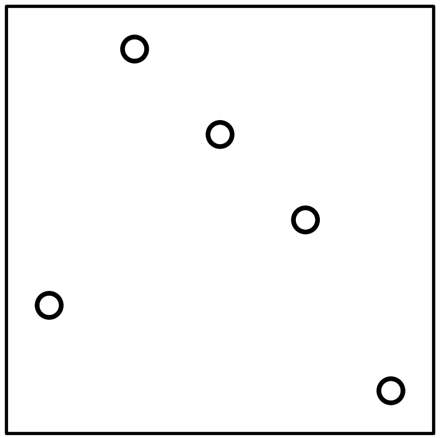




Complete Baxter permutation

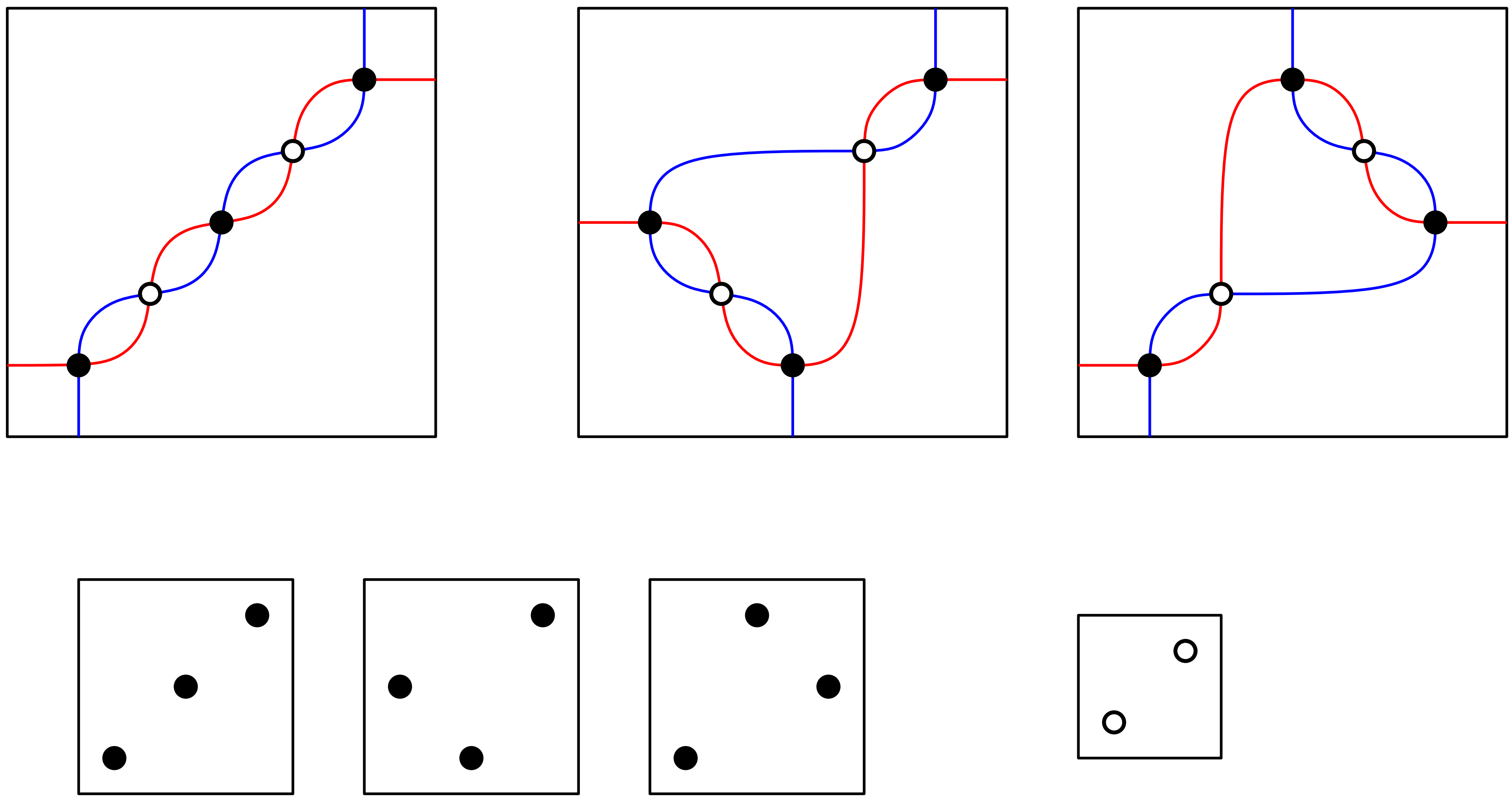


BLACK:
Reduced Baxter perm.
(the odd part)



WHITE:
The even part

In contrast: the even (WHITE) part doesn't determine the odd (BLACK) part uniquely.



Reduced Baxter permutations (BLACK) are now usually referred to as “Baxter permutations”. They can be defined as $(2-41-3, 3-14-2)$-avoiding permutations.

This class is well studied, and many combinatorial structures are known to be in bijection with (reduced) Baxter permutations. (Felsner, Fusy, Noy, Orden. *Bijections for Baxter Families and Related Objects* (2011).)

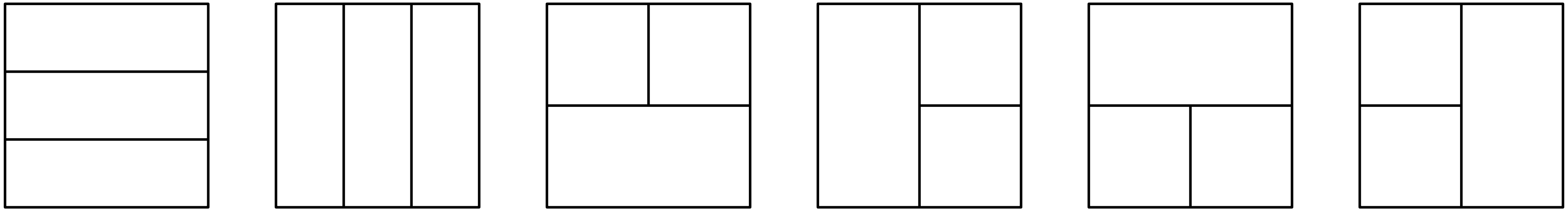
The $n$th Baxter number ($B_n$) is the number of Baxter permutations of site $n$. The generating function:

$$B(t) = x + 2x^2 + 6x^3 + 22x^4 + 92x^5 + 422x^6 + 2074x^7 + \ldots$$

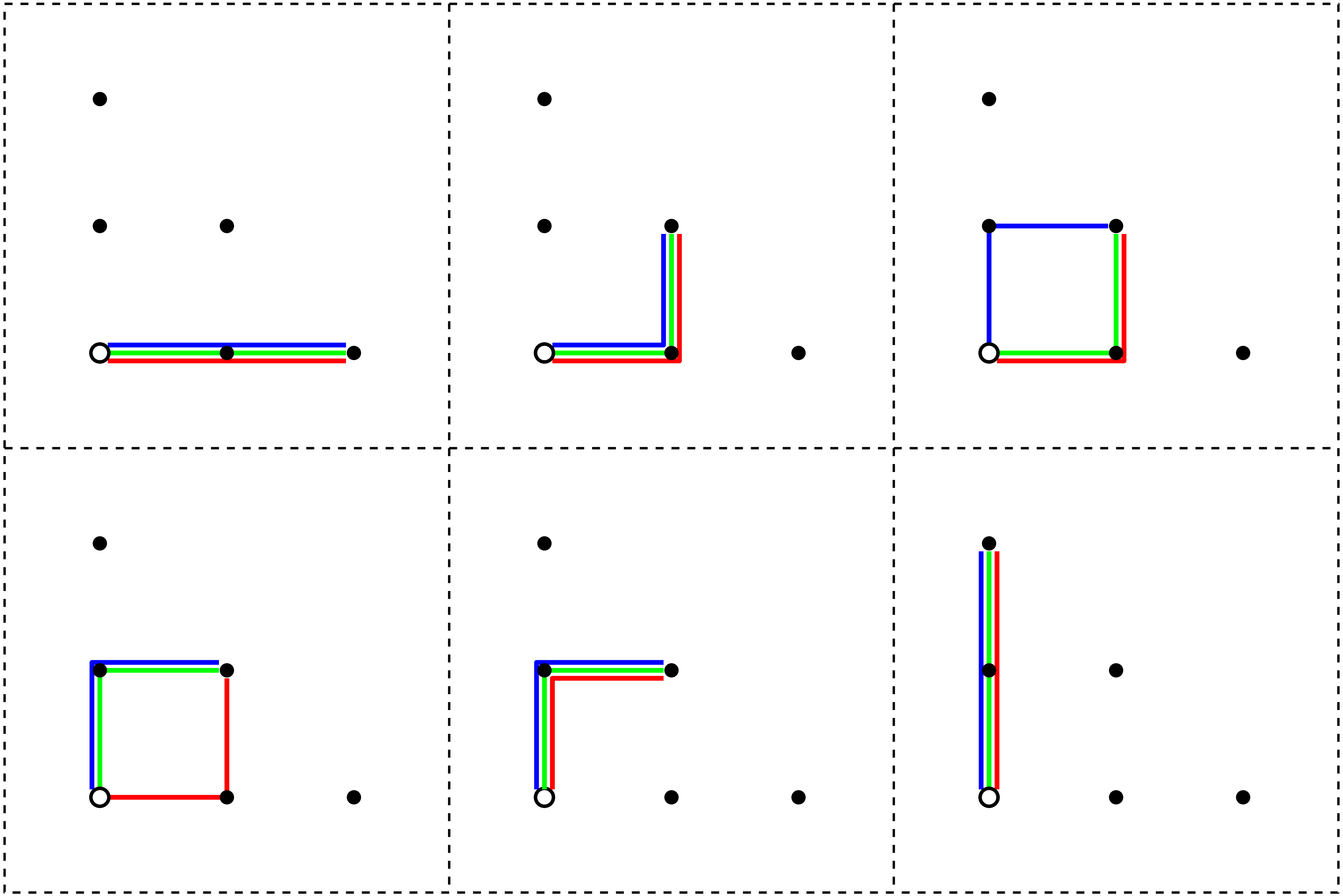
The explicit formula for Baxter numbers (Chung, Graham, Hoggatt, Kleiman 78; Mallows 79):

$$b_n = \sum_{k=0}^{n-1} \frac{\binom{n+1}{k}\binom{n+1}{k+1}\binom{n+1}{k+2}}{\binom{n+1}{0}\binom{n+1}{1}\binom{n+1}{2}}$$

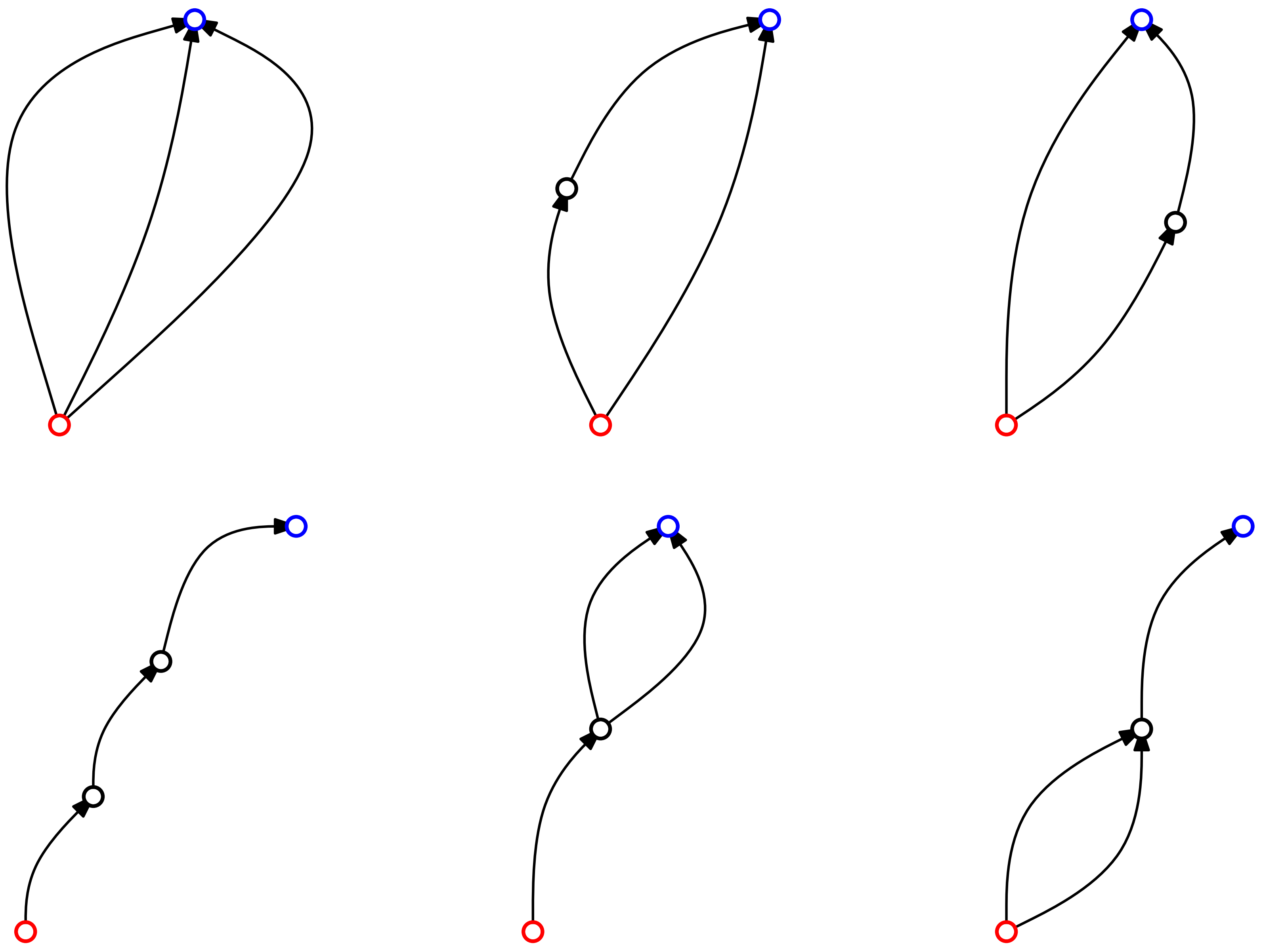
Planar floorplans with $n$ rectangles:

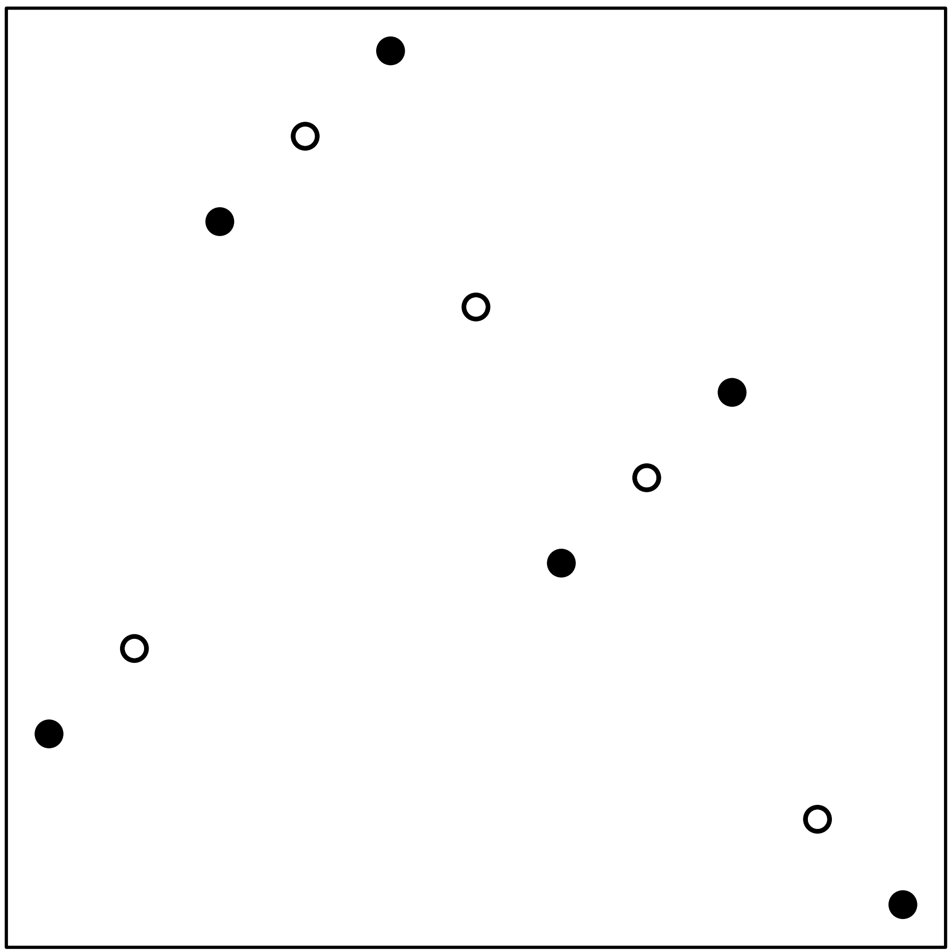


Triples of non-crossing $\{(1,0),(0,1)\}$-paths from $(0,0)$ to a point on $x+y=n-1$

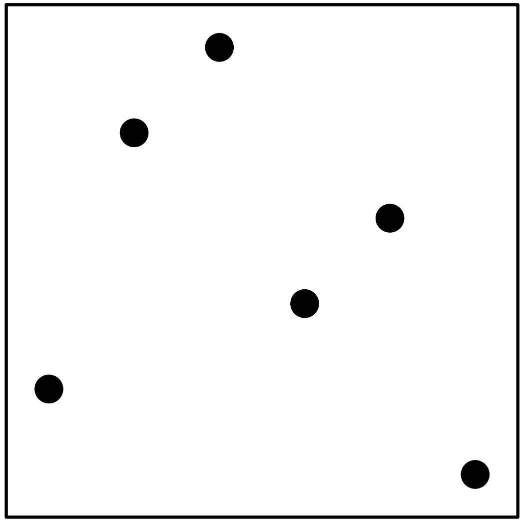


# Plane bipolar orientations with $n$ edges





Complete Baxter permutation



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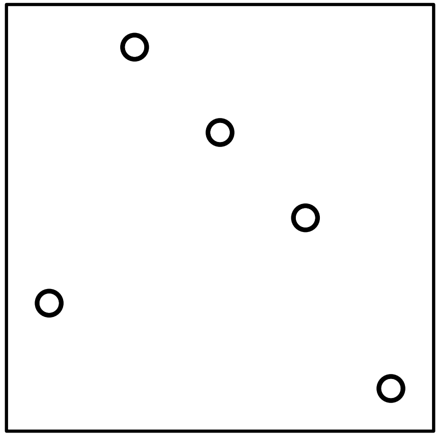
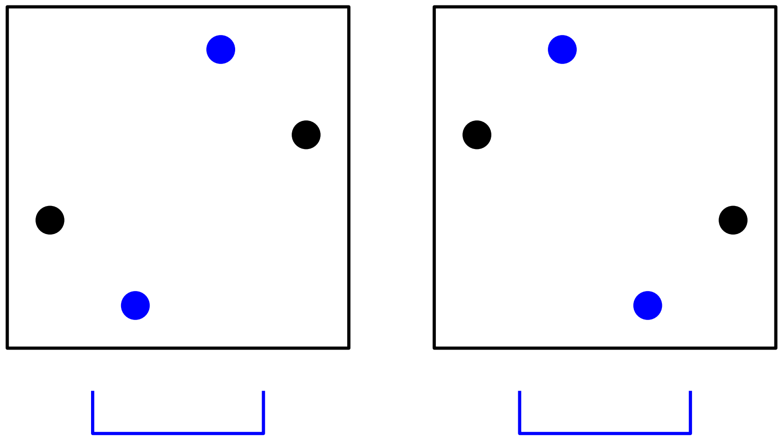


WHITE:
The even part

Denote by $\mathcal{E}$ the set of permutations that can be obtained as the even (WHITE) part of a complete Baxter permutation; by $\mathcal{E}_n$, such permutations of size $n$.

Characterization by forbidden patterns:

$$\mathcal{E} = \mathrm{Av}(2-14-3, 3-41-2).$$



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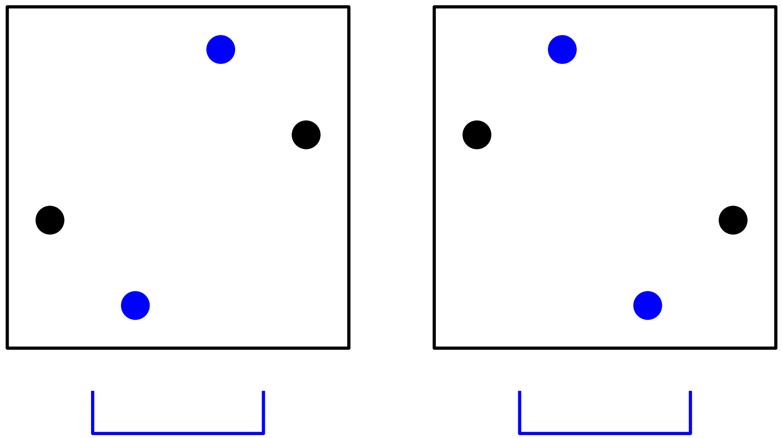
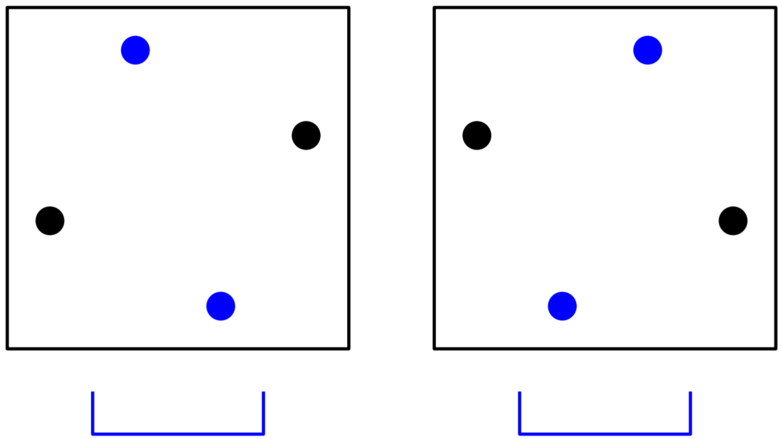
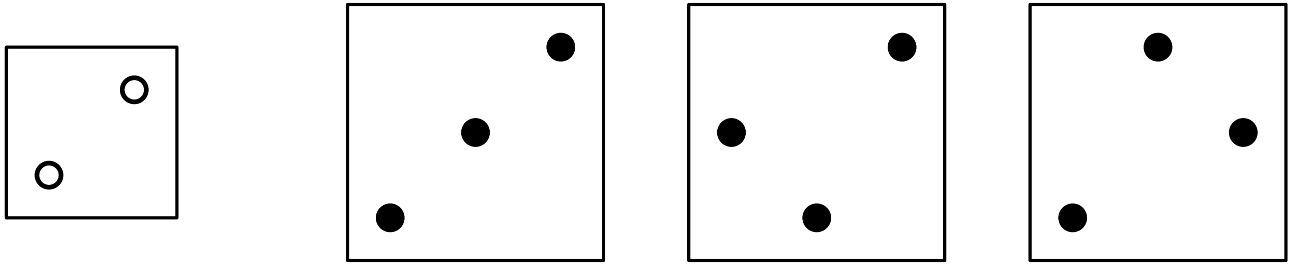


(Reduced Baxter permutations $= \mathrm{Av}(2-41-3, 3-14-2)$.)



Which Baxter pemutations have the same even part?



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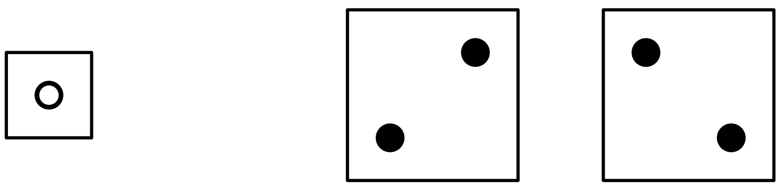

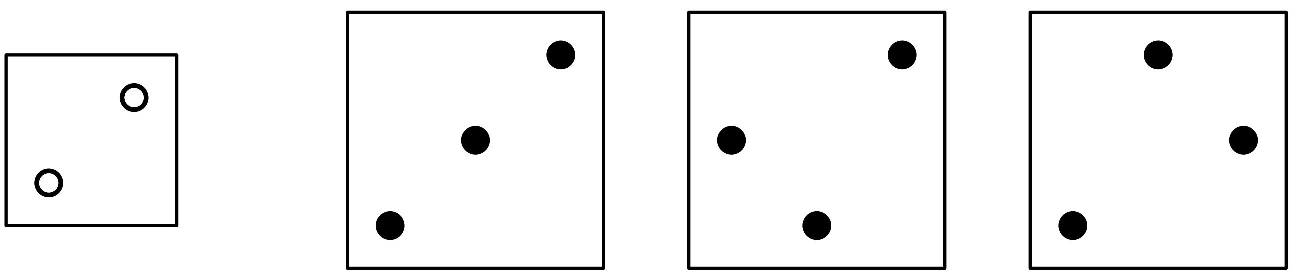

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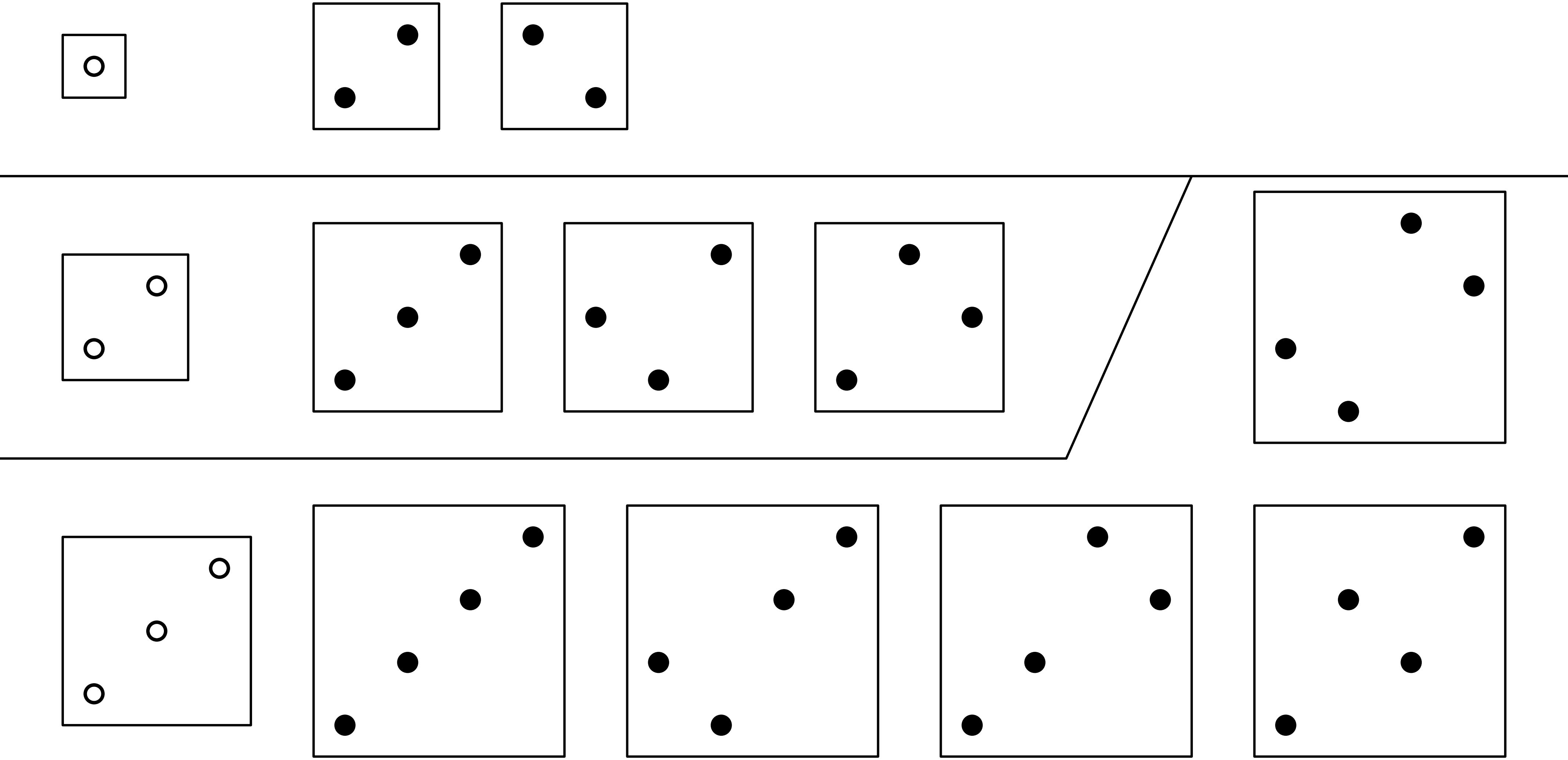

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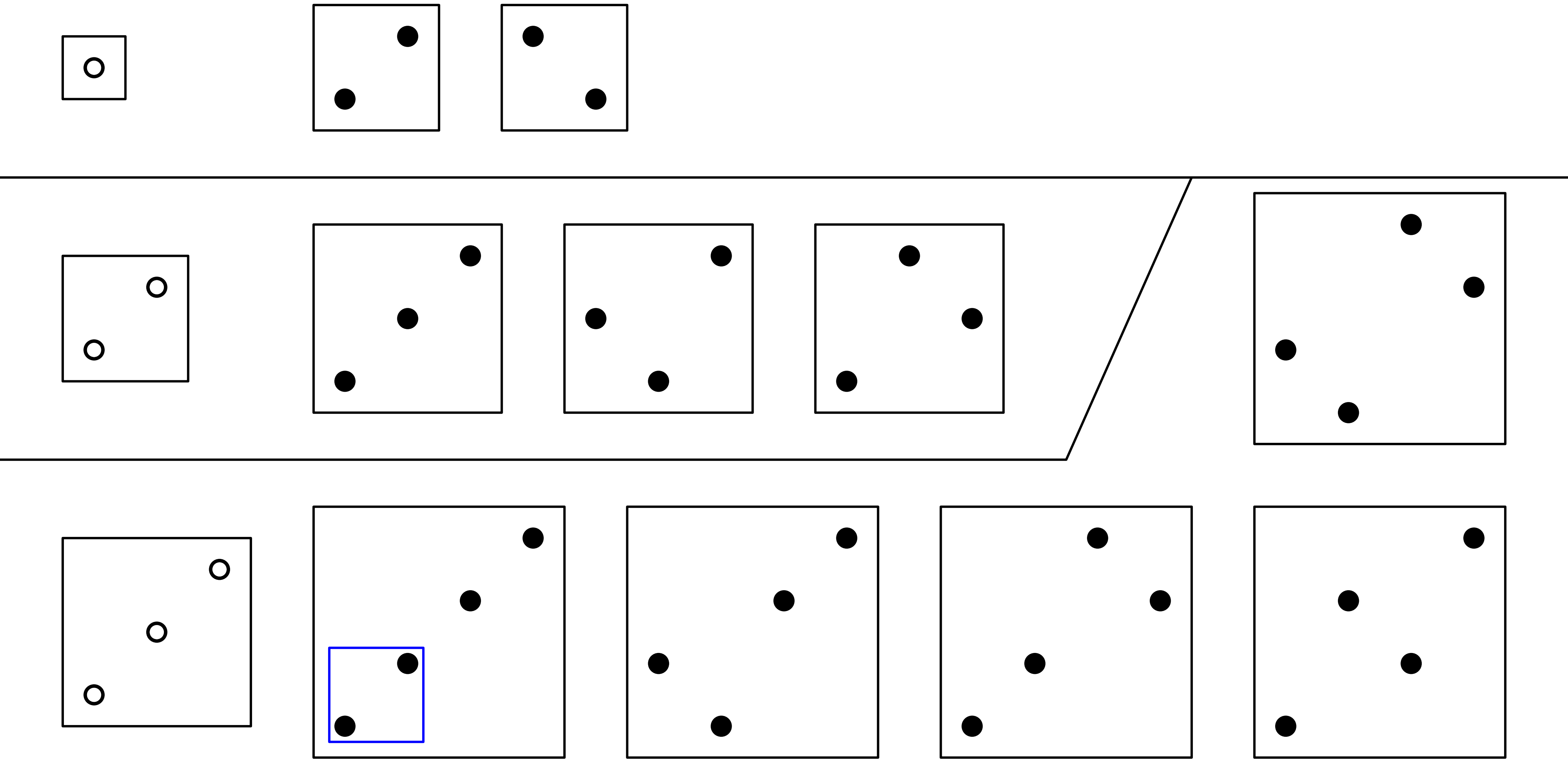

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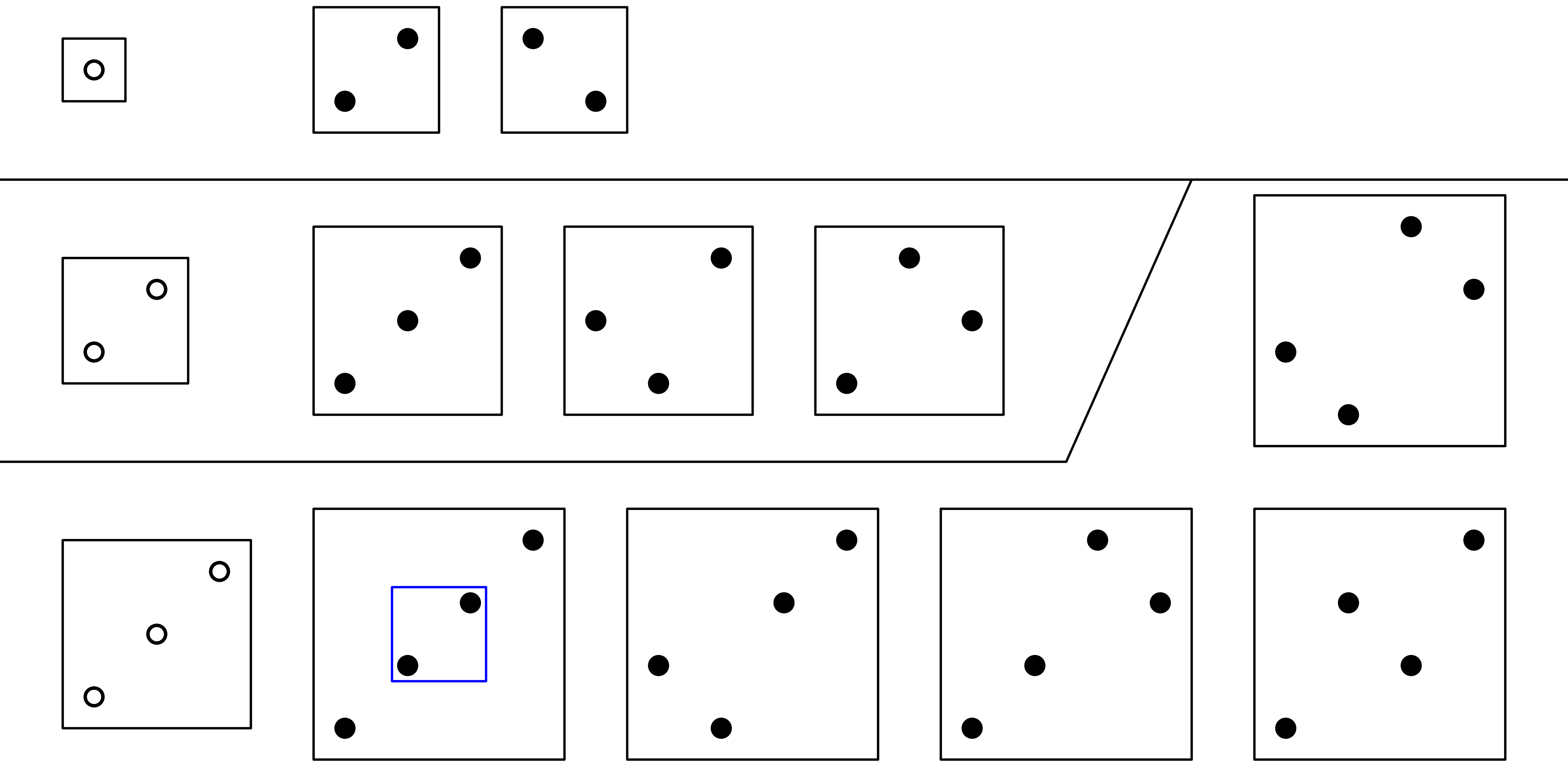

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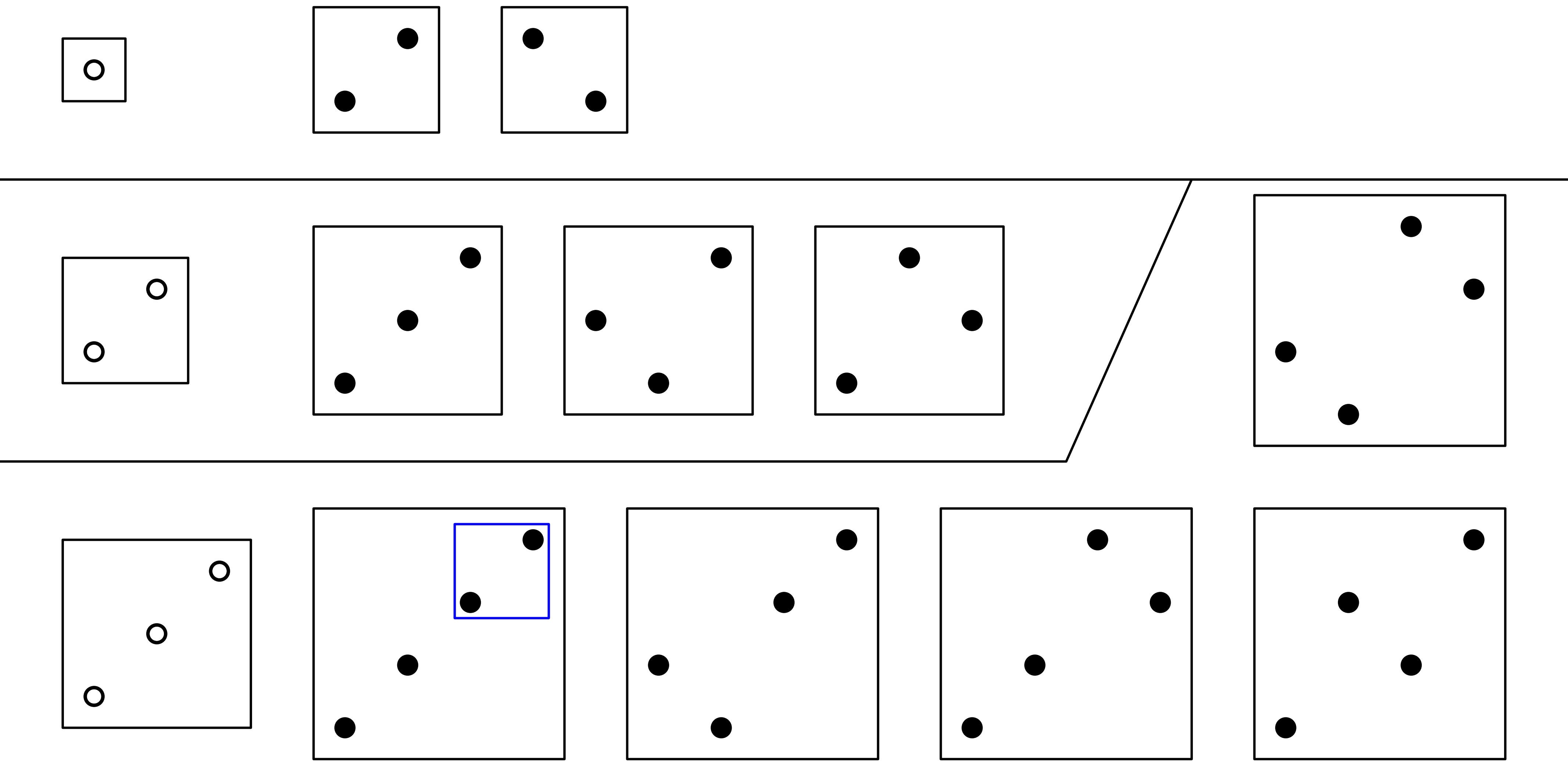
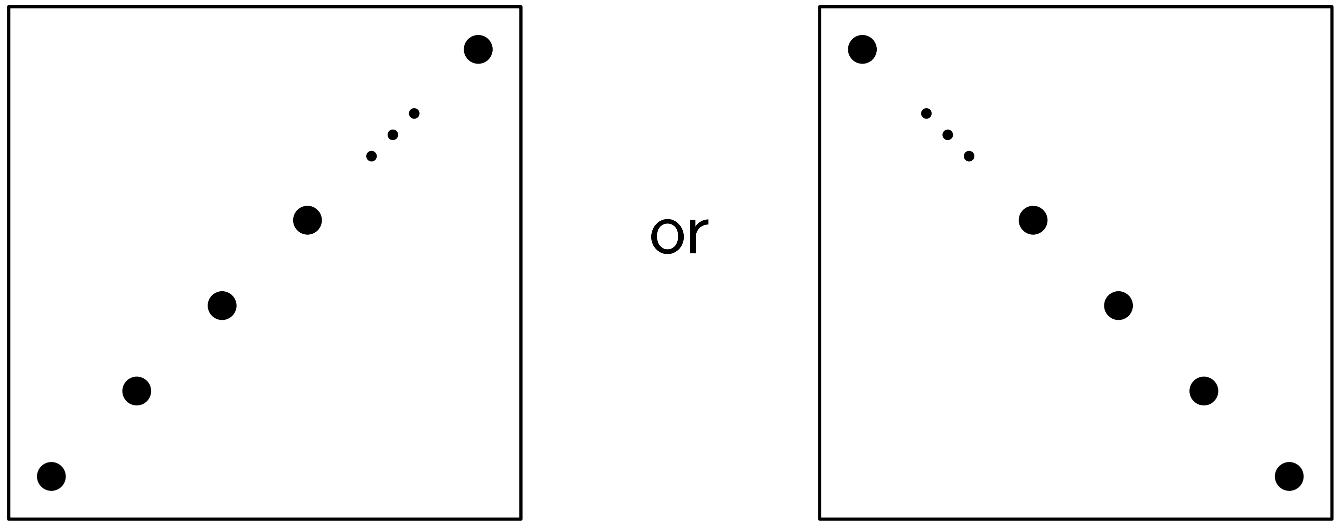
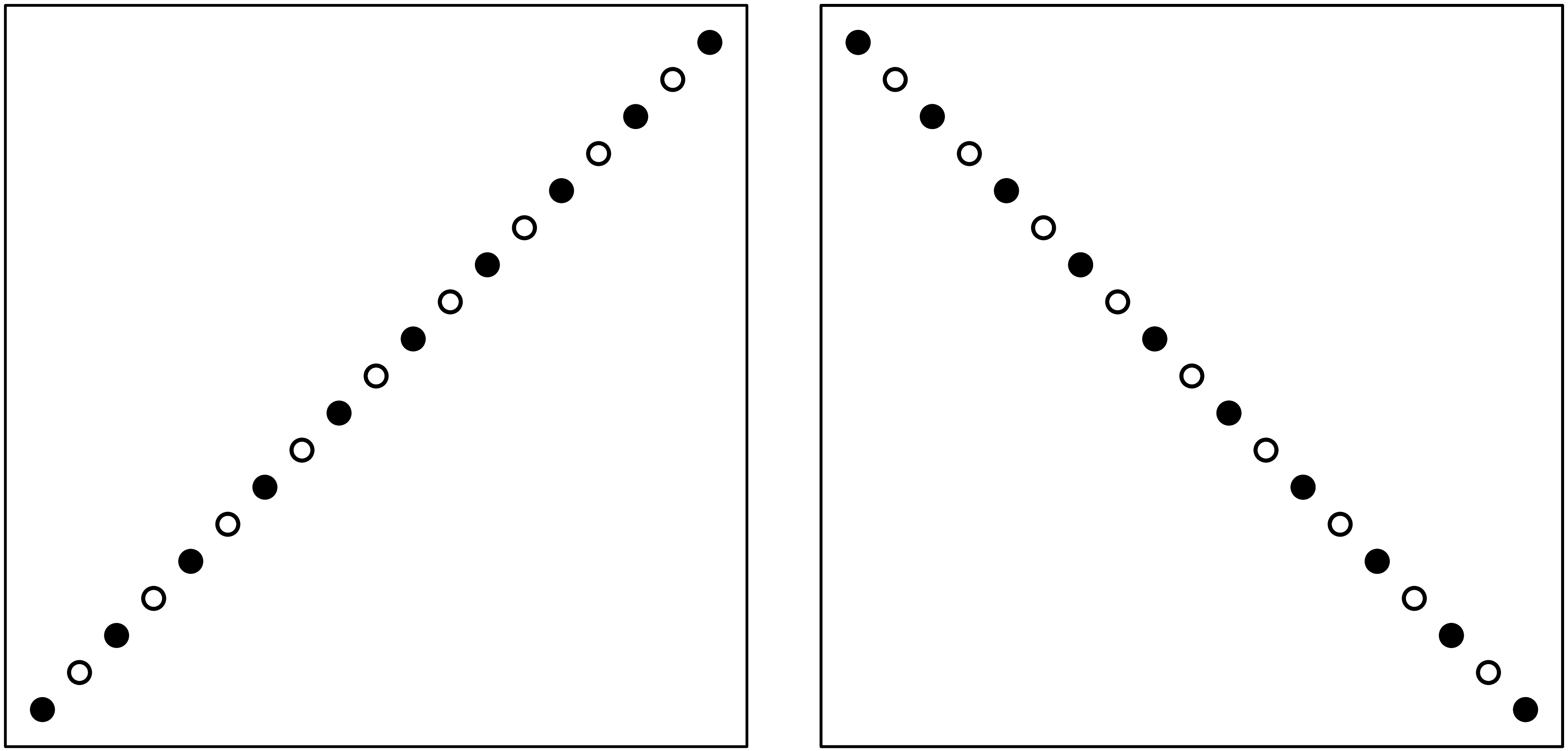


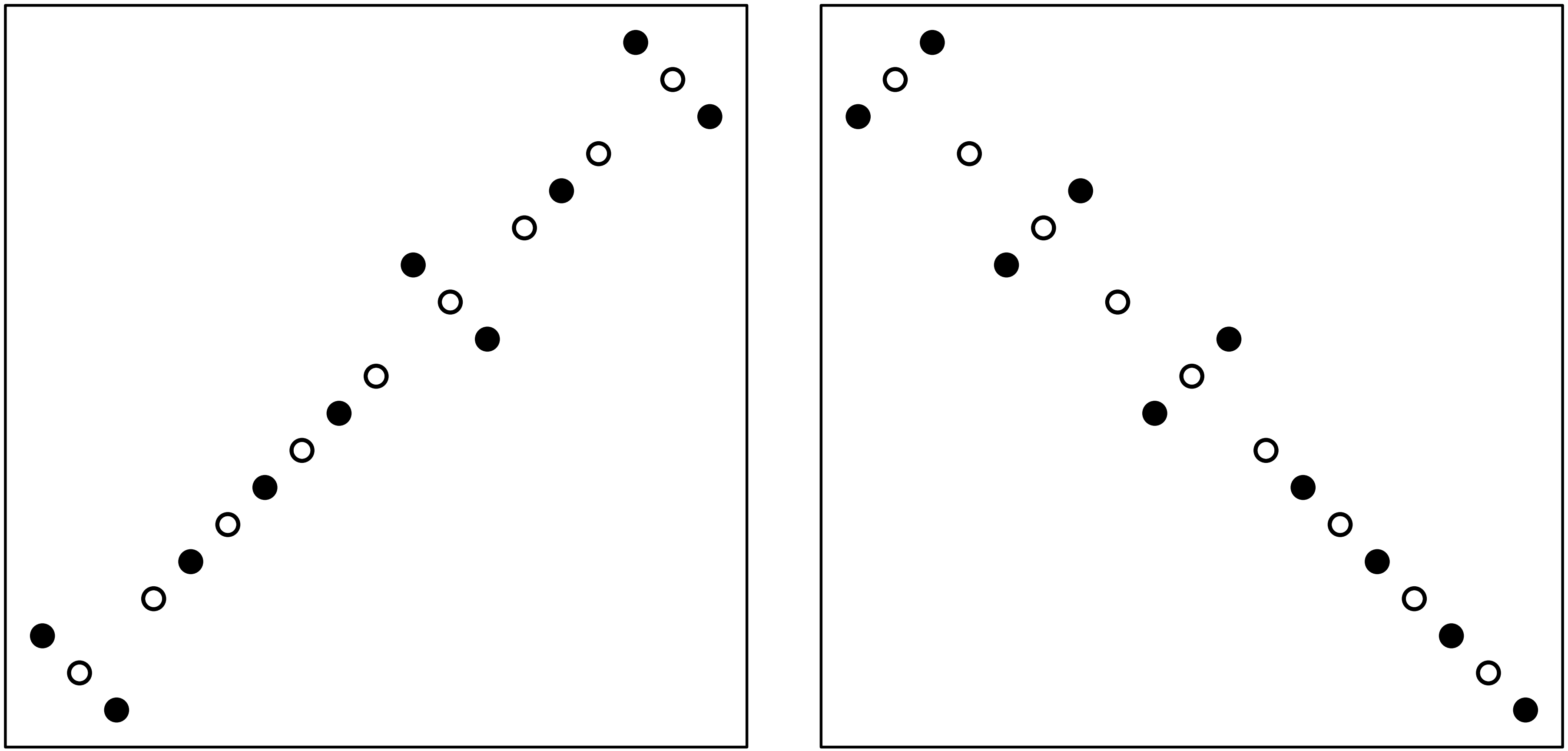


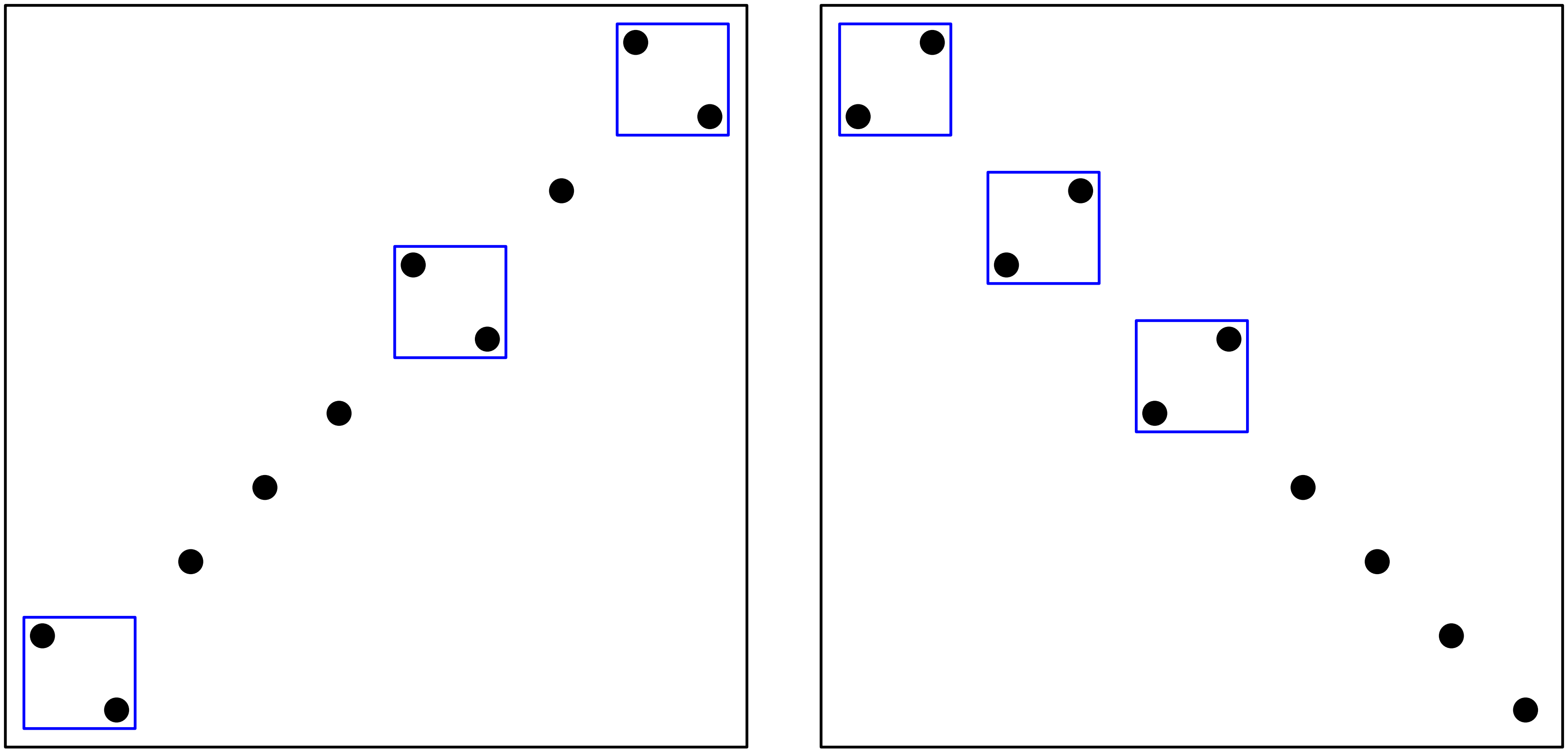

$A_n$ $D_n$

An ascending (resp., descending) *F-block* is either $A_n$ (resp, $D_n$), or a permutation obtained from it by one or several flips of disjoint pairs of adjacent points.

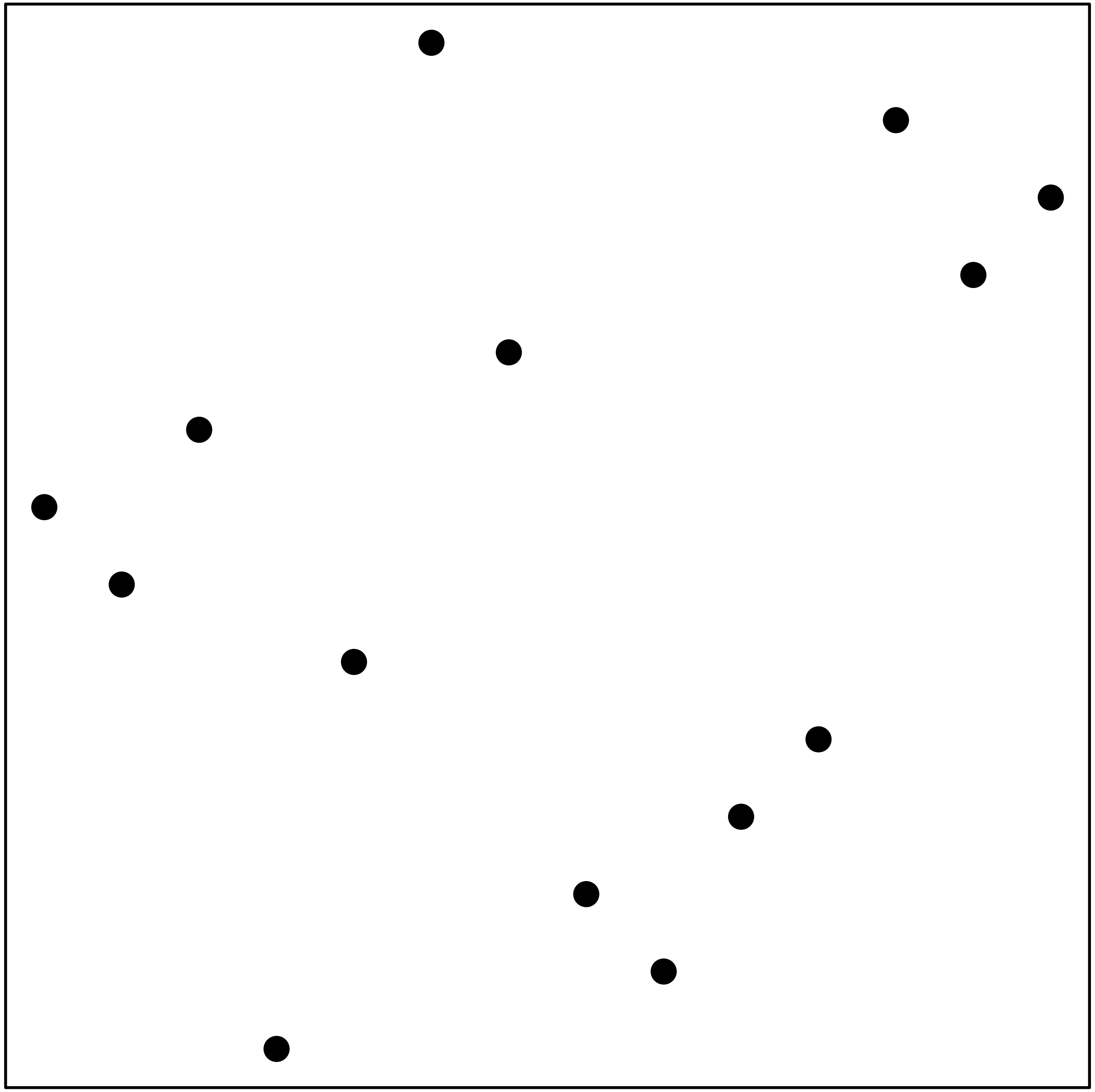
$A_n$ and $D_n$ are *trivial* F-blocks.

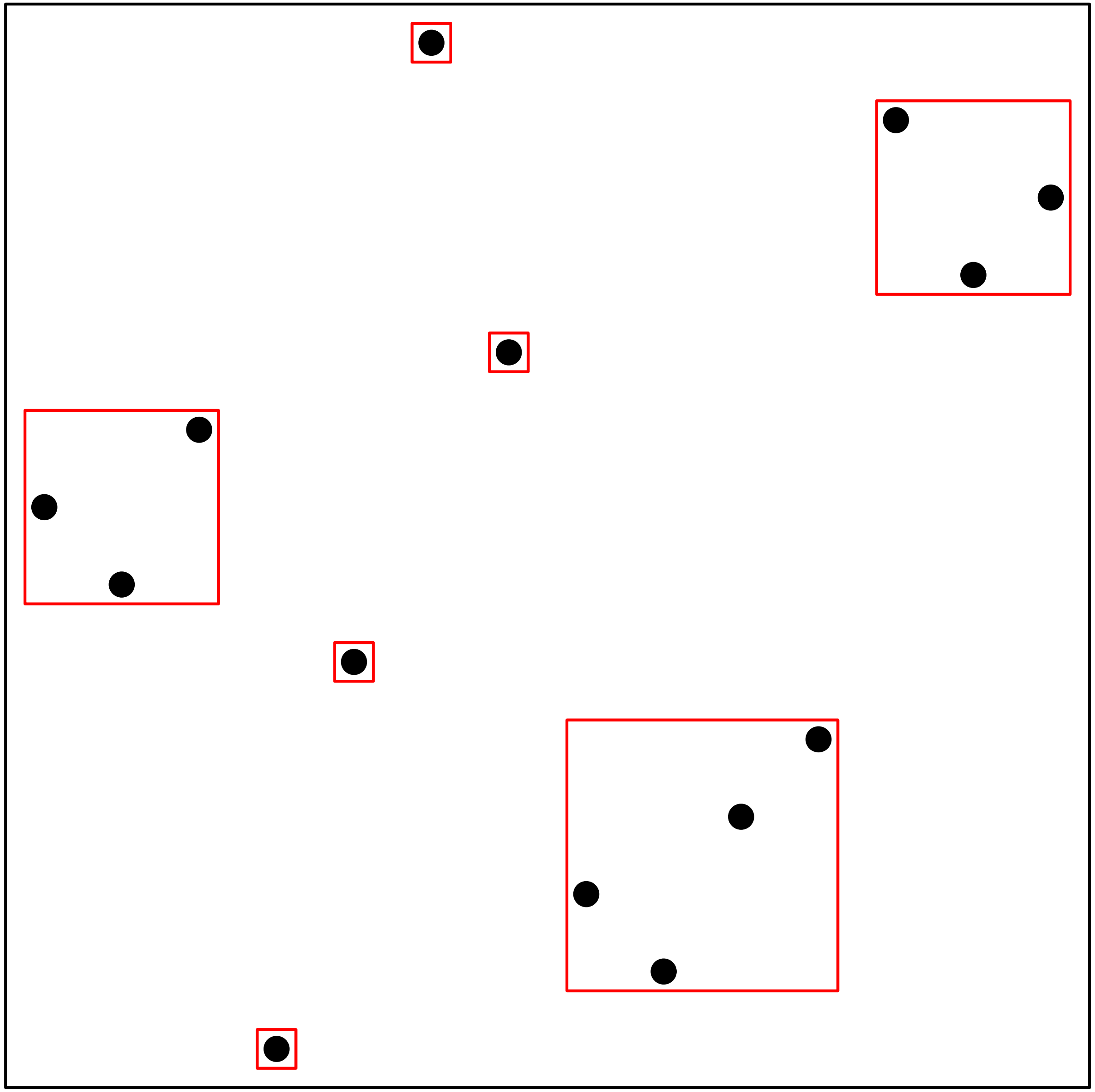


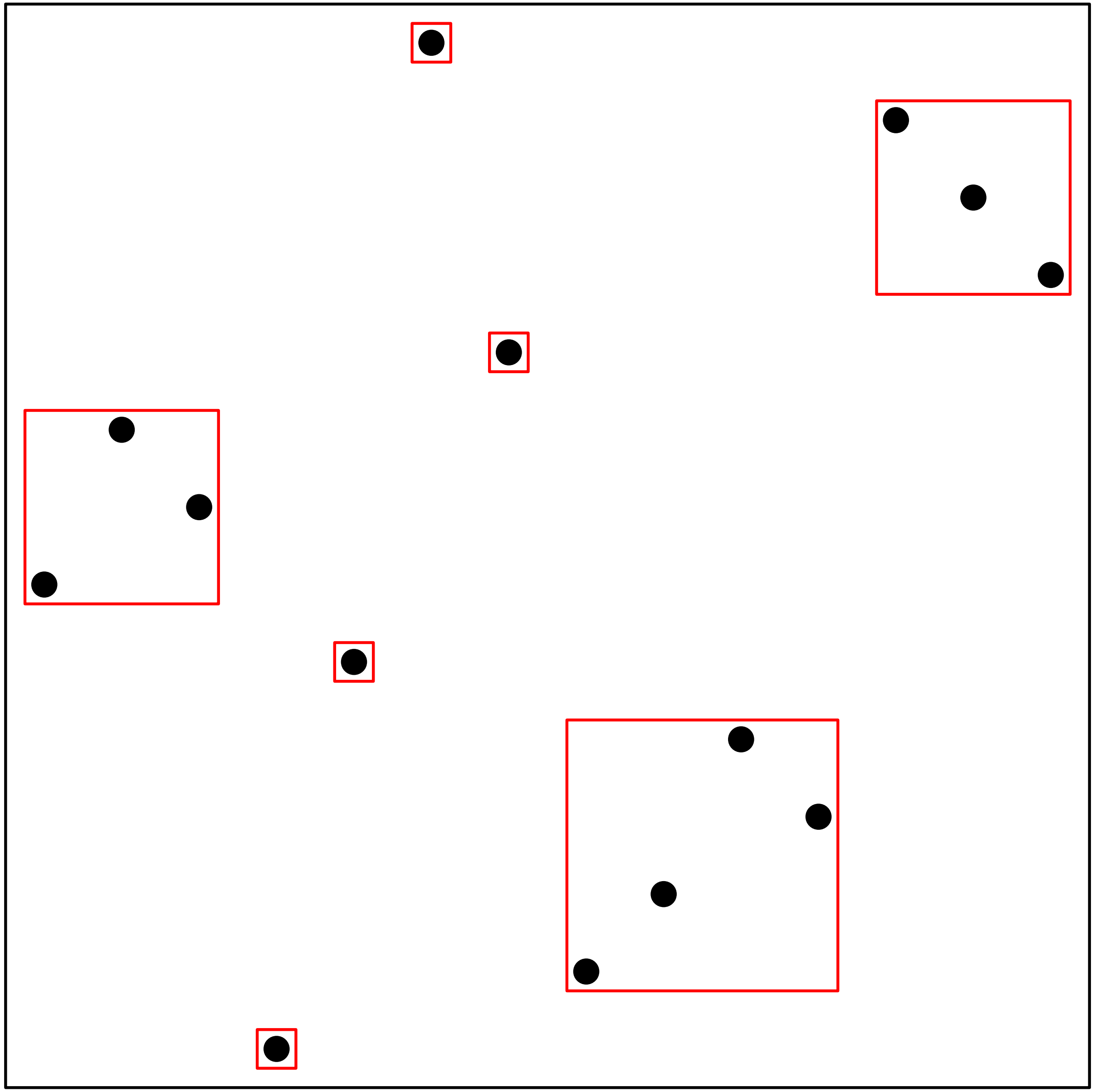


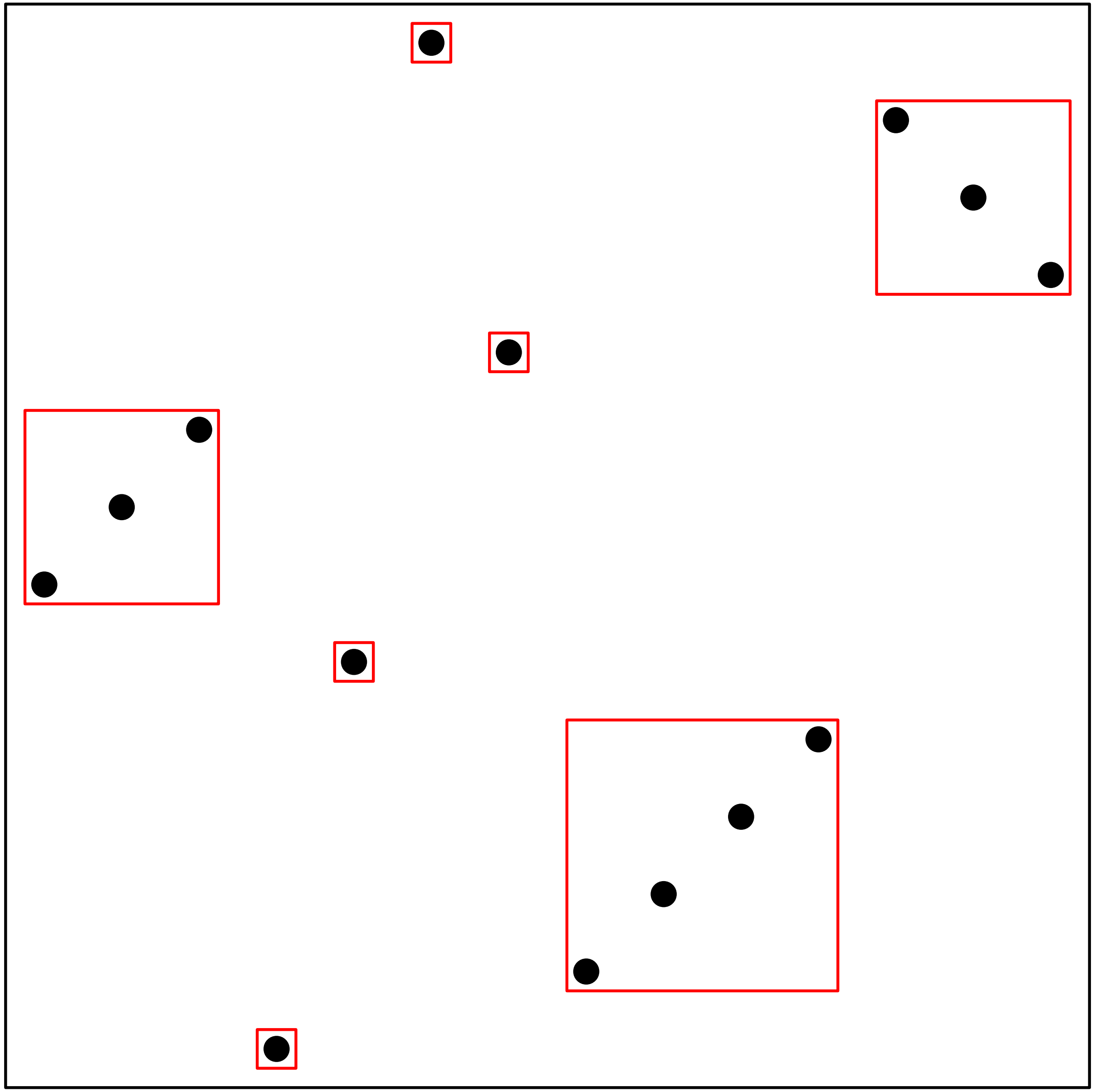


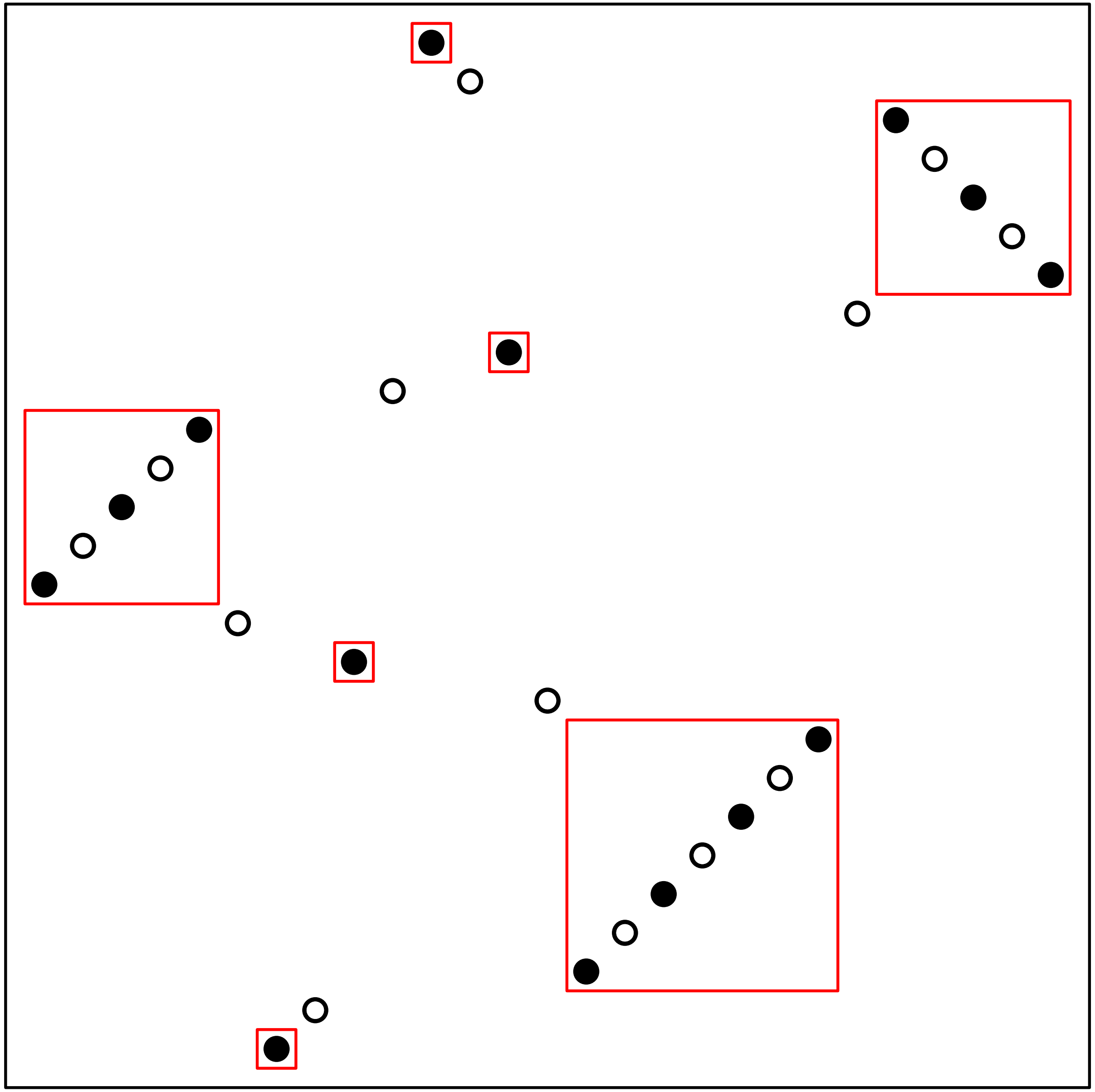
improper pairs



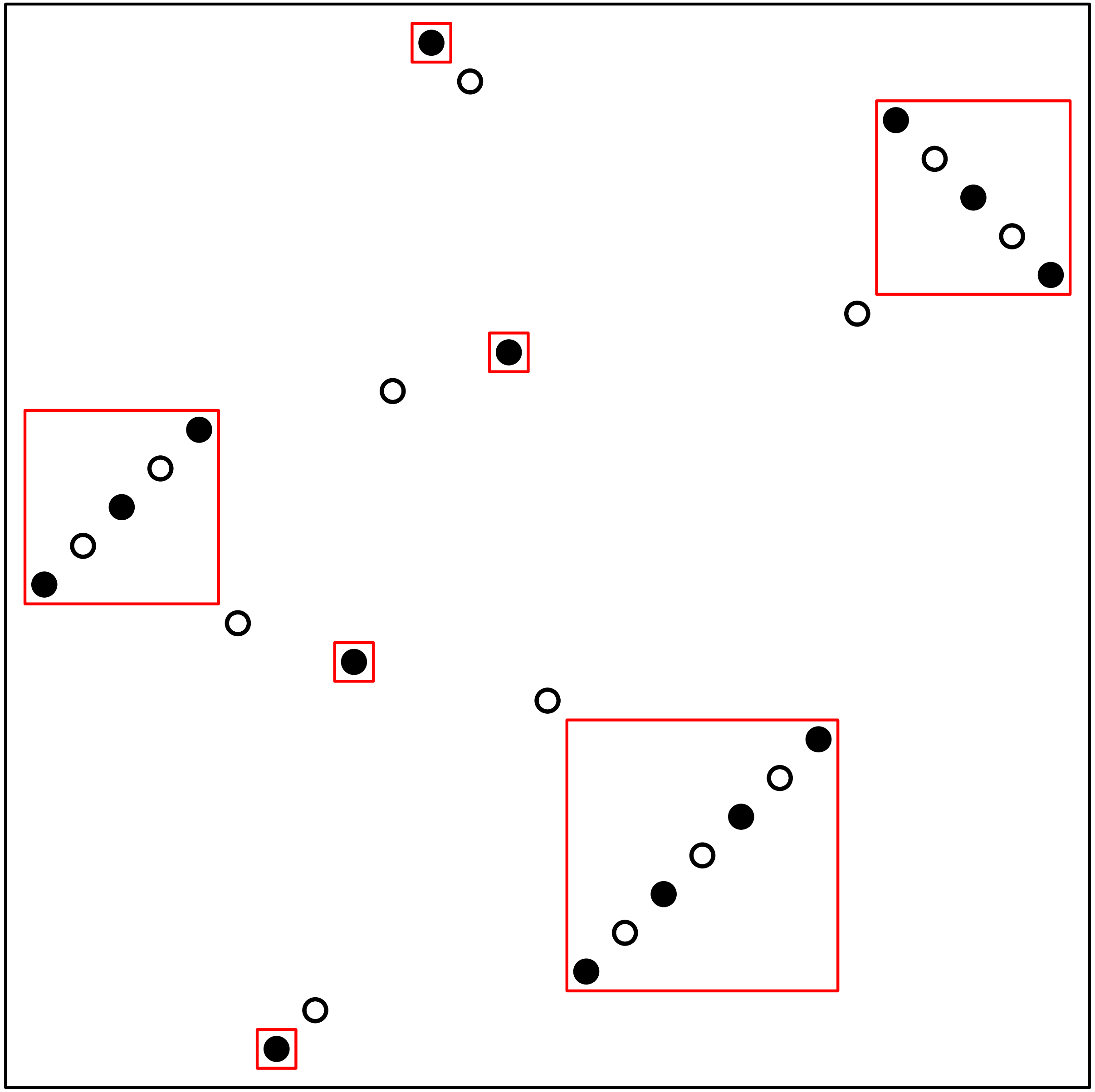








1. Two BLACK (= reduced Baxter) permutations have the same WHITE permutation if and only if they can obtained from each other by replacing some F-blocks by eqivalent F-blocks.



2. The number of WHITE permutations of size $n$ is equal to the number of BLACK ($=$ reduced Baxter) permutations of size $n+1$ **without improper pairs.**

Enumeration.

$$A(t) = 1 + x + 2x^2 + 6x^3 + 22x^4 + 88x^5 + 374x^6 + 1668x^7 + \dots$$

## Enumeration.

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$$A(t) = \frac{1}{t} B(t(1-t)),$$

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$$a_n = \sum_{i=0}^{\lfloor (n+1)/2 \rfloor} (-1)^i \binom{n+1-i}{i} b_{n+1-i}.$$

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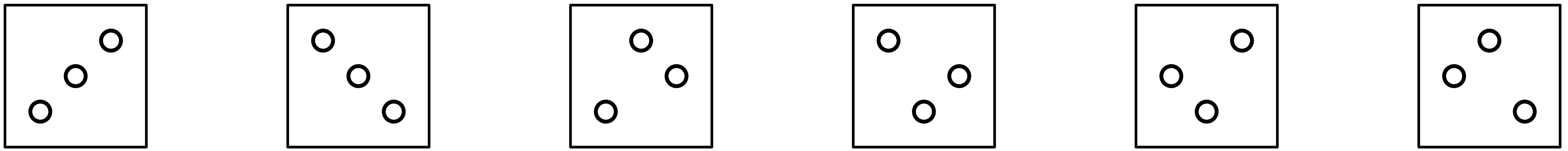
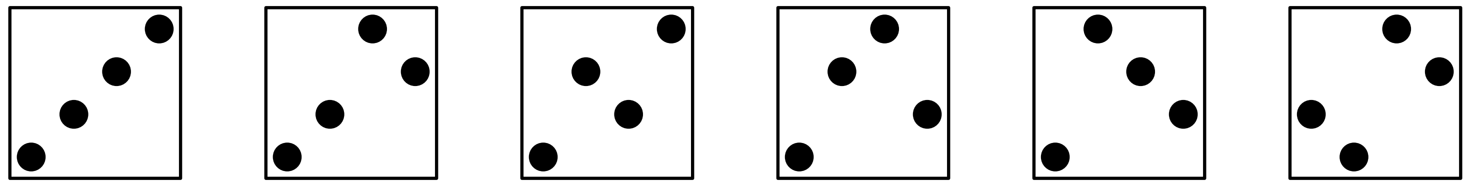
3. $b_{n+1,\,i} = \binom{n+1-i}{i} b_{n+1-i}$.

$n = 3$



$n = 3$

$b_{4,0} = b_4 = 22$



2 4 2 8 4 2

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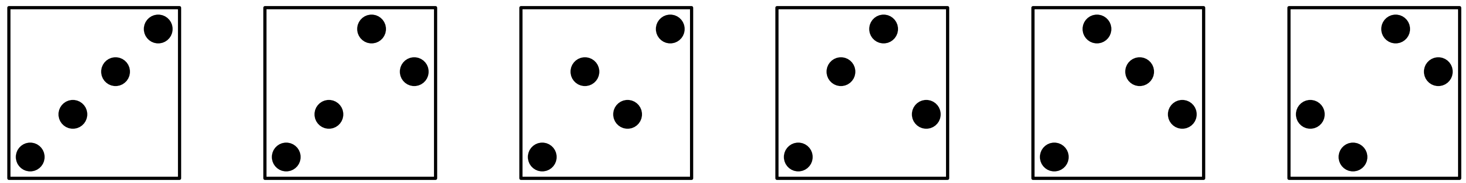
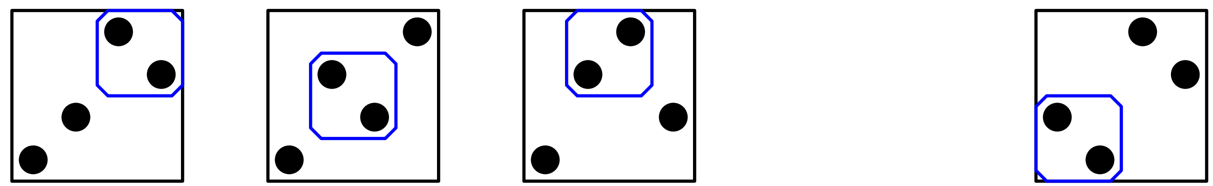
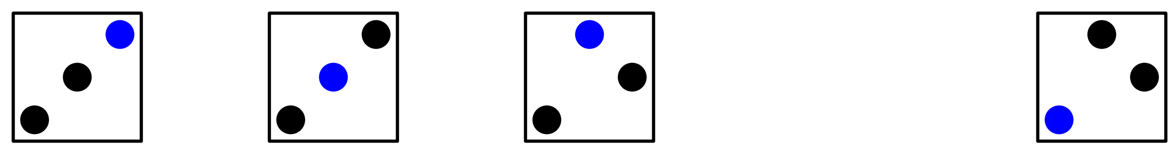


2 4 2 8 4 2

$b_{4,1} = \binom{3}{1} b_3 = 18$



4 2 8 4



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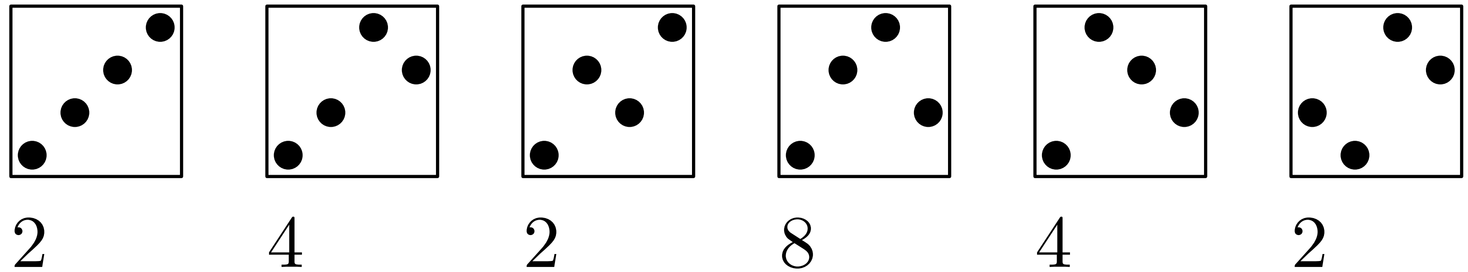


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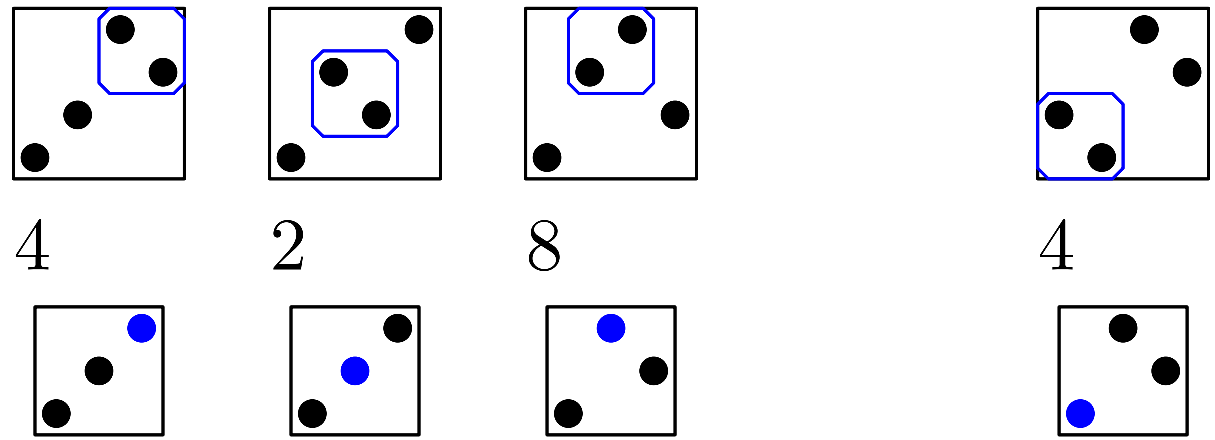
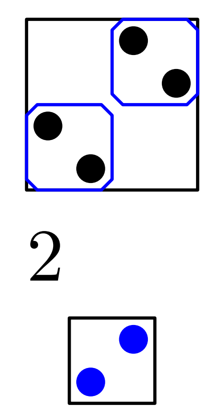
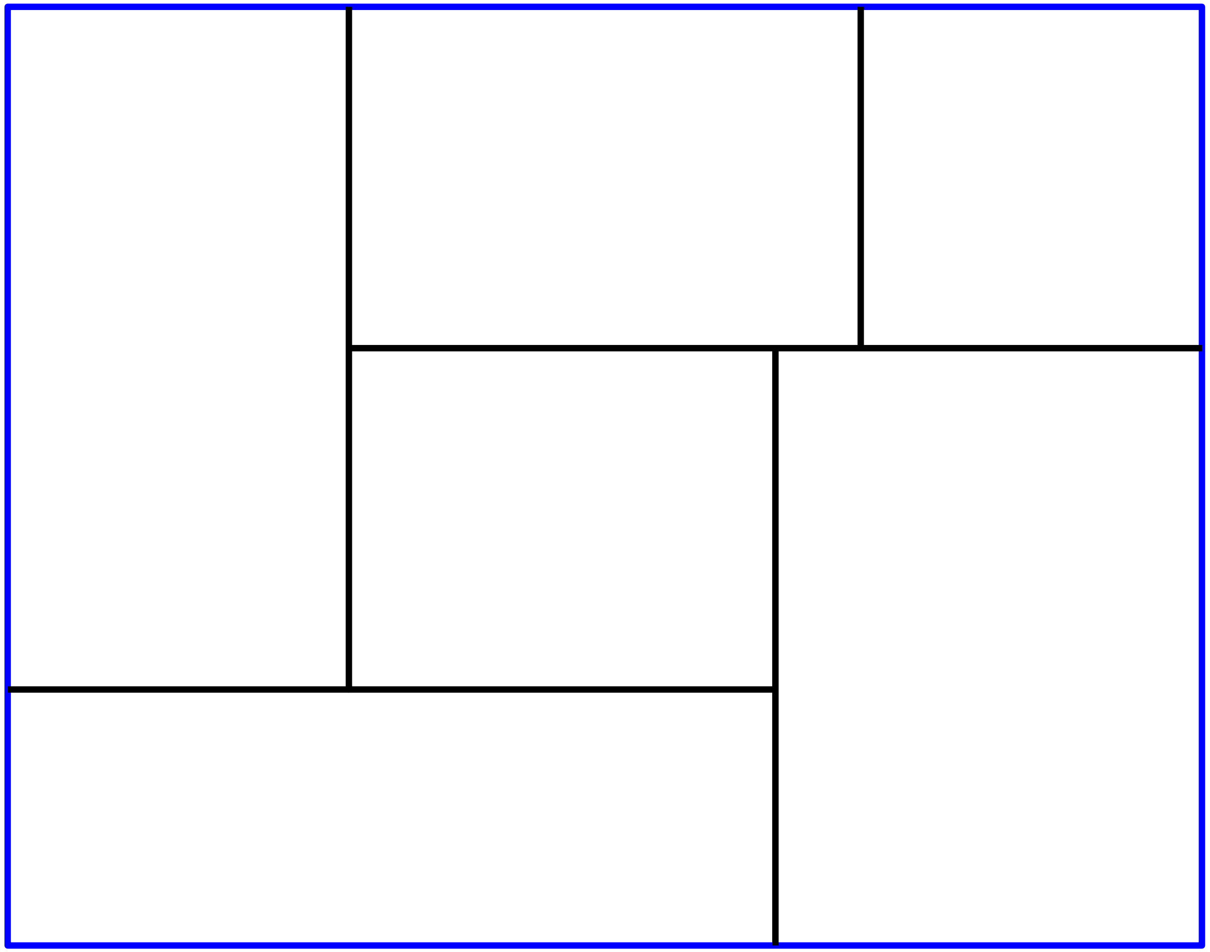



$b_{4,2} = \binom{2}{2} b_2 = 2$



A combinatorial interpretation:
order relations between cuts in planar floorplans.



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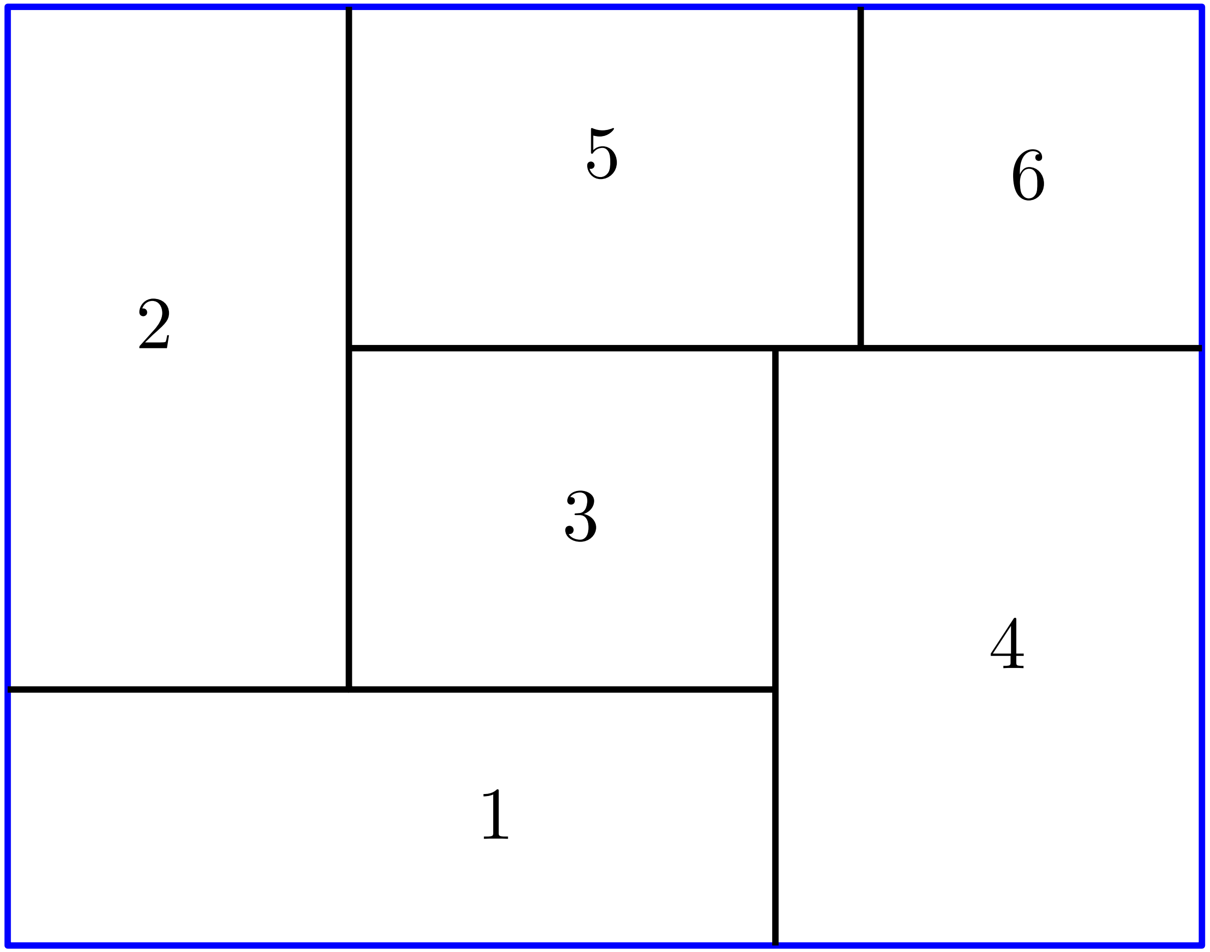

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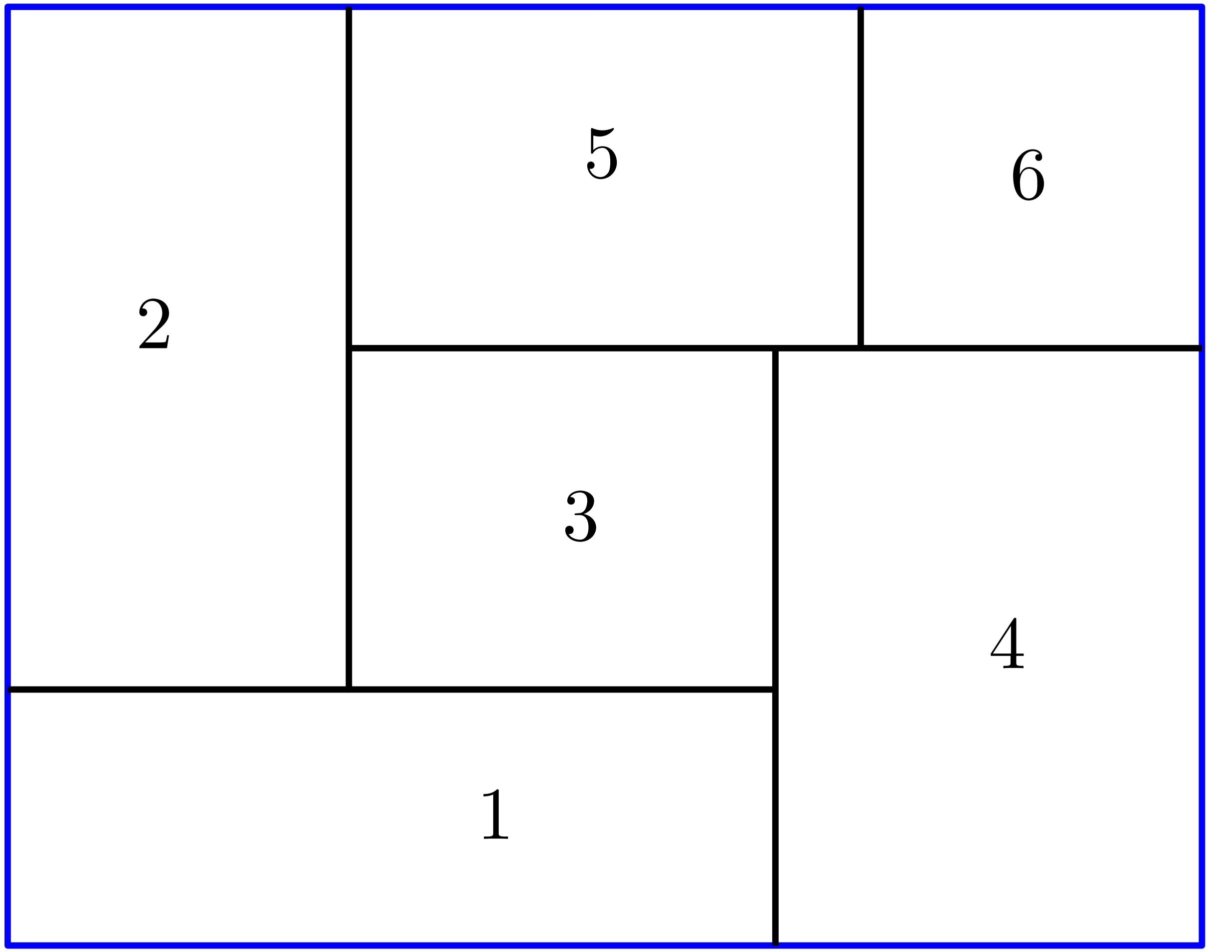


2 5 6 3 1 4

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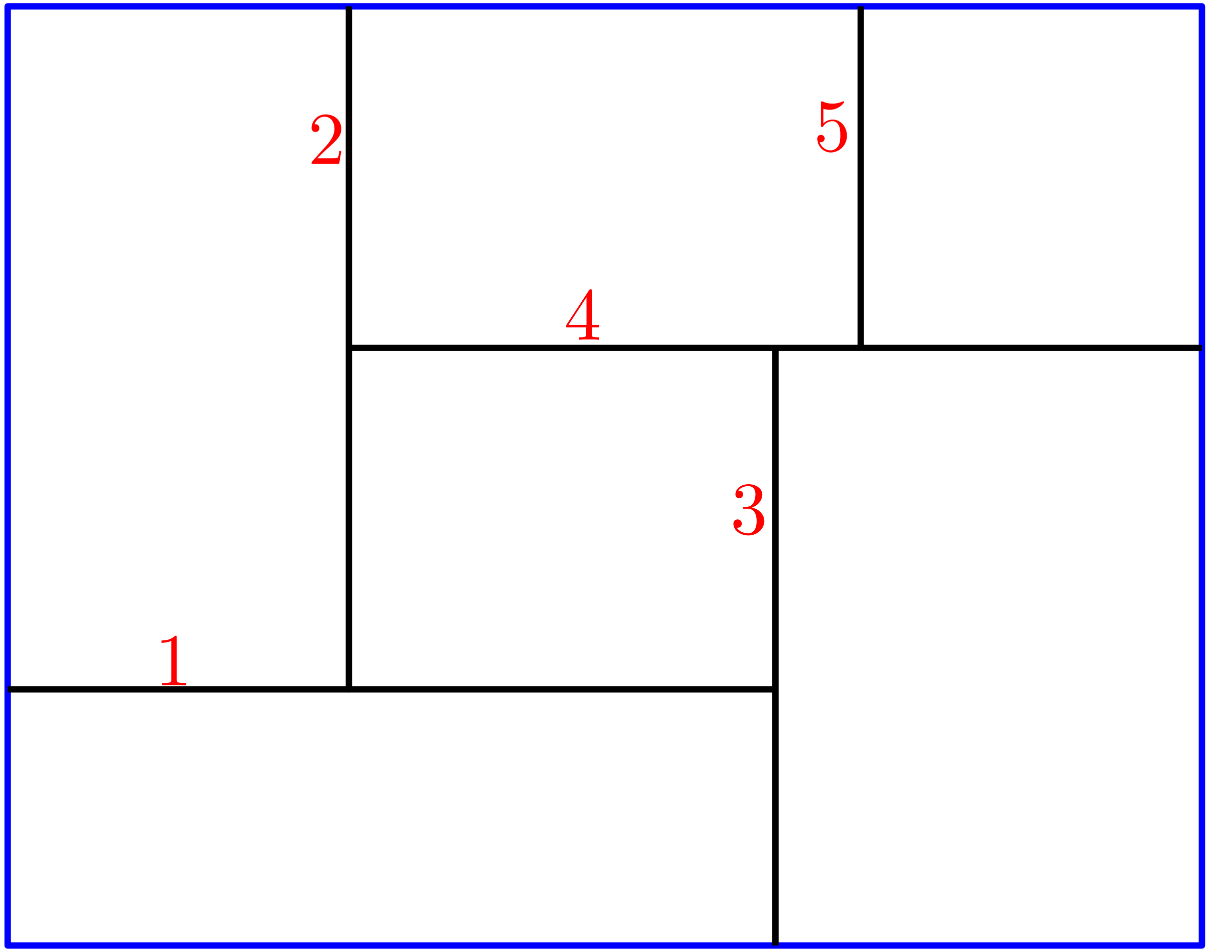


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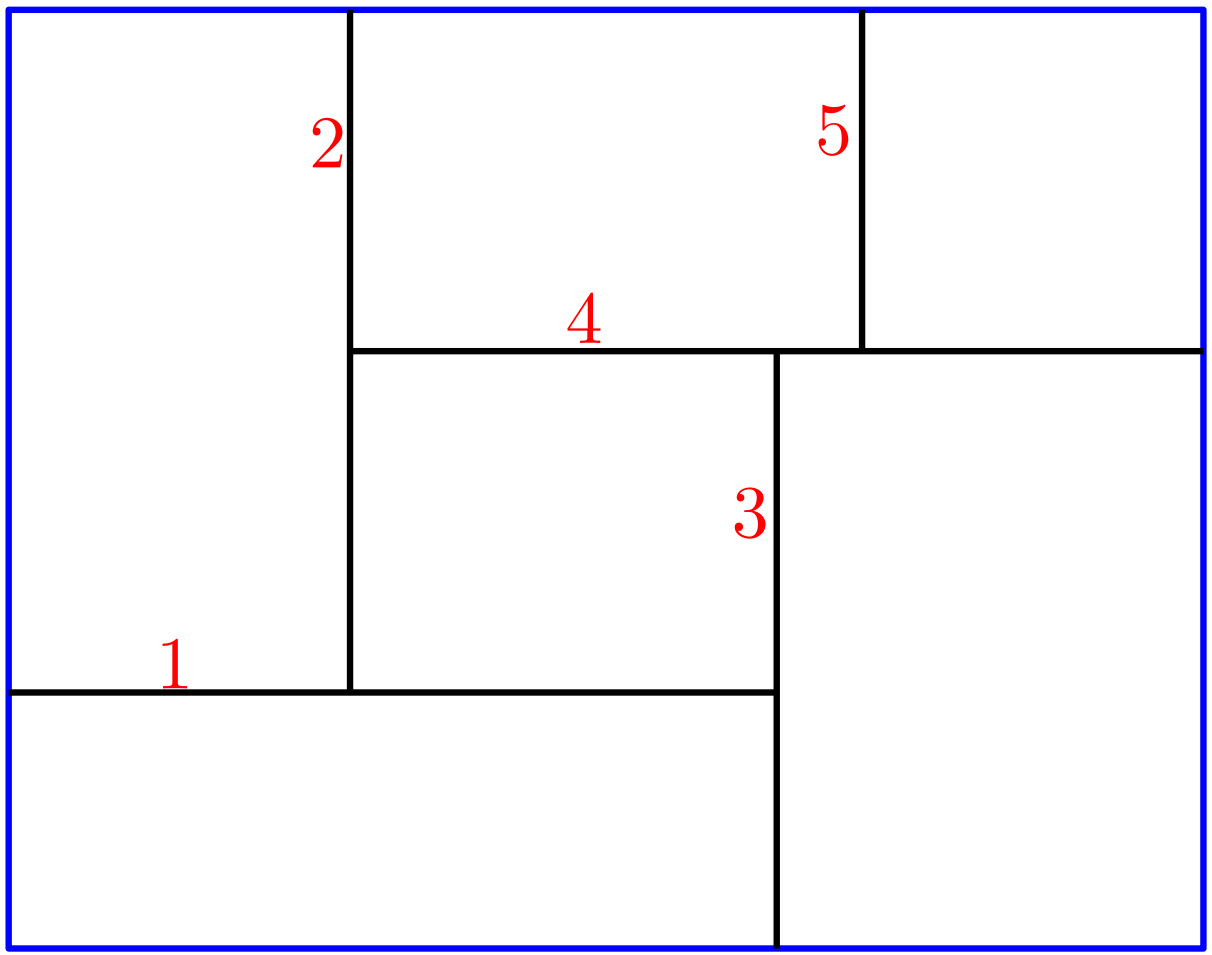


2 5 6 3 1 4

2 5 4 1 3

A combinatorial interpretation:
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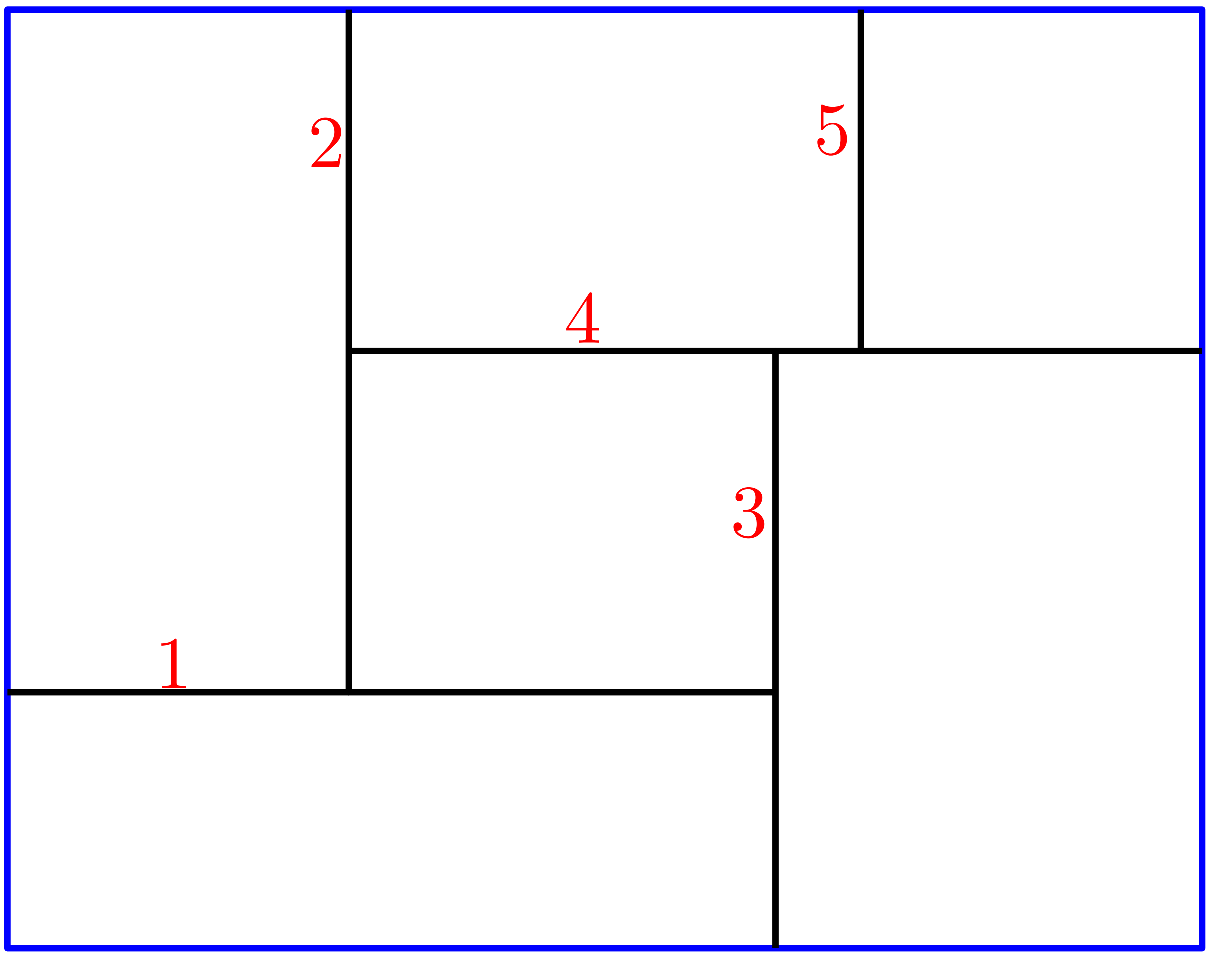
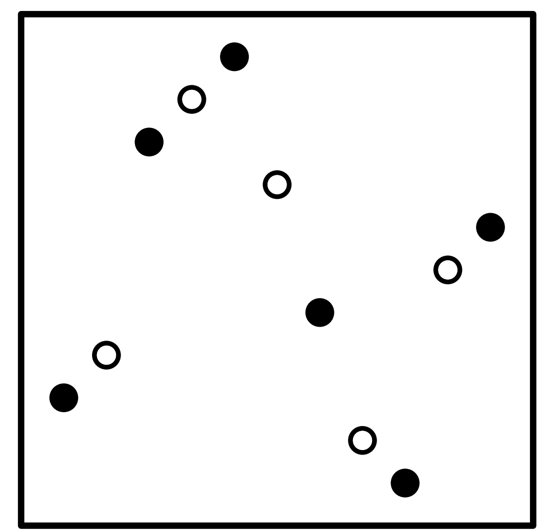
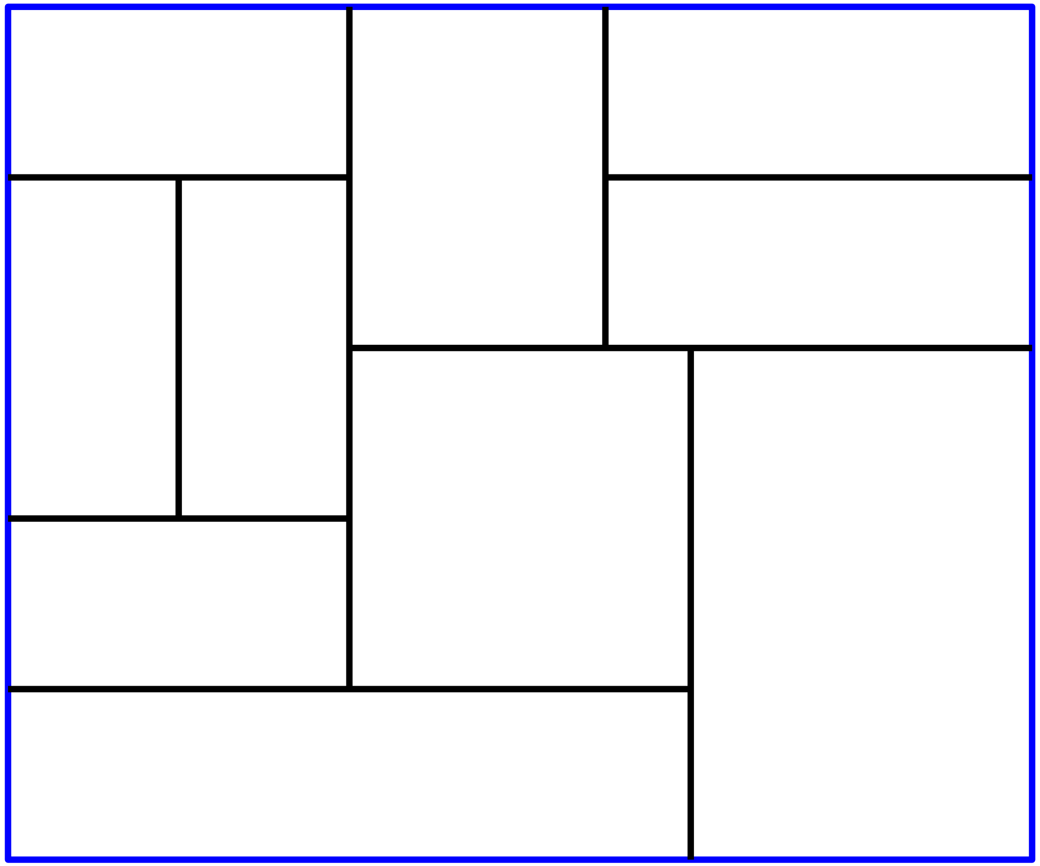
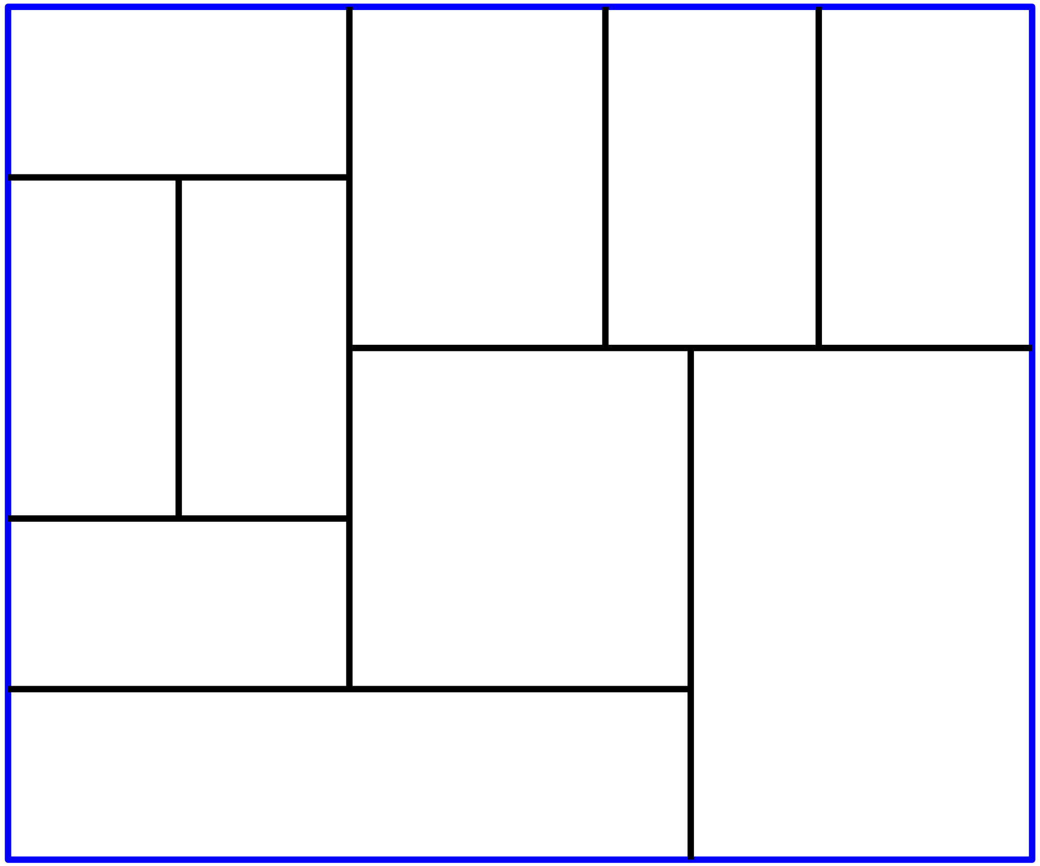


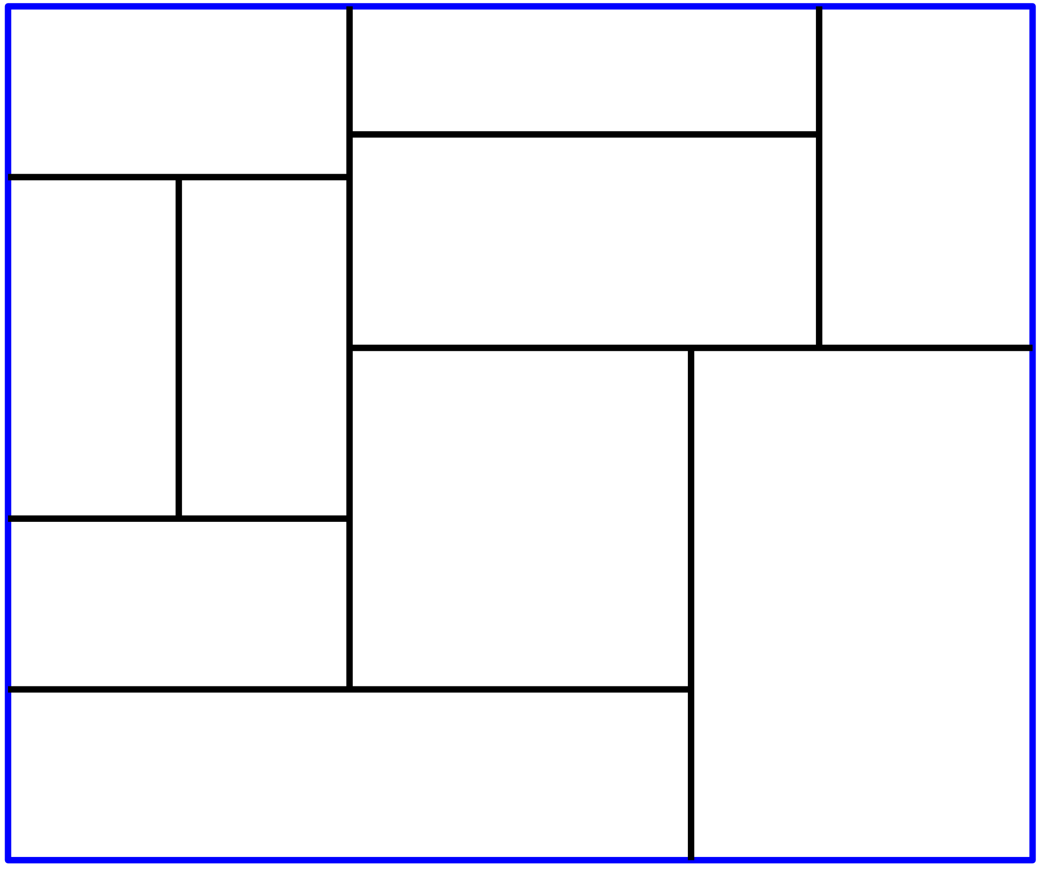

$2\ 5\ 6\ 3\ 1\ 4$

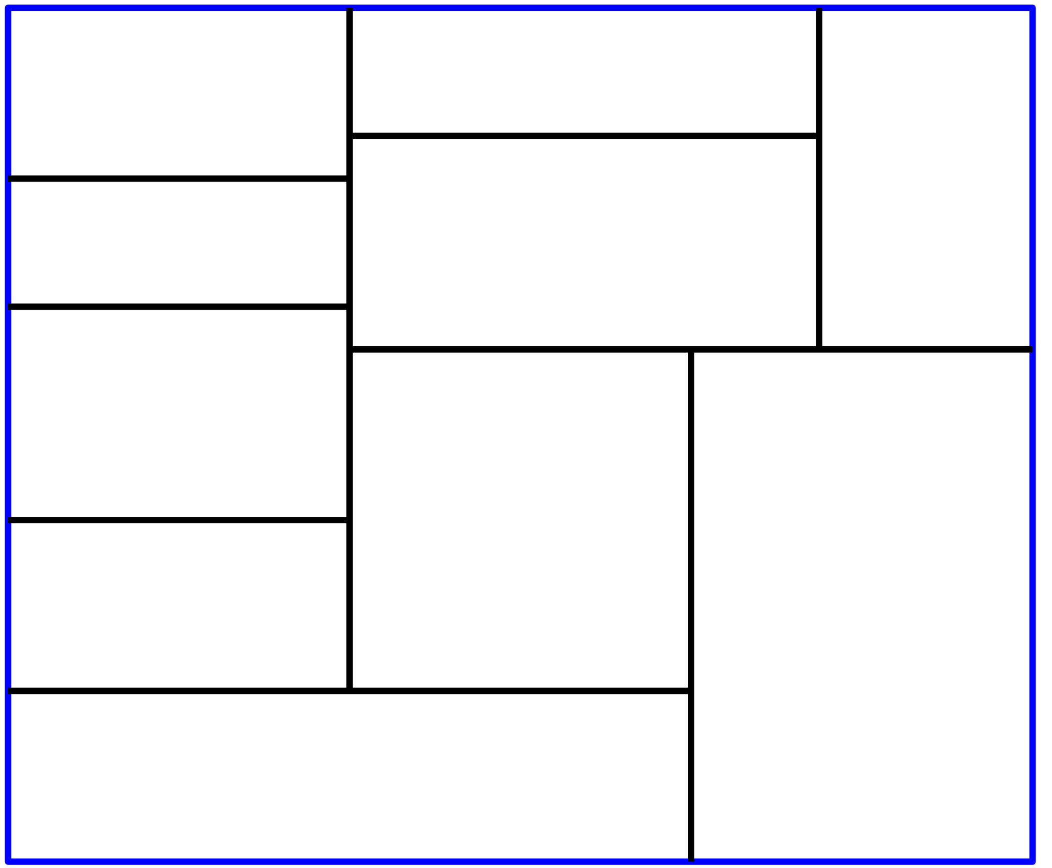
$2\ 5\ 4\ 1\ 3$

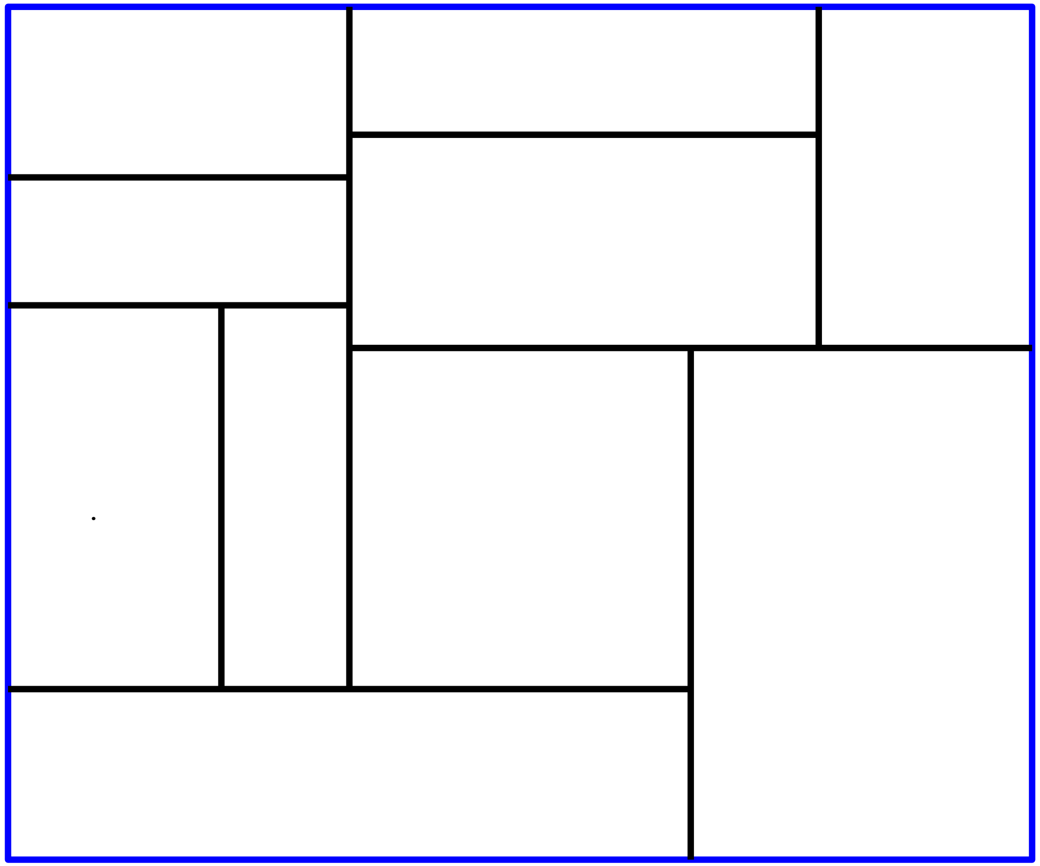


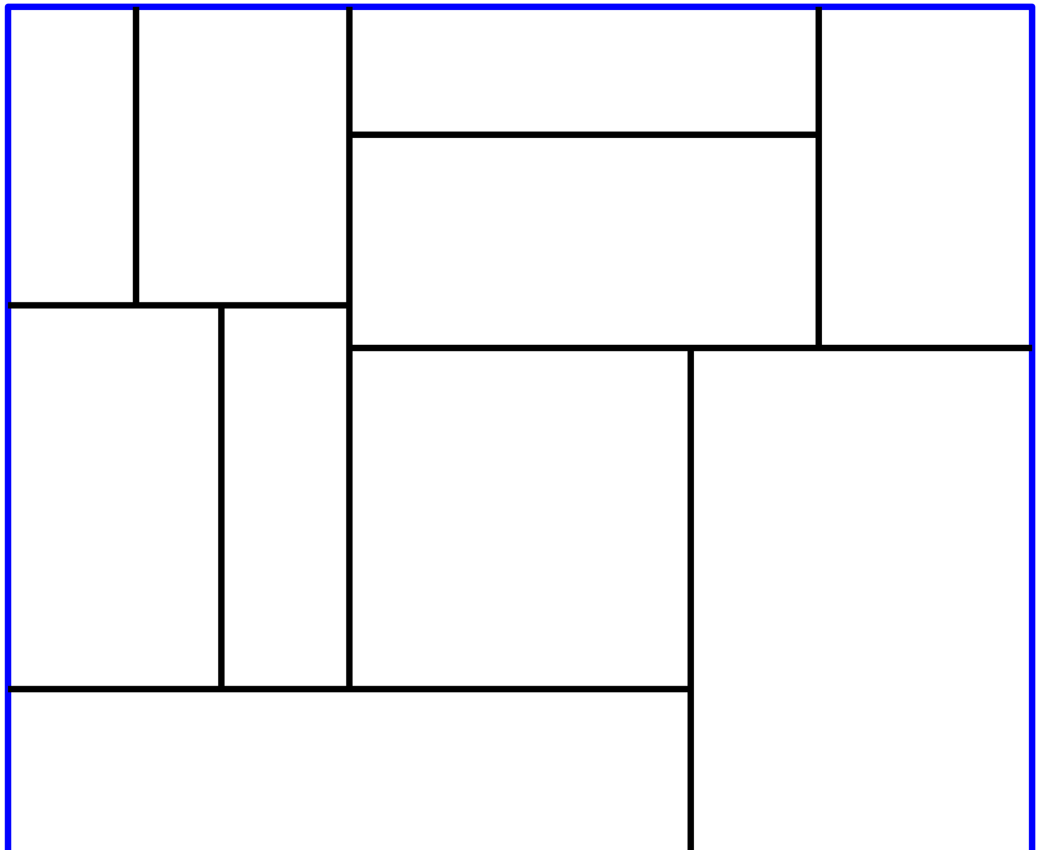


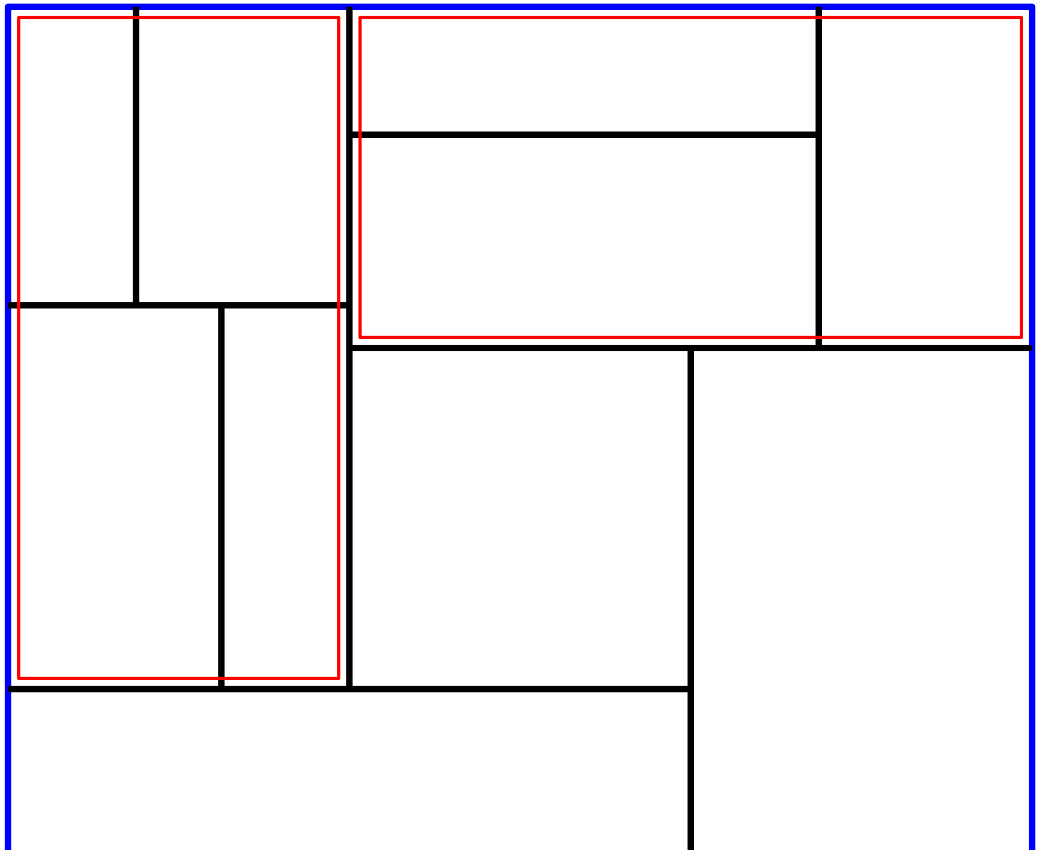


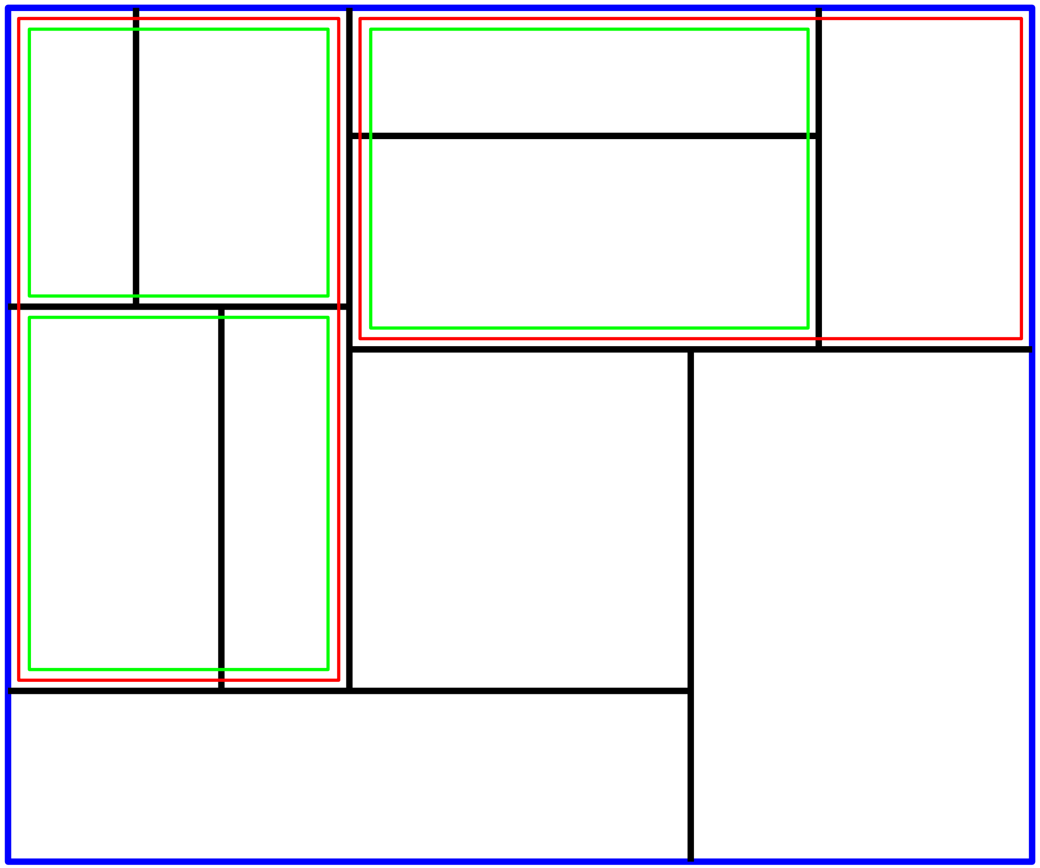


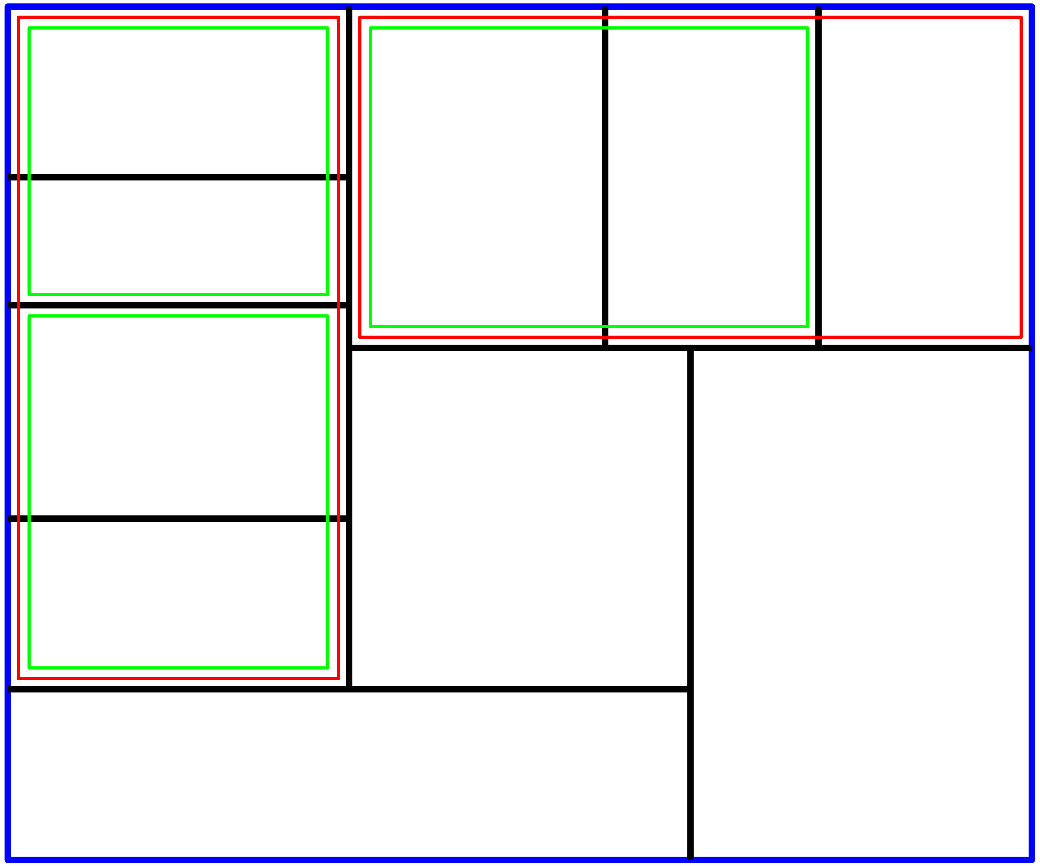


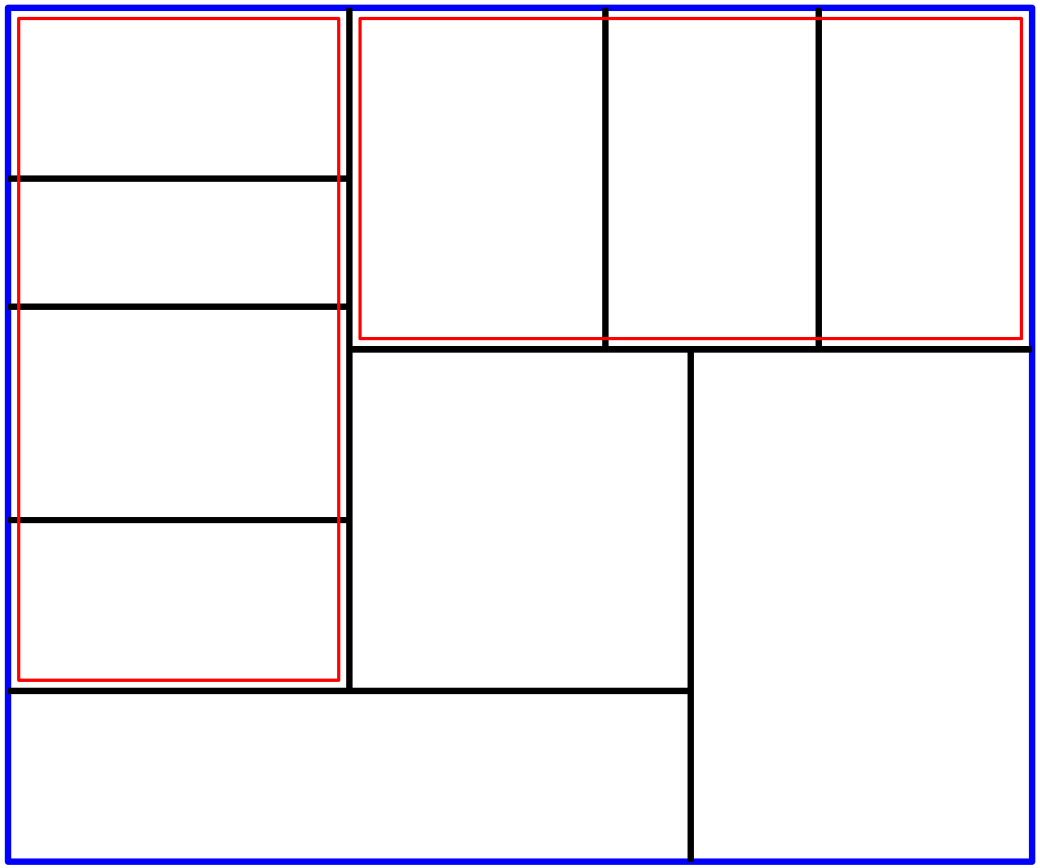




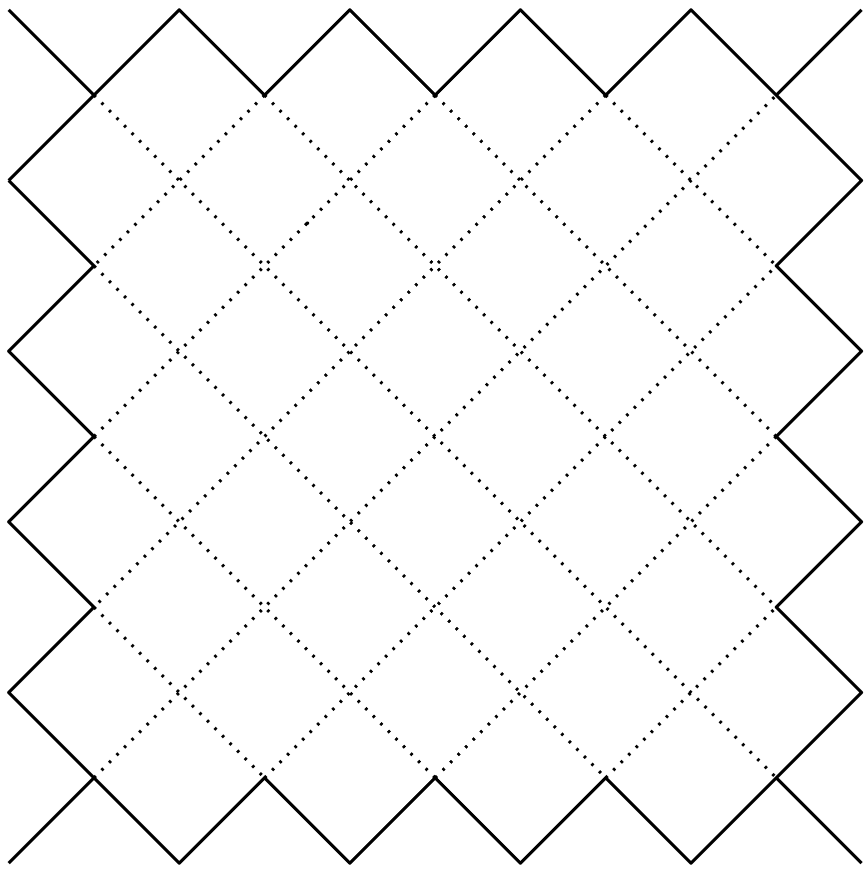








# Aztec Diamonds (Hal Canary)



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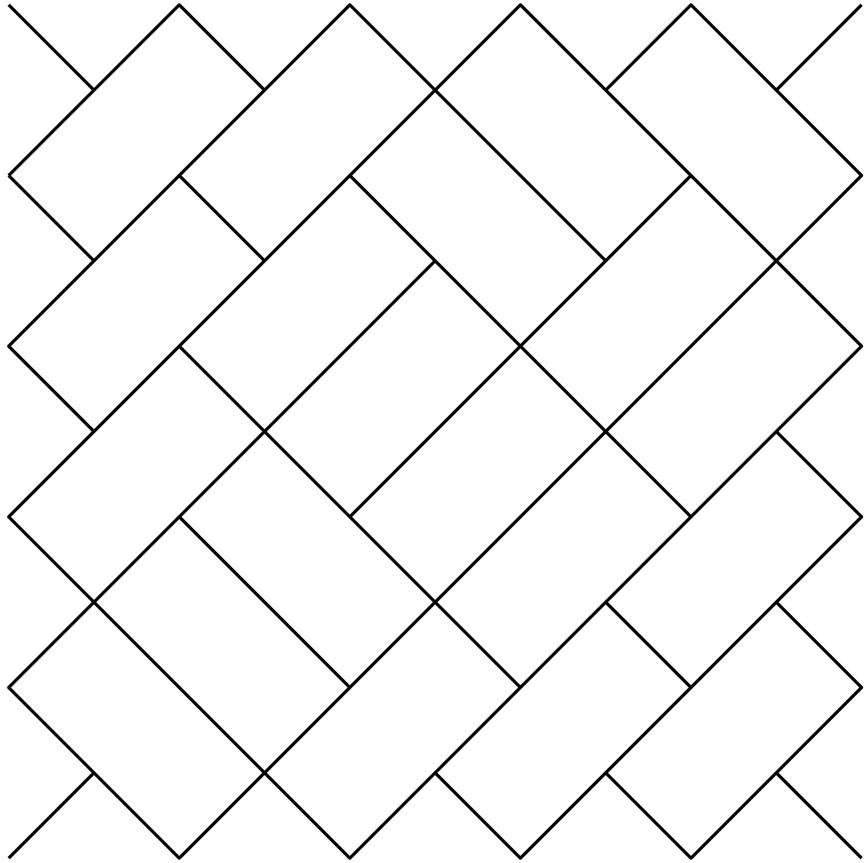

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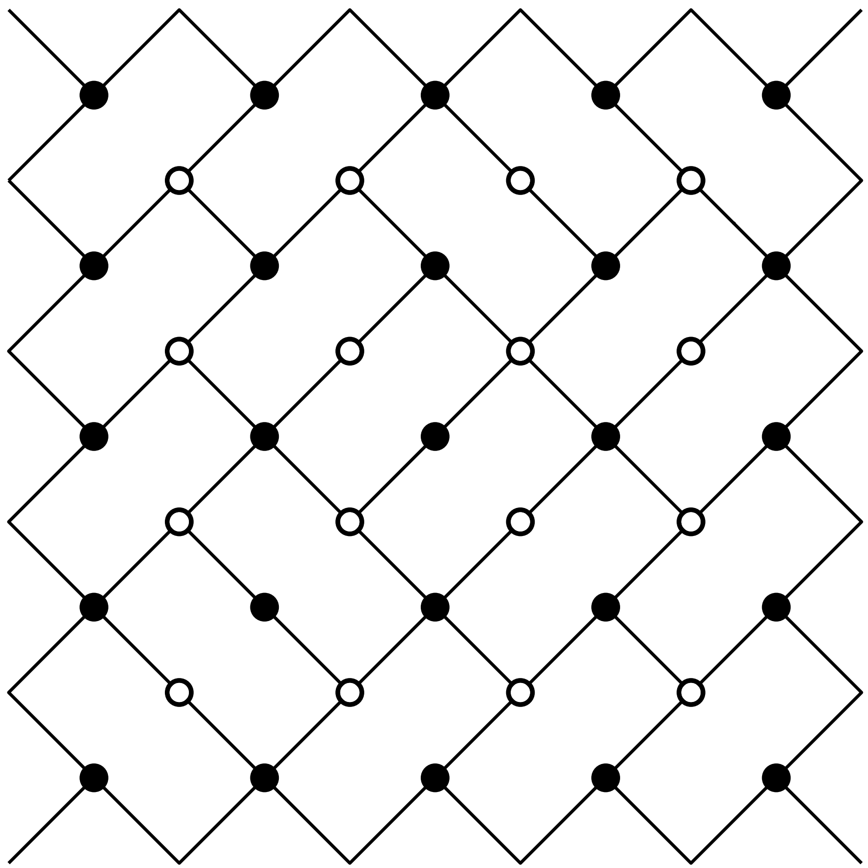

Two ASMs: L: $3 \to 0,\ 2 \to -1,\ 4 \to 1$

S: $3 \to 0,\ 2 \to 1,\ 4 \to -1.$

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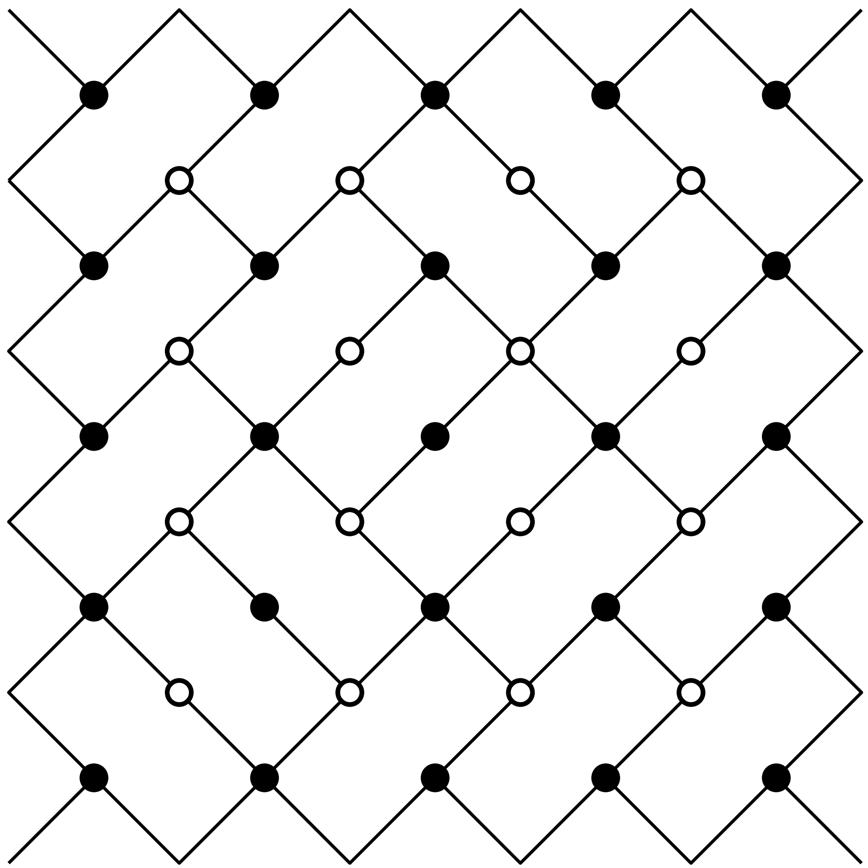

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0 | | 0 | | 1 | | 0 | | 0 |
| | 0 | | 0 | | 1 | | 0 | |
| 0 | | 0 | | 0 | | 0 | | 1 |
| | 0 | | 1 | | $-1$ | | 1 | |
| 0 | | 1 | | $-1$ | | 1 | | 0 |
| | 0 | | 0 | | 1 | | 0 | |
| 1 | | $-1$ | | 1 | | 0 | | 0 |
| | 1 | | 0 | | 0 | | 0 | |
| 0 | | 1 | | 0 | | 0 | | 0 |

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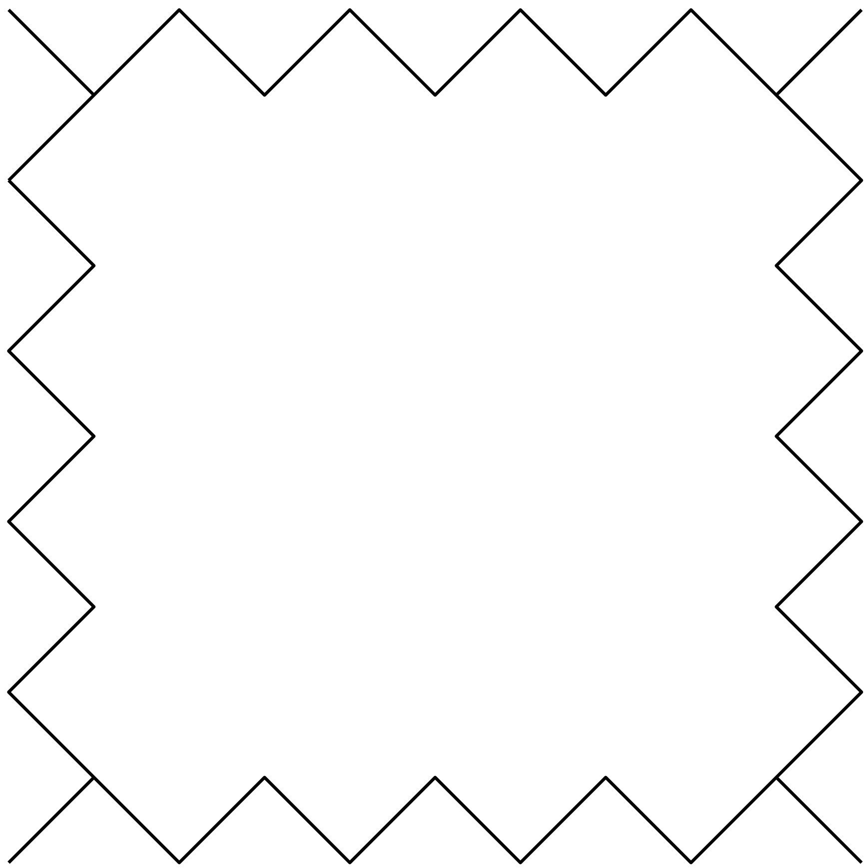
# Aztec Diamonds (Hal Canary)

If the L-ASM is a permutation matrix, then it is the matrix of a Baxter permutation



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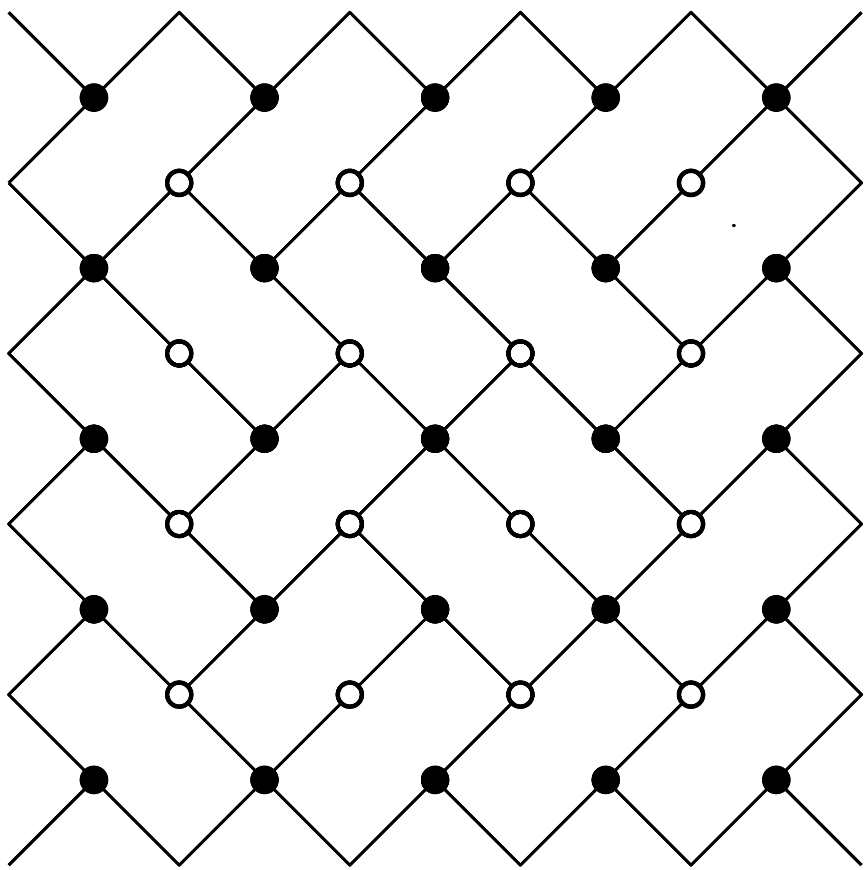

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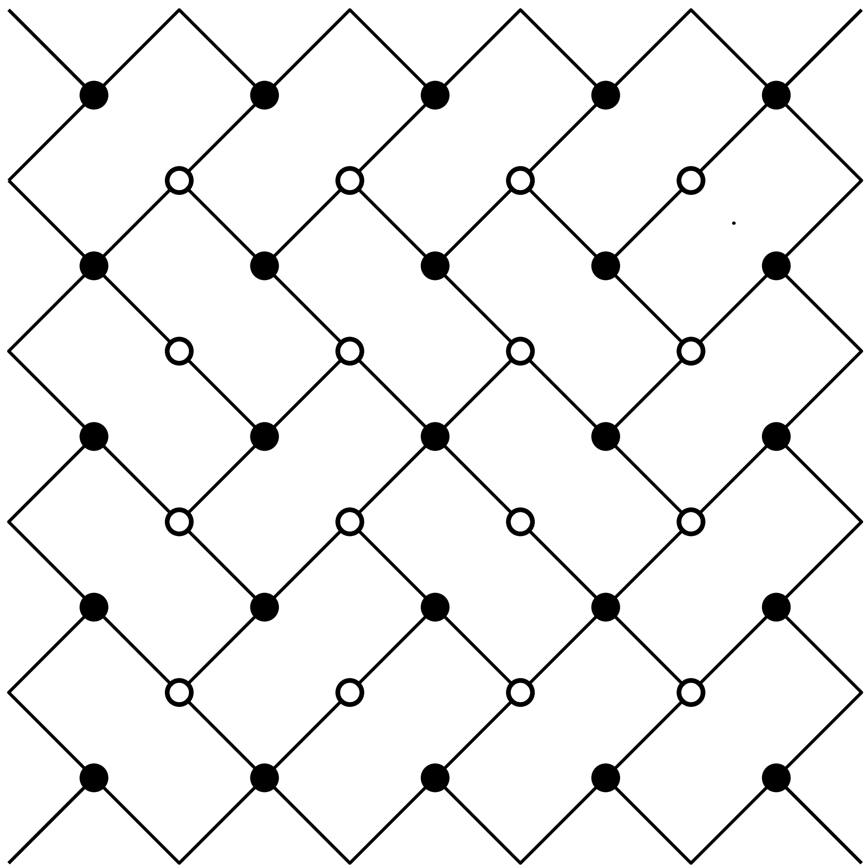

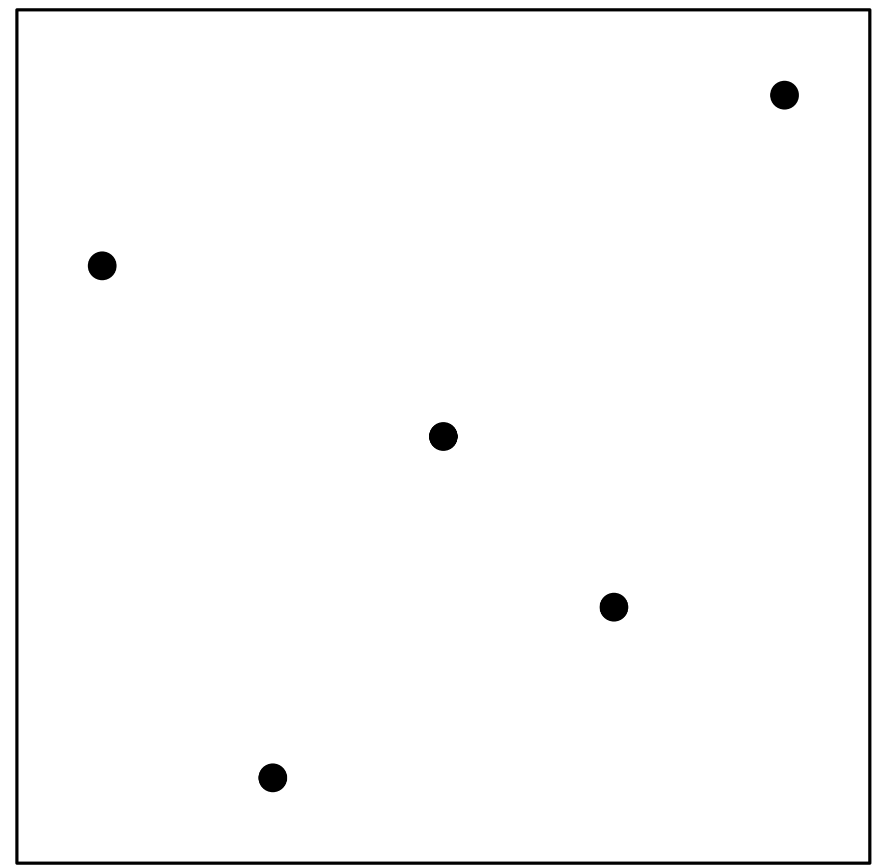

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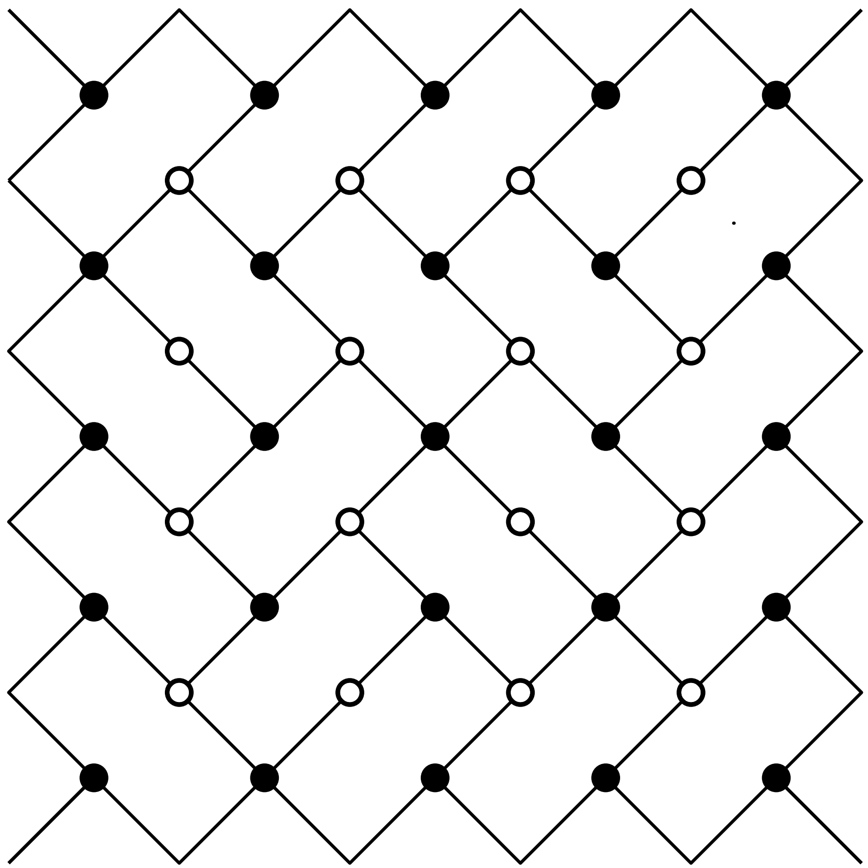
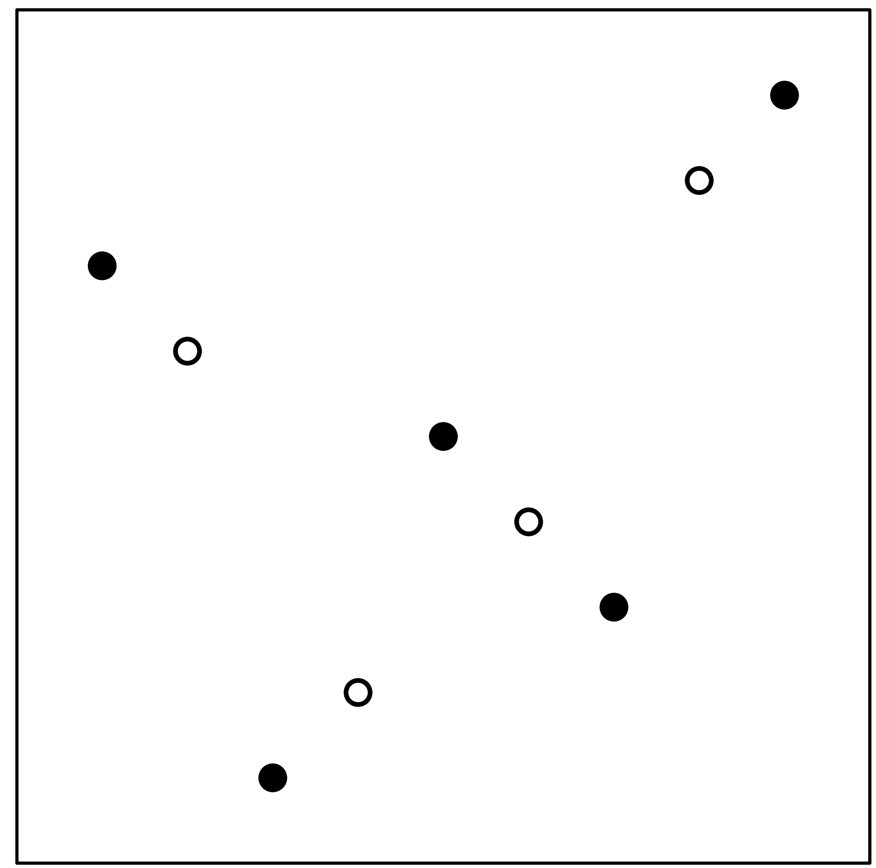
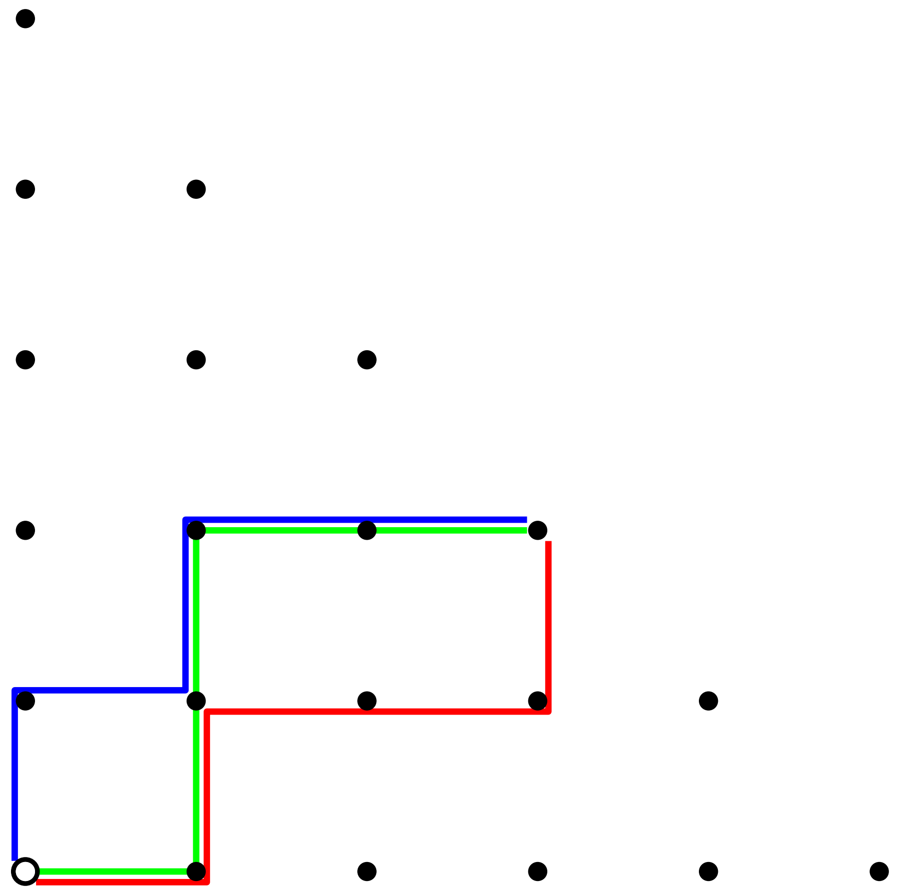


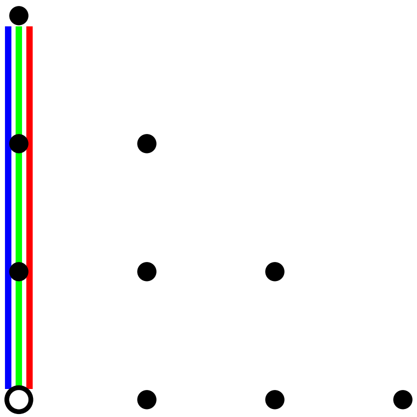


... in this case, the S-AMS is the matrix of a $(2-14-3, 3-41-2)$ permutation. The combined matrix is the matrix of a complete Baxter permutation.

An open question:

Triples of non-crossing monotone paths from $(0, 0)$ to a point on $x + y = n - 1$





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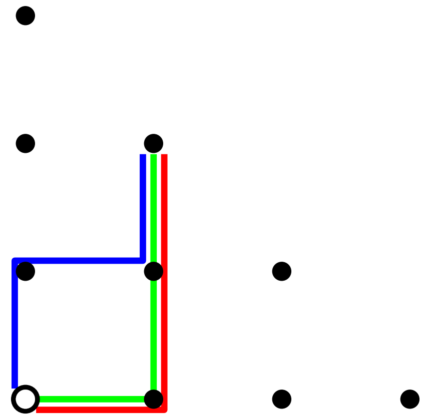

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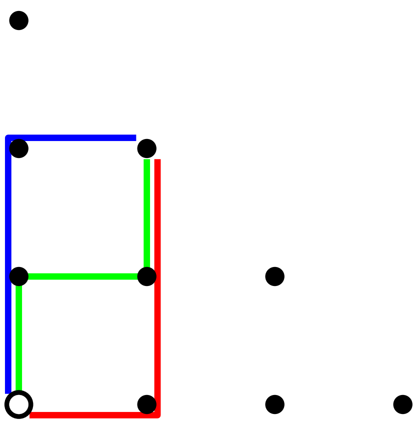

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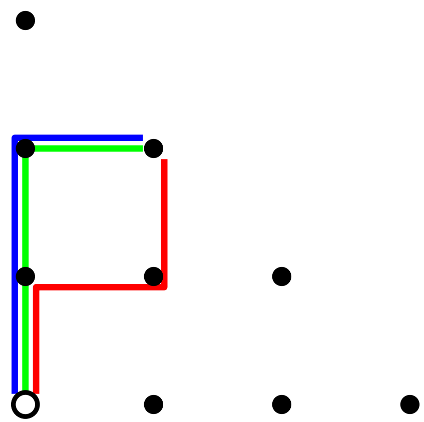

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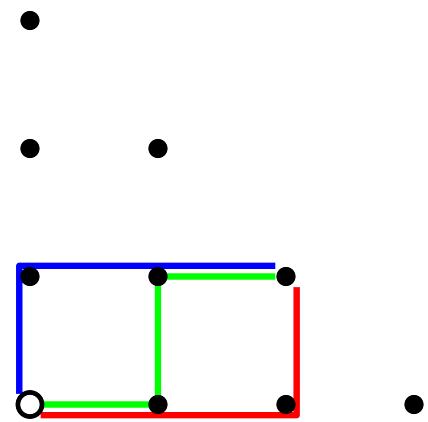

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