Using Algebraic Geometry for Solving Differential Equations

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MAX PLANCK INSTITUTE
FOR MATHEMATICS IN THE SCIENCES

Overview

Motivation

Assume that you want to solve the following differential equations, how would you proceed?

$$20y^3 + y^2 + 20y\,y' - 25y'^2 + y' = 0$$

$$\{-8y'^3 + 27y = 0, z^5 - y^3 = 0, -5z^4z' + 3y^2y' = 0\}$$

$$\sqrt{x}\,y'' - y^{3/2} = 0$$

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It highly depends on the solution space you are working with.

Differential algebra

Let K be a field of characteristic zero, R be a differential ring and

$$R\{y_1, \ldots, y_n\} = R[y_1, y_1', y_1'', \ldots, y_n, y_n', y_n'', \ldots]$$

be the ring of differential polynomials in the differential indeterminates $y_1, \ldots, y_n$ with coefficients in R.

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be the ring of differential polynomials in the differential indeterminates $y_1, \dots, y_n$ with coefficients in R. Given $F \in R\{y_1, \dots, y_n\}$, we define the order of F in y_i as the highest occurring derivative of y_i in F.
We define for a finite set of differential polynomials

$$\mathcal{S} = \{F_1 = 0, \dots, F_M = 0\} \subset R\{y_1, \dots, y_n\} \tag{1}$$

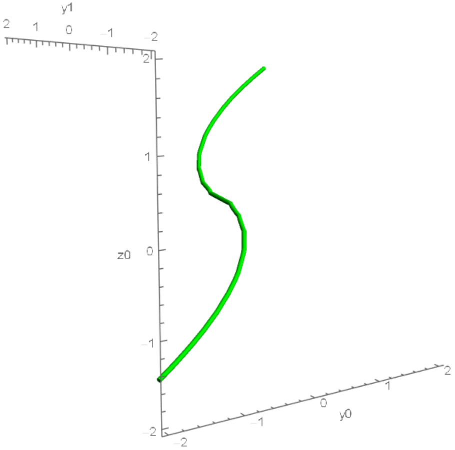
the corresponding algebraic set as

$$\mathbb{V}_K(\mathcal{S}) = \{a \in K^{m+n} \mid F_1(a) = \cdots = F_M(a) = 0\}$$

where $m = m_1 + \cdots + m_n$ and m_i is the order of $\mathcal{S}$ in y_i and $K \subset R$ is a field.

The corresponding algebraic set $\mathbb{V}_{\mathbb{R}}(\mathcal{S})$ of the following system defines a space curve.

$$\mathcal{S} = \{-8y'^3 + 27y = 0, z^5 - y^3 = 0, -5z^4z' + 3y^2y' = 0\}.$$



Algebraic structures

$K(x)$ ... rational functions

$K[[x]]$ ... formal power series

$K((x)) = K[[x]][x^{-1}]$ ... formal Laurent series

$K\{\{x\}\}$ ... algebraic functions, i.e. $y(x)$ such that $Q(x, y(x)) = 0$ for a $Q \in K[x, y] \setminus K[x]$

$K\langle\langle x\rangle\rangle = \bigcup_{n\in\mathbb{N}^*} K((x^{1/n}))$ ... formal Puiseux series (expanded around zero)

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$$K(x) \subset K\{\{x\}\} \subset K\langle\langle x\rangle\rangle.$$

Let $y(x) \in K((x^{1/n}))$ be such that there is no $m \mid n$ and $y(x) \in K((x^{1/m}))$. Then n is called the ramification index of $y(x)$.

Algebraic structures

$\sqrt{x} + \frac{1}{\sqrt[3]{x^5}}$ ... algebraic function (seen as formal Puiseux series: ramification index 6 and order $-3/5$)

$\sum_{i \geq 1} \frac{1}{i} x^{i/6}$ ... formal Puiseux series with ramification index 6, order $1/6$

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Puiseux's theorem

Let $F \in \mathbb{C}(x)[y]$. Then every solution $y(x)$ of $F(y) = 0$ is a formal Puiseux series and convergent.

Moreover, all solutions can be computed (via the Newton-polygon method).

Main goal

Given: $F \in \mathbb{Q}[y, y']$ (or $\mathcal{S} \subset \mathbb{Q}\{y_1, \ldots, y_n\}$ of dimension one).

Goal: Find the rational / algebraic / formal Puiseux series solutions of $F(y, y') = 0$ (or $\mathcal{S}$) and analyze the following properties:

- Existence and uniqueness of solutions.
- Convergence.
- Necessary field extensions.

First order autonomous AODEs

Consider first order algebraic ordinary differential equations (AODEs) with constant coefficients, i.e.

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For a non-constant solution $y(x)$ of $F(y, y') = 0$, the pair $(y(t), y'(t))$, or $(y(t^n), \frac{d}{dt}y(t^n))$ in case of formal Puiseux series, is a parametrization of the corresponding plane curve $\mathbb{V}_{\mathbb{C}}(F)$, called a solution parametrization.

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Necessary condition

Let $K \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$. A necessary condition on the existence of a non-constant rational solution in $K(x)$ (or formal Puiseux series solution in $K\langle\langle x \rangle\rangle$) is that $\mathbb{V}_K(F)$ is not finite and admits a rational (local) parametrization.

Rational solutions

It is well-known that a curve $\mathbb{V}_{\mathbb{C}}(F)$ admits a (bi-)rational parametrization $P(t)$ iff it is of genus zero. In the affirmative case, we can compute $P(t) \in K(t)^2$ in an optimal field $K \subset \mathbb{C}$.

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Theorem [R. Feng, X.S. Gao; 2004]

Assume that $\mathbb{V}_{\mathbb{C}}(F)$ has a birational parametrization $P(t) = (p(t), q(t)) \in K(t)^2$. Then $F(y, y') = 0$ has a (non-constant) rational solution iff

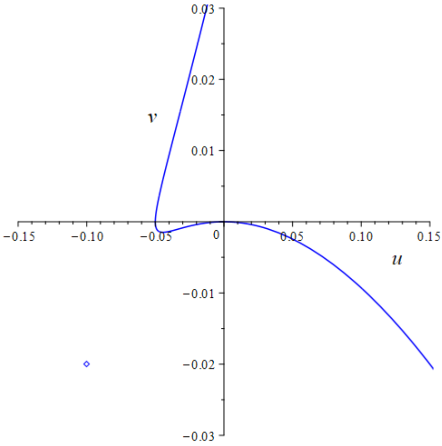
$$q(t) = a\,p'(t) \;\text{ or }\; q(t) = a\,p'(t) \cdot (t - b)^2$$

for some $a, b \in K$, $a \neq 0$. In the affirmative case, $p(a\,x + c)$ or $p(b - \frac{1}{a\,x+c})$ defines all rational solutions and they are in $K(x)$.

Example

Consider

$$F(y, y') = 20y^3 + y^2 + 20y\,y' - 25y'^2 + y' = 0.$$



The corresponding curve $\mathbb{V}_{\mathbb{C}}(F)$ has the rational parametrization

$$(p(t), q(t)) = \left(\frac{(1 + 6\,t)\,t}{(t+1)^2}, -\frac{(1 + 11\,t)\,t^2}{(t+1)^3} \right) \in \mathbb{Q}(t)^2.$$

Since $q(t) = -p'(t)\,t^2$, we obtain the solutions

$$y(x) = p(\tfrac{1}{x-c}) = \frac{x - c + 6}{(x - c + 1)^2} \in \mathbb{Q}(c, x).$$

Places

Local parametrizations of the plane curve $\mathbb{V}_{\mathbb{C}}(F)$ exist around every curve point $(y_0, p_0) \in \mathbb{C}_\infty^2$. Let $P(t), Q(t) \in \mathbb{C}((t))^2$ be such local parametrizations. The relation

$$P(t) \sim Q(t) \text{ iff } P(s(t)) = Q(t) \text{ for some } s(t) \in \mathbb{C}[[t]], \mathrm{ord}_t(s(t)) = 1$$

is an equivalence relation such that $P(0) = Q(0)$. The equivalence classes of irreducible local parametrizations are called places, centered at the common curve-point $P(0)$. Places can be seen as the algebraic counterpart to branches.

Necessary and sufficient condition

Let $[(p(t), q(t))]$ be a place containing a solution parametrization $(y(t^n), \frac{d}{dt}y(t^n)) \in K((t))^2$. Then

$$m = \mathrm{ord}_t(p'(t)) - \mathrm{ord}_t(q(t)) + 1 > 0. \qquad (3)$$

Note that (3) is independent of the representative of the place.

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Key Theorem [J. Cano, J.R. Sendra, F.; 2019]

Let $\mathcal{P}$ be a place of $\mathbb{V}_{\mathbb{C}}(F)$. Then $\mathcal{P}$ contains a solution parametrization if and only if (3) holds for $m \in \mathbb{N}^*$.
In the affirmative case, there are exactly m solution parametrizations in $\mathcal{P}$.

Implicit function theorem

Implicit function theorem

Let $(y_0, p_0) \in \mathbb{V}_K(F)$ be a finite point such that $\frac{\partial F}{\partial y'}(y_0, p_0) \neq 0$ and $p_0 \neq 0$. Then there is a unique formal power series solution $y(x) \in K[[x]]$ of $F(y, y') = 0$.

If the implicit function theorem is applicable, then (3) is fulfilled with $m = 1$.

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If the implicit function theorem is applicable, then (3) is fulfilled with $m = 1$.

We call the exceptional curve points

$$\mathbb{V}_K(F) \cap \left(\mathbb{V}_K(\frac{\partial F}{\partial y'}) \cup \mathbb{V}_K(y') \cup \mathbb{V}_K(\mathrm{lc}_{y'}(F)) \right)$$

where the implicit function theorem does not hold critical points.

Algorithm

Algorithm arising from the proof of Theorem 1

Given $F \in \mathbb{Q}[y, y']$ irreducible.

1) Compute a generic power series solution (by the implicit function theorem).
2) Compute the critical points $(y_0, p_0) \in \mathbb{V}_{\mathbb{C}}(F)$.
3) For every critical point compute a representative $(p(t), q(t))$ of every place at (y_0, p_0) and determine m.
4) Take $s(t) = s_1 t + s_2 t^2 + \cdots$ with s_i undetermined and compute them from

$$p'(s(t))\, s'(t) = m\, t^{m-1}\, q(s(t)). \tag{4}$$

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Equation (4) is called the associated differential equation and can be solved for example with the Newton-polygon method for differential equations. Note that in every step we can ensure convergence.

Results

Theorem 1 (Convergence)

Let $F \in \mathbb{Q}[y, y']^*$. Then all formal Puiseux series solutions $y(x)$ of $F(y, y') = 0$, expanded around a finite point or infinity, are convergent.

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Theorem 2 (Existence, Uniqueness)

Let $N = 2(\deg_{y'}(F) - 1)\deg_y(F) + 1$ and $\varphi(x) \in \mathbb{C}[x^{1/n}, x^{-1/n}]$ be a truncated solution of $F(y, y') = 0$, where the first N terms are computed. Then there exists exactly one $y(x) \in \mathbb{C}\langle\langle x \rangle\rangle$ with $F(y, y') = 0$ extending $\varphi(x)$.

Puiseux series solutions with real / rational coefficients

Places can be represented by a local parametrization of the form $(\alpha\, t^n, q(t)) \in K((t))^2$ with coefficients in an optimal field $K \subset \mathbb{C}$, called rational Puiseux parametrizations.

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Necessary condition

Let $K \in \{\mathbb{Q}, \mathbb{R}\}$. If $y(x) \in K((x^n))$ is a solution of $F(y, y') = 0$, then the rational Puiseux parametrization $(p(t), q(t))$ of the place $[(y(t^n), \frac{d}{dt}(y(t^n)))]$ has coefficients in K.

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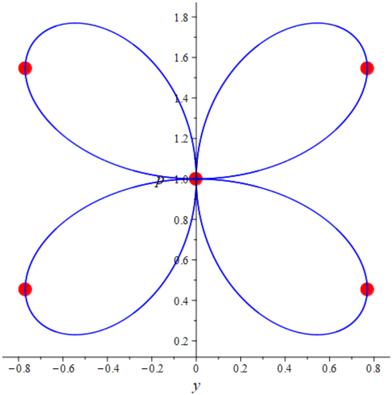
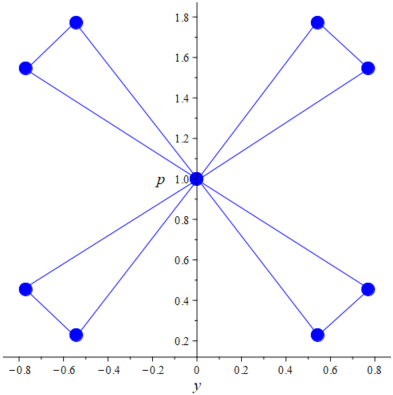
Recall that every non-constant solution is of the form $y(x) = p(s(x^{1/n}))$. If $(p(t) = \alpha\, t^n, q(t)) \in K((t))^2$ is a rational Puiseux parametrization, then $y(x) \in K(s_1)((x^{1/n}))$ with $s_1^n = \frac{n\, q_1}{p_1} \in K$. Hence, after computing s_1, we know whether $y(x)$ has coefficients in $K \in \{\mathbb{Q}, \mathbb{R}\}$.

Example

Consider $F(y, y') = ((y' - 1)^2 + y^2)^3 - 4(y' - 1)^2 y^2 = 0.$





The generic power series solution is given as $y_0 + p_0\,x + \mathcal{O}(x^2)$ with $(y_0, p_0) \in \mathbb{C}^2$ and $F(y_0, p_0) = 0$ or, in case we are interested in solutions with real / rational coefficients, by the topological graph of $\mathbb{V}_{\mathbb{C}}(F)$.

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The critical curve-points are

$$\begin{aligned} \mathcal{B} \quad = \quad & \{(0,1)\} \cup \left\{(\alpha, 0) \,|\, \alpha^6 + 3\alpha^4 - \alpha^2 + 1 = 0\right\} \cup \\ & \left\{ \left(\tfrac{4\beta}{9}, \gamma\right) \,|\, \beta^2 = 3, 27\gamma^2 - 54\gamma + 19 = 0 \right\} \end{aligned}$$

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- At $\mathbf{c}_1 = (0,1)$ there are 4 places defined by

$$\begin{array}{ll} (p(t),q(t)) = (2t^2, 1+2t-\frac{3t^2}{2}+\mathcal{O}(t^3)) & \text{suitable with } m=2 \\ (-2t^2, 1-2t-\frac{3t^2}{2}+\mathcal{O}(t^3)) & \text{suitable with } m=2 \\ (t, 1+\frac{t^2}{2}+\frac{3t^4}{16}+\mathcal{O}(t^6)) & \text{suitable with } m=1 \\ (t, 1-\frac{t^2}{2}-\frac{3t^4}{16}+\mathcal{O}(t^6)) & \text{suitable with } m=1 \end{array}$$

For $(p(t), q(t))$ the associated differential equation is

$$s(t)\, s'(t) = t \left(1 + 2s(t) - \frac{3s(t)^2}{2}\right)$$

with the solutions

$$s_1(t) = \frac{t}{\sqrt{2}} + \frac{t^2}{3} + \frac{\sqrt{2}t^3}{36} + \mathcal{O}(t^4),$$
$$s_2(t) = \frac{-t}{\sqrt{2}} + \frac{t^2}{3} - \frac{\sqrt{2}t^3}{36} + \mathcal{O}(t^4).$$

By considering all places at $\mathbf{c}_1$ we obtain

$$\left\{ \begin{array}{l} p(s_1(x^{1/2})) = x + \frac{2\sqrt{2}x^{3/2}}{3} + \frac{x^2}{3} + \mathcal{O}(x^{5/2}), \\ p(s_2(x^{1/2})) = x - \frac{2\sqrt{2}x^{3/2}}{3} + \frac{x^2}{3} + \mathcal{O}(x^{5/2}), \\ x + \frac{2\sqrt{2}ix^{3/2}}{3} - \frac{x^2}{3} + \mathcal{O}(x^{5/2}),\ x - \frac{2\sqrt{2}ix^{3/2}}{3} - \frac{x^2}{3} + \mathcal{O}(x^{5/2}), \\ x + \frac{x^3}{6} + \frac{17x^5}{240} + \mathcal{O}(x^6),\ x - \frac{x^3}{6} + \frac{17x^5}{240} + \mathcal{O}(x^6) \end{array} \right\}$$

- For $\mathbf{c}_\alpha = (\alpha, 0)$ we obtain the rational Puiseux parametrizations

$$\left(\alpha + t, \left(\frac{11}{19}\alpha^5 + \frac{36}{19}\alpha^3 + \frac{4}{19}\alpha\right) t + \mathcal{O}(t^2)\right),$$

which are not suitable.

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  which are not suitable.
- Let $\mathbf{c}_{\beta,\gamma} = \left(\frac{4\beta}{9}, \gamma\right)$, where $\beta^2 = 3$, and $27\gamma^2 - 54\gamma + 19 = 0$. Then the places represented by

$$\left(\frac{4\beta}{9} - t^2, \gamma - \frac{\sqrt[4]{27}}{3} t + \mathcal{O}(t^2)\right)$$

  are suitable with $m = 2$ leading to eight Puiseux series solutions given by

$$\frac{4\beta}{9} + \gamma x \pm \frac{2\sqrt{-\gamma\,\beta}}{3\sqrt{3}} x^{3/2} + \left(\frac{5\gamma}{32} - \frac{143}{864}\right) \beta x^2 + \mathcal{O}(x^{5/2}).$$

  where four are real $((\beta, \gamma) \in \{(-\sqrt{3}, 1 + \frac{2\sqrt{6}}{9}), (-\sqrt{3}, 1 - \frac{2\sqrt{6}}{9})\})$.

Algebraic solutions

Theorem 3

Let $F \in \mathbb{Q}[y, y']$ be irreducible with a non-constant algebraic solution $y(x) \in \mathbb{C}\{\{x\}\}$. Then all non-constant formal Puiseux series solutions of $F(y, y') = 0$ are algebraic over $\mathbb{C}(x)$.

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Moreover, if $Q(x, y) \in \mathbb{C}[x, y]$ is the minimal polynomial of $y(x)$, then all non-constant formal Puiseux series solutions are given by $Q(x + c, y)$.

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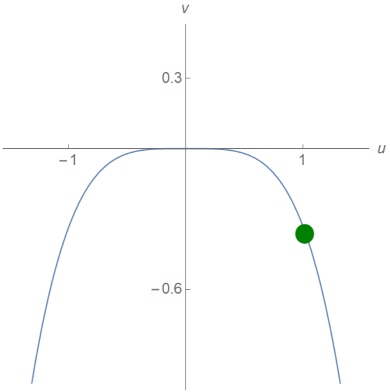
Based on this theorem, we just compute one (non-constant) formal Puiseux series solution and check whether it is algebraic.

Example

Consider $F(y, y') = y^4 + 3y' = 0$ and the initial value $(1, -1/3) \in \mathbb{V}_{\mathbb{Q}}(F)$.



By the implicit function theorem, we obtain the formal power series solution

$$y(x) = 1 - \frac{x}{3} + \frac{2x^2}{9} - \frac{14x^3}{81} + \mathcal{O}(x^4)$$

with the minimal polynomial $Q(x, y) = x\,y^3 - 1$. All solutions, namely $z(x) = \frac{\zeta}{\sqrt[3]{x+c}}$ for $\zeta^3 = 1$, are then determined by $Q(x + c, y)$.

Simple systems

By using algebraic and differential reduction (here we use the Thomas decomposition), differential systems $\mathcal{S} \subset \mathbb{Q}\{y_1, \ldots, y_n\}$ can be decomposed into a finite collection of simple subsystems $(\mathcal{S}_k, \mathcal{U}_k)$ representing a set of equalities

$$\mathcal{S} = \{G_1 = 0, \ldots, G_M = 0\} \subset \mathbb{Q}\{y_1, \ldots, y_n\}$$

and inequalities

$$\mathcal{U} = \{U_1 \neq 0, \ldots, U_N \neq 0\} \subset \mathbb{Q}\{y_1, \ldots, y_n\}.$$

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and inequalities

$$\mathcal{U} = \{U_1 \neq 0, \ldots, U_N \neq 0\} \subset \mathbb{Q}\{y_1, \ldots, y_n\}.$$

The simple subsystems have as algebraic equations the same zeros as the given system. In particular, the decompositions has the same solution set, i.e.

$$\mathrm{Sol}_{\mathbb{C}\langle\langle x \rangle\rangle}(\mathcal{S}) = \dot{\bigcup} \, \mathrm{Sol}_{\mathbb{C}\langle\langle x \rangle\rangle}(\mathcal{S}_k, \mathcal{U}_k).$$

Simple systems

Simple systems have in particular the following properties:

- $G_1, \ldots, G_M, U_1, \ldots, U_N$ have pairwise distinct leading variables (they are in triangular form);
- $G_1, \ldots, G_M$ are pairwise differentially reduced and $U_1, \ldots, U_N$ are reduced with respect to the G_i's.

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Systems $\mathcal{S}$, where $\mathbb{V}_{\mathbb{C}}(\mathcal{S})$ is of dimension one, can be decomposed into simple subsystems leading to constant solution components and to simple subsystems of the form

$$\left\{ \begin{array}{rcll} G_1(y_1, y_1') & = & 0, & \\ G_s(y_1, y_1', y_2, \ldots, y_s) & = & 0, & s \in \{2, \ldots, n\}, \\ U(y_1) & \neq & 0, & \end{array} \right. \tag{I}$$

where the leading variables (w.r.t. the ordering $y_1 < y_1' < \cdots < y_n < y_n' < \cdots$) are $\operatorname{lv}(G_1) = y_1', \operatorname{lv}(G_s) = y_s$ and $U \in \mathbb{Q}[y_1] \setminus \{0\}$.

Results

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Theorem 4 (Convergence)

Let $\mathcal{S} \subset \mathbb{Q}\{y_1, \ldots, y_n\}$ be such that $\mathbb{V}_{\mathbb{C}}(\mathcal{S})$ is of dimension one. Then every component of a formal Puiseux series solution, expanded around a finite point or at infinity, is convergent or can be chosen arbitrarily.

Algebraic solutions

Computations with Puiseux series vectors are an algorithmically intricate problem. For algebraic solutions, however, computations simplify.

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Computations with Puiseux series vectors are an algorithmically intricate problem. For algebraic solutions, however, computations simplify. For a system of the type (I)

$$\begin{cases} G_1(y_1, y_1') & = & 0, & \\ G_s(y_1, y_1', y_2, \dots, y_s) & = & 0, & s \in \{2, \dots, n\}, \\ U(y_1) & \neq & 0, & \end{cases}$$

and a polynomial relation $P_1(x, y_1) = 0$ with $P_1 \in \mathbb{C}[x, y_1]$ and $\mathrm{lv}(P_1) = y_1$, we can again compute a decomposition into finitely many algebraic simple subsystems of the type

$$\{ \quad G_s(x, y_1, \dots, y_s) \;=\; 0, \qquad s \in \{1, \dots, n\}, \tag{II}$$

where $G_s \in K[x, y_1, \dots, y_s]$ with $\mathrm{lv}(G_s) = y_s$.

Algebraic solutions

Combining this observation with Theorem 3, we get:

Corollary

Let $(\mathcal{S}, \mathcal{U})$ be a simple system of the form (I) such that $G_1 \in \mathbb{C}[y_1, y_1']$ is irreducible with an algebraic solution

$$y_1(x) \in \mathbb{C}\langle\langle x \rangle\rangle \setminus \mathbb{C}.$$

Then all formal Puiseux series solutions of $(\mathcal{S}, \mathcal{U})$ are algebraic over $\mathbb{C}(x)$.

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The description of the algebraic solutions can be done either as

- algebraic simple subsystems of the form (II), namely $\{G_1(x, y_1) = 0, \ldots, G_n(x, y_1, \ldots, y_n) = 0\}$; or
- the minimal polynomials $\{Q_1(x, y_1) = 0, \ldots, Q_n(x, y_n) = 0\}$.

Summary

Algorithm arising from the proof of Theorem 4

Given $\mathcal{S} \subset \mathbb{Q}\{y_1, \ldots, y_n\}$ such that $\mathbb{V}_{\mathbb{C}}(\mathcal{S})$ is of dimension one.

1) Compute a Thomas decomposition of $\mathcal{S}$.
2) For every simple subsystem involving no derivatives, there are only constant solutions. For the simple subsystems $(\tilde{\mathcal{S}}, \tilde{\mathcal{U}})$ of the type (I), check whether $G_1(y_1, y_1')$ has an algebraic solution $y_1(x) \in \mathbb{C}\langle\langle x \rangle\rangle$.
3) In the affirmative case, compute a Thomas decomposition of $(\tilde{\mathcal{S}} \cup \{Q_1\}, \tilde{\mathcal{U}})$ where Q_1 is the minimal polynomial of $y_1(x)$.
4) The algebraic solutions are then given as algebraic simple systems (or can be expressed as a vector of minimal polynomials).

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The solutions of (5) and (6) are the same: Let $z_1(x) = \zeta\, x^{9/10},\ z_2(x) = -\zeta\, x^{9/10}$ with $\zeta^5 = 1$. Then $(y_i(x), z_i(x))$ is a solution (but neither $(y_1(x), z_2(x))$ nor $(y_2(x), z_1(x))$).

The algebraic simple system (6),

$$\{Q_1(x,y) = y^2 - x^3,\ G_2(x,y,z) = z^5 - x^3\,y\},$$

leads to the vector of minimal polynomials

$$\{Q_1(x,y) = y^2 - x^3,\ Q_2(x,z) = z^{10} - x^9\}. \tag{7}$$

The system (7), however, has $(y_1(x), z_2(x))$ and $(y_2(x), z_1(x))$ as solutions.

Further results

- A generalization to parametric differential equations $F \in K[y, y']$ where $K = \mathbb{Q}(a_1, \ldots, a_m)$ for some variables $a_1, \ldots, a_m$ of the above results is generically possible. For particular choices of a_i or when the a_i are functions in x, generalizations are not straight-forward anymore. In these cases, some results can still be recovered, but connections to classical unsolved questions appear (e.g. Hilbert's irreducibility problem).

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- By using methods from algebraic geometry, we present a procedure for transforming a given system of radical differential equations $\mathcal{S} \subset K_m(x)$ with a radical tower

$$K_0 = \mathbb{C}(y_1, \ldots, y_n) \subseteq K_1 \subseteq \cdots \subseteq K_m,$$

where $K_i = K_{i-1}(\delta_i)$, $\delta_i^{e_i} \in K_{i-1}$ for some $e_i \in \mathbb{N}$, can be transformed into a system of AODEs. Solutions are in one-to-one correspondence and standard-techniques are applicable to the transformed system.

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$$\begin{cases} s' = f(s), \\ y = p(s). \end{cases} \tag{8}$$

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In the case of input-output equations $F \in \mathbb{Q}(u, u')[y, y']$, finding a rational parametrization over $K(u, u')$ is a necessary but not a sufficient condition for finding a realization with $f, p \in K(s, u)$. We give an algorithm for deciding the existence of complex and real realizations, i.e. when $K \in \{\mathbb{C}, \mathbb{Q}\}$.

Open problems

- For $F(y, y') \in \mathbb{Q}[y, y']$, we expect similar results (existence, uniqueness and convergence) for more general type of solutions such as transseries.

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- etc.

References and Acknowledgments

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