Relations between the shape of a permutation and the shape of the base poset derived from the corresponding Lehmer codes

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1. Lehmer Codes and Weak Bruhat Order

2. Denoncourt's Work

3. Relations Between ω and M_ω

4. Relations Between $\Delta(\omega)$ and M_ω

1. Lehmer Codes and Weak Bruhat Order

$\omega = \omega(1)\omega(2)\cdots\omega(n) \in S_n$

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Lehmer Code

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$$\mathbf{c}(\omega) = (c_1(\omega), c_2(\omega), \cdots, c_n(\omega))$$

Lehmer Code

$\Longleftrightarrow$

$c_1(\omega)$:
the number of i (≥ 1) such that $\omega(1) > \omega(i)$

$c_2(\omega)$:
the number of i (≥ 2) such that $\omega(2) > \omega(i)$

$$\vdots$$

$c_n(\omega)$:
the number of i $(\geq n)$ such that $\omega(n) > \omega(i)$

Example

$\omega = 423615$

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Endow a product order on Lehmer Codes

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For $\omega, \tau \in S_n \quad \omega \leq \tau \Longleftrightarrow \mathrm{Inv}(\omega) \subset \mathrm{Inv}(\tau)$

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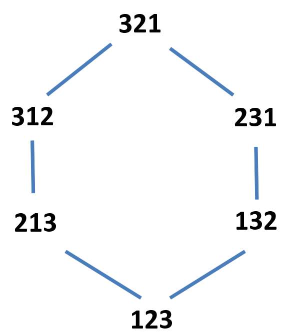

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$\Lambda_\omega := [e, \omega]$ Interval in Weak Bruhat Order

Theorem (Denoncourt 2011)
1. $\mathbf{c}$ *is an order preserving map*

2. $\mathbf{c}(\Lambda_\omega)$ *is a distributive lattice in* $\mathbb{N}^n$

Definition (Denoncourt 2011)
For $\omega \in S_n$, i *with* $c_i(\omega) \neq 0$ *and* $1 \leq x \leq c_i(\omega)$, *define* $m_{i,x}(\omega) \in \mathbb{N}^n$ *s.t.*

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4. *For* $j > i$ *with* $\omega(i) < \omega(j)$, j*-th coordinate of* $m_{i,x}(\omega)$ *is*
$$\max\{0, x - c_{i,j}(x)\}$$
where $c_{i,j}$ *is the number of* $i \leq k \leq j$ *s.t.*
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The join-irreducibles of $\mathbf{c}(\Lambda_\omega)$ *are*

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$$m_{1,4} = (4, 0, \mathbf{3}, 0, \ , 0, 0) \qquad 53\mathbf{7}1642$$

$$m_{1,4} = (4, 0, 3, 0, \mathbf{2}, 0, 0) \qquad 5371\mathbf{6}42$$

$$m_{1,4} = (4,0,3,0,2,0,0) \quad m_{1,3} = (3,0,2,0,1,0,0)$$

$$m_{1,2} = (2,0,1,0,0,0,0) \quad m_{1,1} = (1,0,0,0,0,0,0)$$

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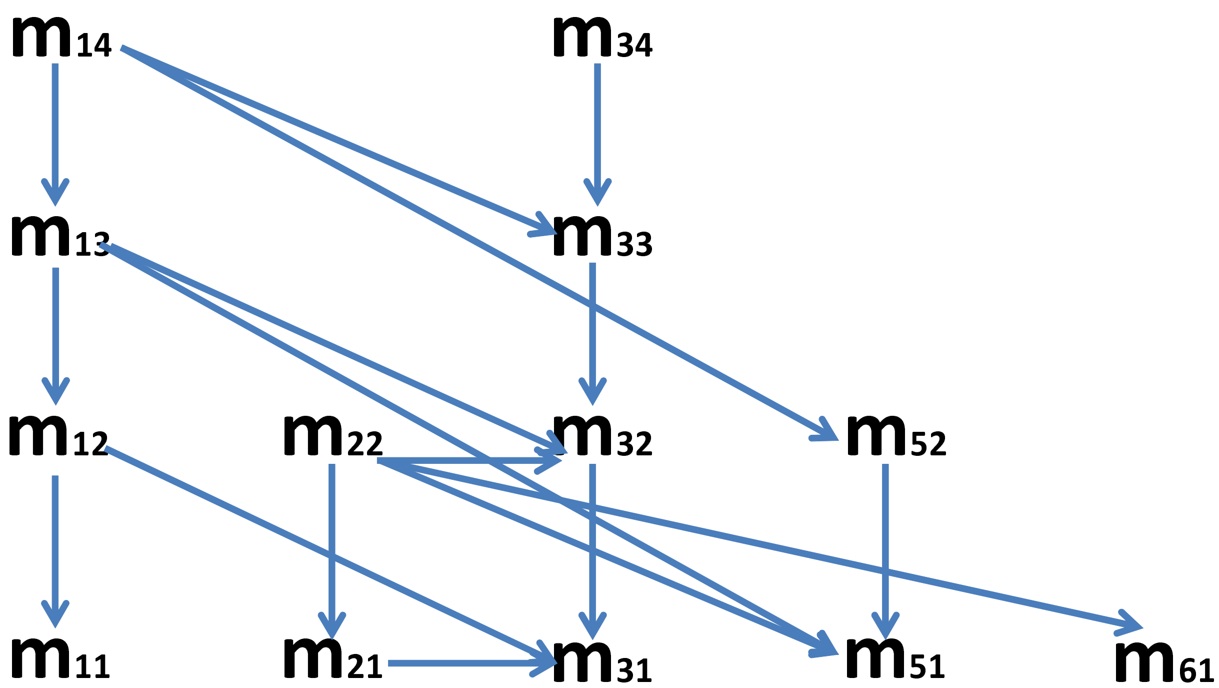

m_{14}
m_{34}
m_{13}
m_{33}
m_{12}
m_{22}
m_{32}
m_{52}
m_{11}
m_{21}
m_{31}
m_{51}
m_{61}

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$\Longrightarrow$ *M_ω is disjoint union of chains.*

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Definition
P, Q Posets
P is called to be Q free poset iff
there are no subposets $R \subset P$ s.t. $R \simeq Q$.

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A poset P is B_2-free iff P has no 4 elements isomorphic to Boolean algebra of rank 2.

Theorem (T. 2011)
ω is a 3412-3421-avoiding permutation
$\iff M_\omega$ *is a B_2 free poset.*

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A relation between the shape of a permutation and the shape of the base poset derived from the Lehmer codes, arXiv:1111.3094.

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Example
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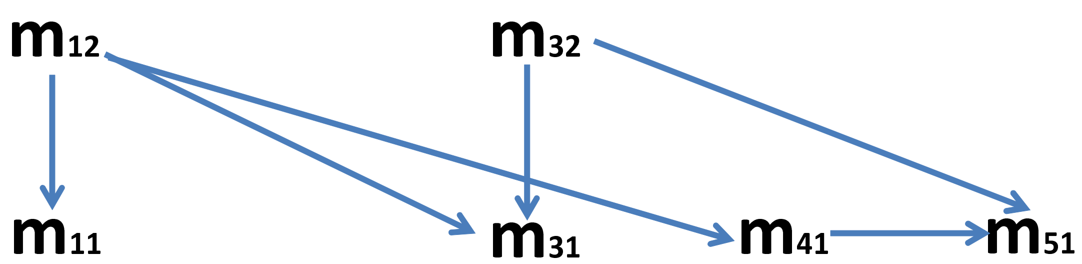

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Definition

For $\omega \in S_n$ *Consider a graph* $G(\omega)$ *s.t.*

1. *vertex set* $\{i \mid \exists j > i, s.t. \omega(i) > \omega(j)\}$
2. *Connect* i *and* j $(i < j)$ *if* $\exists k > j$ *s.t.* $st(\omega(i)\omega(j)\omega(k)) = 231$.

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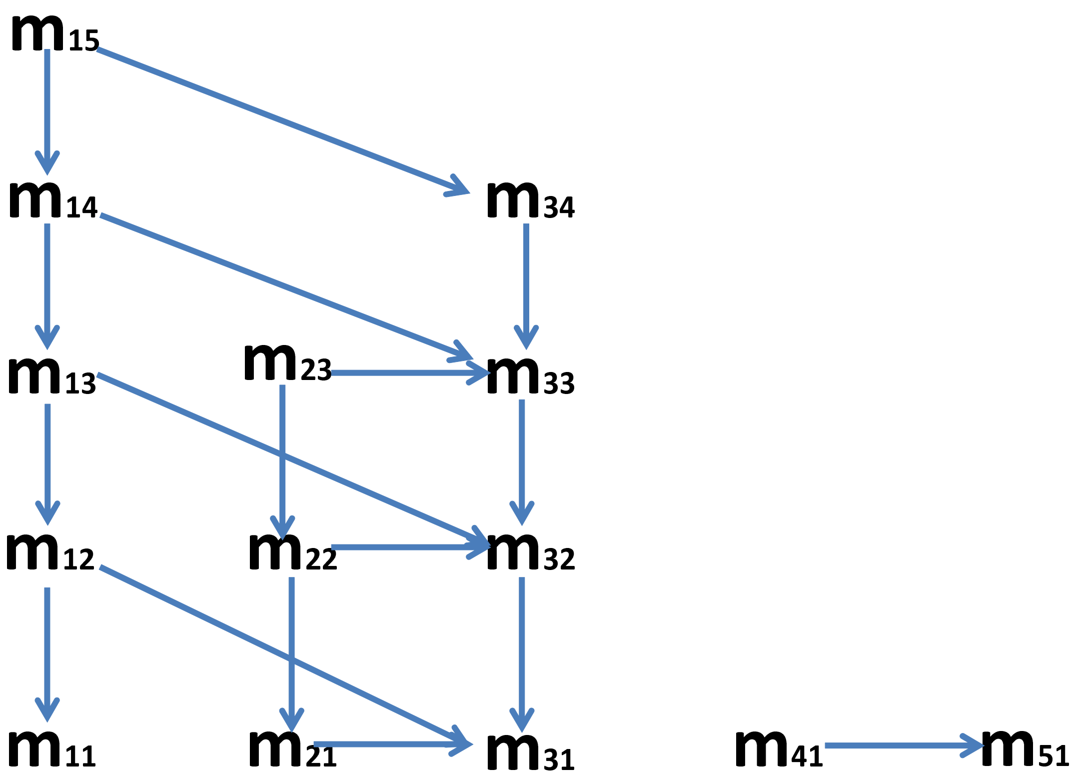

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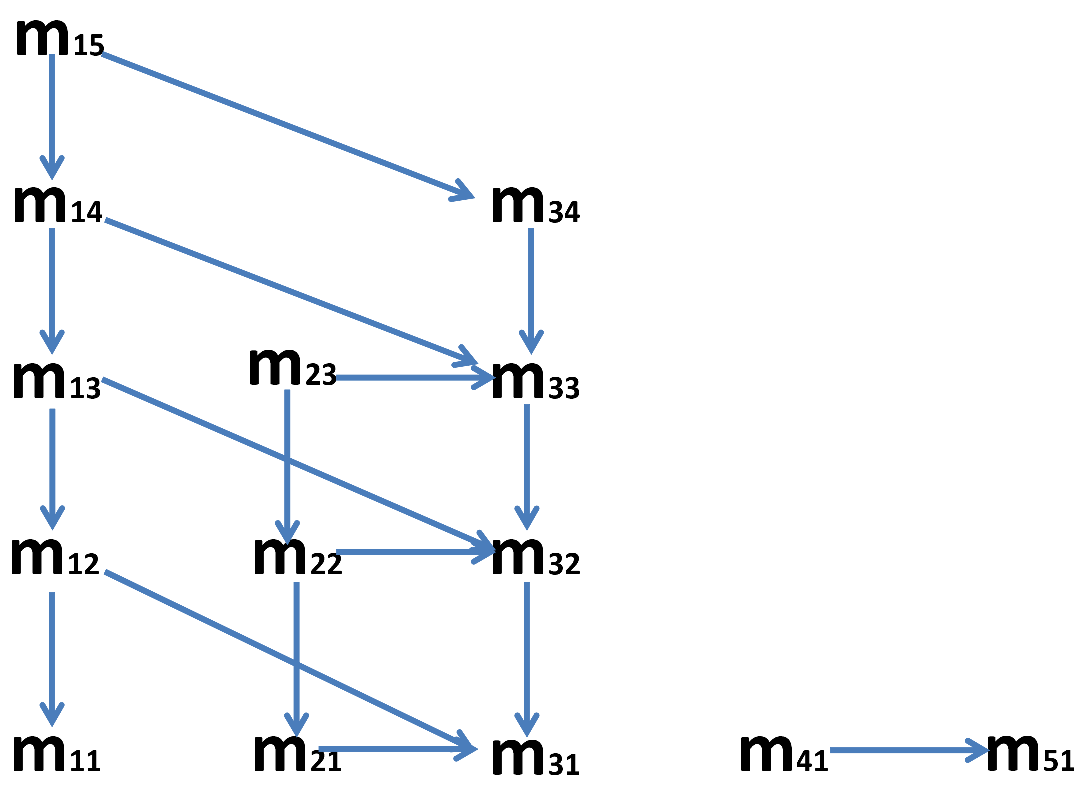

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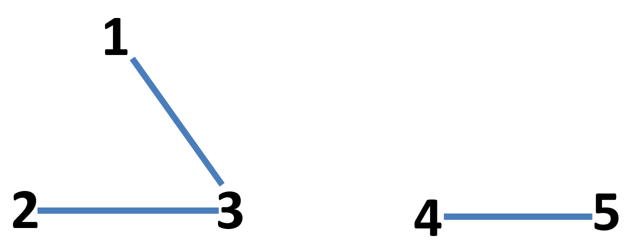
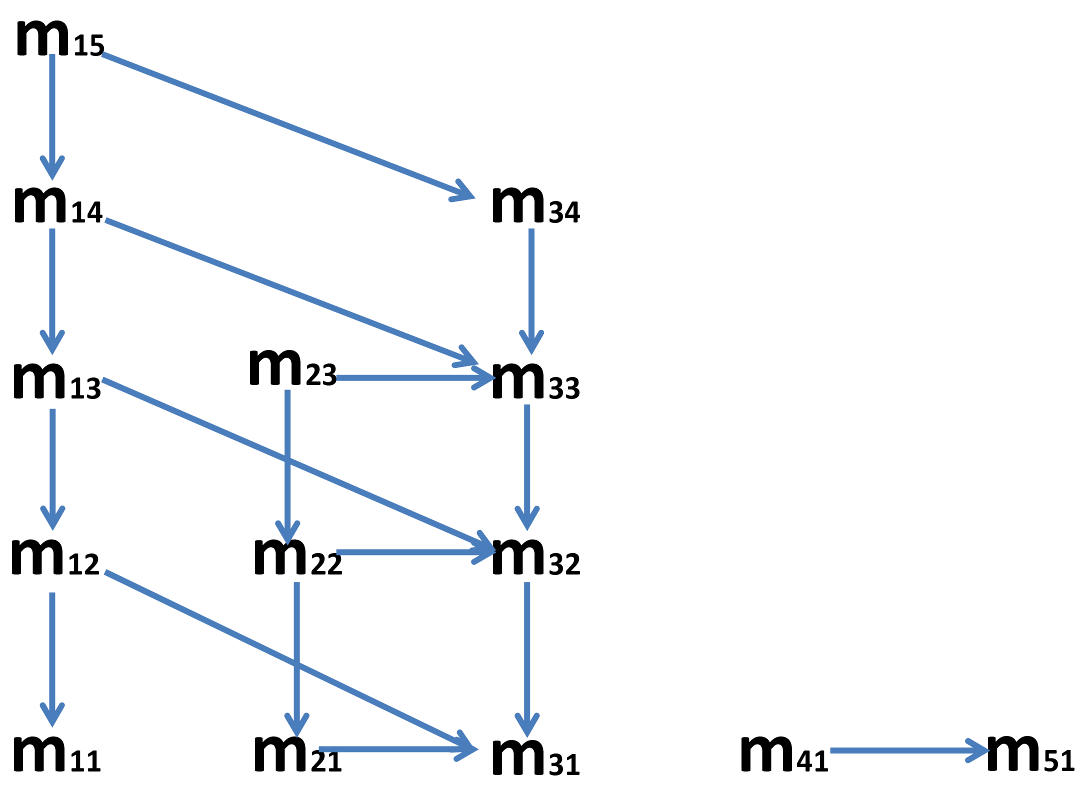
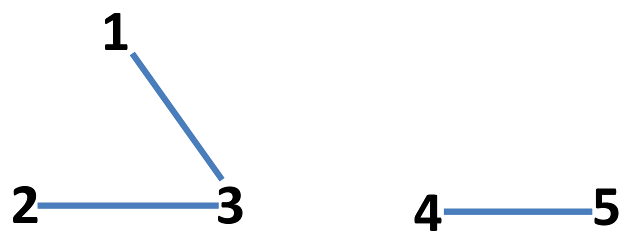


1
2
3
4
5





Theorem

The number of components of M_ω *equals to that of* $G(\omega)$

Problem

1. When M_ω becomes tree?

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$(i,j) \leq (k,l) \iff i \leq k < l \leq j$

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Lemma
$\sharp\Delta(\omega) = \sharp\mathrm{Inv}(\omega)$

Remark (Motivation of $\Delta(\omega)$)
S_n is the Weyl group of type A_{n-1}

$\{\alpha_1, \alpha_2, \cdots, \alpha_{n-1}\}$ *fundamental roots*

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$\Delta^+ (\Delta^-)$ *positive roots (negative roots)*

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It is known that

$$\tilde{\Delta}(\omega) \simeq \Delta(\omega) \text{ as a poset}$$

We define a map

$$\begin{aligned} \Phi_\omega : \quad & M_\omega \to \Delta(\omega) \\ \Phi_\omega(m_{i,x}) & := (i, j_x) \end{aligned}$$

where $(i, j_1), (i, j_2), \cdots (i, j_x), \cdots) \in \mathrm{Inv}(\omega)$ with $j_1 < j_2 < \cdots < j_x < \cdots$

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Theorem (T)
Φ_ω *is a poset isomorphism if and only if* ω *is a 321-avoiding permutation.*

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Remark
A 321-avoiding permutation is a fully commutative element.

Problem

Are there natural generalizations of this fact to Weyl groups ?