Faster algorithms for symmetric polynomials

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based on joint works with
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MAX team seminar
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Part I: Computing critical points for invariant algebraic systems

Part II: Deciding the emptiness of invariant algebraic sets over real fields
Let $\mathbb{K}$ be a field and $f_1, \ldots, f_s$ be polynomials in $\mathbb{K}[x_1, \ldots, x_n]$. 

Solve $f_1 = \cdots = f_s = 0 \Rightarrow$ solution set in $\mathbb{K}^n$. 

- Exact/Symbolic methods: compute an algebraic data-structure which can be exploited to extract global information on its solutions in $\mathbb{K}$.
- Determines the dimension of the solution set in $\mathbb{K}^n$.

- Algebraic sets: the solution set of the ideal $I = \langle f_1, \ldots, f_s \rangle \subset \mathbb{K}[x_1, \ldots, x_n]$. 


Polynomial system solving

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Solve $f_1 = \cdots = f_s = 0 \iff$ solution set in $\overline{\mathbb{K}}^n$

- $\mathbb{K} = \mathbb{Q}$ and $\overline{\mathbb{K}} = \mathbb{R}$ or $\overline{\mathbb{K}} = \mathbb{C}$
- or $\mathbb{K} = \overline{\mathbb{K}}$ a prime field
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**Exact/Symbolic methods**: compute an algebraic data-structure which

- can be exploited to extract global information on its solutions in $\overline{\mathbb{K}}$
- determines the dimension of the solution set in $\overline{\mathbb{K}}^n$
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**Algebraic sets**: the solution set of the ideal $I = \langle f_1, \ldots, f_s \rangle \subset \mathbb{K}[x_1, \ldots, x_n]$
$W \subset \overline{K}^n$: a non-empty algebraic set given as the zero set of $I = \langle f_1, \ldots, f_s \rangle$
Dimension of algebraic sets

$W \subset \overline{\mathbb{K}}^n$: a non-empty algebraic set given as the zero set of $I = \langle f_1, \ldots, f_s \rangle$

$E_d$: a generic $d$-dimensional affine subspace of $\overline{\mathbb{K}}^n$
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**Geometric definition:**
\[ \dim(W) = \text{the maximum integer } d \text{ s.t. } 0 < \text{card}(W \cap E_d) < \infty \]
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**Folklore procedure :**
- compute a Gröbner basis of $I$
- deduce the Hilbert series $\frac{N(t)}{(1-t)^d}$ of $I$
Zero-dimensional algebraic sets

$W \subset \overline{\mathbb{K}}^n$: a non-empty algebraic set given as the zero set of $I = \langle f_1, \ldots, f_s \rangle$

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**Representation of zero-dimensional sets:** using univariate polynomials

\[ v(t) = 0, \quad x_i = v_i(t)/v_0(t) \quad (1 \leq i \leq n), \quad v_0(t) = \frac{\partial v(t)}{\partial t} \]
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Example: Consider $W \subset \overline{K}^2$ with $v(t) = t^2 - t$, $v_1 = t$, $v_2 = 3t - 1$, and $v_0 = 2t - 1$. Then

$x_1 = t/(2t - 1) \quad \text{and} \quad x_2 = (3t - 1)/(2t - 1)$
**Zero-dimensional algebraic sets**

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If $v(t) = 0$, then $t = 0$ or $t = 1$. Then $x_1 = \frac{0}{2 \cdot 0 - 1} = 0$ and $x_2 = \frac{3 \cdot 0 - 1}{2 \cdot 0 - 1} = 1$
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If $v(t) = 0$, then $t = 0$ or $t = 1$. Then $x_1 = \frac{1}{2t-1} = 1$ and $x_2 = \frac{3t-1}{2t-1} = 2$. 
Zero-dimensional algebraic sets

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If $v(t) = 0$, then $t = 0$ or $t = 1$. Thus $W = \{(0, 1), (1, 2)\}$. 
Zero-dimensional algebraic sets

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Normally, we have $(v, v_1, \ldots, v_n)$, then exploit information for $W$. 
Critical points

Minimize: \( \phi(x_1, x_2, x_3) = x_1 x_2 x_3 - 3(x_1 + x_2 + x_3) \) subject to

\[
g(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - 6 = 0.
\]
Minimize: $\phi(x_1, x_2, x_3) = x_1x_2x_3 - 3(x_1 + x_2 + x_3)$ subject to

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The minima satisfy

$$x_1^2 + x_2^2 + x_3^2 - 6 = 0$$

and

$$\text{rank} \begin{bmatrix} 2x_1 & 2x_2 & 2x_3 \\ x_2x_3 - 3 & x_1x_3 - 3 & x_1x_2 - 3 \end{bmatrix} < 2.$$
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- **optimization**

- **real algebraic geometry (decide the emptiness over the reals)**
  (will see in the 2nd half of the talk)

- ...
Let $\phi$ and $f = (f_1, \ldots, f_s)$ be polynomials in $\mathbb{K}[x_1, \ldots, x_n]$ with $s \leq n$ s.t.

**Assumption (A):** the Jacobian matrix of $f$ has **full rank** at any solution of $f$
Let $\phi$ and $f = (f_1, \ldots, f_s)$ be polynomials in $\mathbb{K}[x_1, \ldots, x_n]$ with $s \leq n$ s.t.

**Assumption (A):** the Jacobian matrix of $f$ has full rank at any solution of $f$

Then, $V(f)$ is smooth and $(n - s)$-equidimensional and the set of critical points of $\phi$ restricted to $V(f)$:

$$W(\phi, f) := \{ \mathbf{x} \in \overline{\mathbb{K}}^n : f(\mathbf{x}) = 0 \text{ and } \text{rank}(\text{jac}(f, \phi)(\mathbf{x})) < s + 1 \}$$
Let $\phi$ and $f = (f_1, \ldots, f_s)$ be polynomials in $\mathbb{K}[x_1, \ldots, x_n]$ with $s \leq n$ s.t.

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$$W(\phi, f) := \{ \mathbf{x} \in \mathbb{K}^n : f(\mathbf{x}) = 0 \text{ and (all } (s + 1) \text{ minors of } \text{jac}(f, \phi))\mathbf{(x)} = 0 \}$$

**Input**: symmetric polynomials $\phi$ and $(f_1, \ldots, f_s)$ in $\mathbb{K}[x_1, \ldots, x_n]$

**Condition**: $f = (f_1, \ldots, f_s)$ satisfies (A) and $W(\phi, f)$ is of zero-dimensional

**Output**: a representation for $W(\phi, f)$
Main result

Suppose $\varphi$ and $f = (f_1, \ldots, f_s)$ are symmetric polynomials in $K[x_1, \ldots, x_n]$

- the Jacobian matrix of $f$ has full rank at any solution of $f$
- the degrees of $f$ and $\varphi$ are at most $d$
- the set $W(\varphi, f) \subset \overline{K}^n$ is finite
Suppose $\phi$ and $f = (f_1, \ldots, f_s)$ are symmetric polynomials in $\mathbb{K}[x_1, \ldots, x_n]$

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**Theorem [Labahn-Safey El Din-Schost-Vu, 2023]**

- There is a randomized algorithm that takes as input $f$ and $\phi$ and outputs a representation for $W(\phi, f)$ with the runtime is $\left(d^s \binom{n+d}{n} \binom{n}{s+1}\right)^{O(1)}$.
- The size of the output of our algorithm is at most $d^s \binom{n+d-1}{n}$.
Previous work

[Labahn-Hubert]
  • scaling invariants and symmetry reduction of dynamical systems

[Busé-Karasoulou]
  • resultant of an equivariant polynomial system

[Riener]
  • deciding the emptiness symmetric semi-algebraic sets, fixed degree

[Riener-Safey El Din]
  • real root finding for equivariant semi-algebraic systems

[Faugère-Rahmany]
  • use SAGBI-Gröbner bases to solve symmetric systems

[Faugère-Svartz]
  • globally invariant systems
Determinantal systems

Given \( f = (f_1, \ldots, f_s) \subset \mathbb{K}[x_1, \ldots, x_n] \) and \( G \in \mathbb{K}[x_1, \ldots, x_n]^{p \times q} \)

- \( \text{wdeg}(x_i) = w_i \geq 1 \) for \( i = 1, \ldots, n \)
- \( \text{wcdeg}(G, j) := \max_{1 \leq i \leq p} (\text{wdeg}(g_{i,j})) = \delta_j \)

Compute \( V_p(f, G) := \{ \mathbf{x} \in \overline{\mathbb{K}}^n : f(\mathbf{x}) = 0 \text{ and } \text{rank}(G(\mathbf{x})) < p \} \)
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**Theorem** [Hauenstein-Labahn-Safey El Din-Schost-Vu, 2021]

Assume that $n = q-p+s+1$ and $E_k(\cdot)$ the $k$-th elementary symmetric function. Then there are at most

$$c = \text{wdeg}(f_1) \cdot \ldots \cdot \text{wdeg}(f_s) \cdot E_{n-s}(\delta_1, \ldots, \delta_q)/\Delta \text{ with } \Delta = w_1 \cdot \ldots \cdot w_n$$

isolated points, counted with multiplicities, in $V_p(f, G)$, which can be computed by a randomized algorithm \texttt{Homotopy_weighted} with runtime being polynomial in $c$.

In classical domains, i.e., $\text{wdeg}(x_i) = 1$ for all $i$.  

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Compute \( V_p(f, G) := \{ x \in \overline{\mathbb{K}}^n : f(x) = 0 \text{ and } \text{rank}(G(x)) < p \} \)

Example: \( \mathbb{K}[x_1, x_2, x_3] \) with \( \text{wdeg}(x_k) = k \)

Consider \( f_1 = x_1^2 - 3x_1x_2 + 3x_3 - 8 \) and \( \text{wdeg}(G) = \begin{pmatrix} 3 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix} \), then

\[ \text{wdeg}(f_1) = 3, \quad \text{wcdeg}(G) = (3, 2, 1) \]

and \( c = 3 \cdot E_2(3, 2, 1)/(1 \cdot 2 \cdot 3) = 3 \cdot (3 \cdot 2 + 3 \cdot 1 + 2 \cdot 1)/(1 \cdot 2 \cdot 3) = 30/6 = 5. \)
Determinantal systems and the critical points problem

Given \( f = (f_1, \ldots, f_s) \subset \mathbb{K}[x_1, \ldots, x_n] \) and \( G \in \mathbb{K}[x_1, \ldots, x_n]^{p \times q} \)
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W(\phi, f) = V_{s+1}(f, \text{jac}(f, \phi))
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isolated points, counted with multiplicities, in \( V_p(f, G) \), which can be computed by a randomized algorithm \texttt{Homotopy_weighted} with runtime being polynomial in \( c \).

\[
W(\phi, f) = V_{s+1}(f, \text{jac}(f, \phi)) \quad \text{Not exploit the symmetry}
\]
Symmetry polynomials

A polynomial $f$ is symmetric (or $S_n$-invariant) if

$$f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = f$$

for all $\sigma \in S_n$
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A set $W \subset \overline{K^n}$ is $S_n$-invariant if
\[ \sigma(x) \in W \text{ for all } \sigma \in S_n \text{ and } x \in W \]
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for all $\sigma \in S_n$ and $x \in W$

Example: $x_1^2 + x_2^2 + x_3^2 + x_4^2 - 6x_1x_2x_3x_4 - 1$ is $S_4$-invariant
Symmetry polynomials

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A set \( W \subset \overline{K}^n \) is \( S_n \)-invariant if
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\sigma(x) \in W \quad \text{for all } \sigma \in S_n \text{ and } x \in W
\]
Example: \( x_1^2 + x_2^2 + x_3^2 + x_4^2 - 6x_1x_2x_3x_4 - 1 \) is \( S_4 \)-invariant

Example: \( x_1^2 + x_2^2 + x_3^2 - 1 \) is \( S_3 \)-invariant

Property: if \( f = (f_1, \ldots, f_s) \) and \( \phi \) are symmetric, then \( W(\phi, f) \) is \( S_n \)-invariant
Data-structure for invariant sets

A list of positive integers $\lambda = (n_1, \ldots, n_1, \ldots, n_r, \ldots, n_r)$ is a partition of $n$ if

$$n_1 \ell_1 + n_2 \ell_2 + \cdots + n_r \ell_r = n$$

with $\ell := \ell_1 + \cdots + \ell_r$ is the length of $\lambda$. 

Example:

$\lambda = (1, 2)$ of $n = 3$, then $n_1 = 1$, $n_2 = 2$, $\ell_1 = 1$, $\ell_2 = 1$, and $\ell = 2$

Denote $C_{\lambda} \subset K_n$ contains $a = (a_1, 1, \ldots, a_1, 1, \ldots, a_r, 1, \ldots, a_r, 1) \in K_n$:

$a_1, 1 \neq a_2, 1$

Example:

$C_{(1, 2)} = \{ (3, 4, 4), (4, 3, 4), (4, 4, 3) \} \subseteq K_3$,

$W \subset K_n$ a $S_n$-invariant set, then $W = \bigsqcup \lambda W_{\lambda}$ with $W_{\lambda} := S_n (W \cap C_{\lambda})$

Example:

Suppose $n = 3$ and $W = \{ (5, 5, 5), (3, 4, 4), (3, 4, 4), (4, 4, 3) \}$. Then $W_{(1, 2)} = \{ (3, 4, 4), (4, 3, 4), (4, 4, 3) \}$, 

$W_{(3)} = \{ (5, 5, 5) \}$, and $W_{(1, 1, 1)} = \emptyset$. 

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Data-structure for invariant sets

A list of positive integers $\lambda = (n_1, \ldots, n_1, \ldots, n_r, \ldots, n_r)$ is a partition of $n$ if

$$n_1\ell_1 + n_2\ell_2 + \cdots + n_r\ell_r = n$$

with $\ell := \ell_1 + \cdots + \ell_r$ is the length of $\lambda$.

**Example:** $\lambda = (1, 2)$ of $n = 3$, then $n_1 = 1, n_2 = 2, \ell_1 = 1, \ell_2 = 1$, and $\ell = 2$.
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Denote $C_\lambda \subset \overline{\mathbb{K}}^n$ contains

$$a = \left(\overbrace{a_1, \ldots, a_1, \ldots, a_1}^{n_1}, \overbrace{a_1, \ell_1, \ldots, a_1, \ell_1}^{n_1}, \ldots, \overbrace{a_r, \ell_r, \ldots, a_r, \ell_r}^{n_r}, \ldots, \overbrace{a_r, \ell_r, \ldots, a_r, \ell_r}^{n_r}\right) \in \overline{\mathbb{K}}^n : a_{i,j} \text{ are distinct}$$
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Denote $C_\lambda \subset \bar{\mathbb{K}}^n$ contains

$$a = (a_{1,1}, \ldots, a_{1,1}, \ldots, a_{1,\ell_1}, \ldots, a_{1,\ell_1}, \ldots, a_{r,1}, \ldots, a_{r,1}, \ldots, a_{r,\ell_r}, \ldots, a_{r,\ell_r}) \in \bar{\mathbb{K}}^n : a_{i,j} \text{ are distinct}$$

**Example:** $C_{(1,2)} = \{(a_{1,1}, a_{2,1}, a_{2,1}) \in \bar{\mathbb{K}}^3 : a_{1,1} \neq a_{2,1}\},$ e.g., $(3, 4, 4) \in C_{(1,2)}$
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A list of positive integers $\lambda = (n_1, \ldots, n_1, \ldots, n_r, \ldots, n_r)$ is a partition of $n$ if

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$$a = (a_1, a_1, \ldots, a_1, a_1, a_1, \ldots, a_r, a_r, \ldots, a_r, a_r, a_r, \ldots, a_r, a_r, \ldots, a_r, a_r, a_r) \in \overline{K}^n : a_{i,j} \text{ are distinct}$$

**Example:** $C_{(1,2)} = \{(a_1, 1, a_2, 1, a_2, 1) \in \overline{K}^3 : a_{1,1} \neq a_{2,1}\}$, e.g., $(3, 4, 4) \in C_{(1,2)}$

$W \subset \overline{K}^n$ a $S_n$-invariant set, then $W = \bigsqcup_{\lambda} W_\lambda$ (disjoint union) with $W_\lambda := S_n(W \cap C_\lambda)$
A list of positive integers \( \lambda = (n_1, \ldots, n_1, \ldots, n_r, \ldots, n_r) \) is a partition of \( n \) if \( n_1 \ell_1 + n_2 \ell_2 + \cdots + n_r \ell_r = n \) with \( \ell := \ell_1 + \cdots + \ell_r \) is the length of \( \lambda \).

**Example:** \( \lambda = (1, 2) \) of \( n = 3 \), then \( n_1 = 1, n_2 = 2, \ell_1 = 1, \ell_2 = 1, \) and \( \ell = 2 \)

Denote \( C_\lambda \subset \overline{\mathbb{K}}^n \) contains

\[
\mathbf{a} = \left(\underbrace{a_{1,1}, \ldots, a_{1,1}}_{n_1}, \ldots, \underbrace{a_1, \ell_1, \ldots, a_1, \ell_1}_{n_1}, \ldots, \underbrace{a_r, 1, \ldots, a_r, 1}_{n_r} \ldots, \underbrace{a_r, \ell_r, \ldots, a_r, \ell_r}_{n_r}\right) \in \overline{\mathbb{K}}^n : a_{i,j} \text{ are distinct}
\]

**Example:** \( C_{(1,2)} = \{(a_{1,1}, a_{2,1}, a_{2,1}) \in \overline{\mathbb{K}}^3 : a_{1,1} \neq a_{2,1}\} \), e.g., \( (3, 4, 4) \in C_{(1,2)} \)

\( W \subset \overline{\mathbb{K}}^n \) a \( S_n \)-invariant set, then \( W = \bigsqcup_\lambda W_\lambda \) (disjoint union) with \( W_\lambda := S_n(W \cap C_\lambda) \)

**Example:** Suppose \( n = 3 \) and \( W = \{(5, 5, 5), (3, 4, 4), (3, 4, 4), (4, 4, 3)\} \). Then

\( W_{(1,2)} = \{(3, 4, 4), (4, 3, 4), (4, 4, 3)\}, W_{(3)} = \{(5, 5, 5)\}, \) and \( W_{(1,1,1)} = \emptyset \)
For a partition $\lambda = (n_1^{\ell_1} \ldots n_r^{\ell_r})$ and $a = (a_{i,j})_{1 \leq i \leq r, 1 \leq j \leq \ell_i}$, the compression mapping:

$$E_\lambda(a) = (E_{i,1}(a_{i,1}, \ldots, a_{i,\ell_i}), \ldots, E_{i,\ell_i}(a_{i,1}, \ldots, a_{i,\ell_i}))_{1 \leq i \leq r} \in \overline{K}^\ell,$$

where $E_{i,j}$'s the $j$-th elementary symmetric function of $a_{i,1}, \ldots, a_{i,\ell_i}$. 

Example:

$\lambda = (1^2, 2^2)$ of $n = 5$, then

$$E(1^2)(3, 4, 4, 5, 5) = (E_{1,1}(3), E_{2,1}(4, 5)) = (3, 9),$$

$W \subset K_n$ $a$-invariant set

$W_{\lambda} := S_n(W \cap C_{\lambda})$, $W'_{\lambda} := E_{\lambda}(W \cap C_{\lambda}) \subset K_{\ell_i}$

Example:

Suppose $n = 3$ and $W = \{(5, 5, 5), (3, 4, 4), (3, 4, 4), (4, 4, 3)\}$. Then

$W(1^2) = \{(3, 4, 4), (4, 4, 3)\}$, $W(3) = \{(5, 5, 5)\}$, and $W(1^3) = \emptyset$.

and

$W'(1^2) = \{(3, 4)\}$, $W'(3) = \{(5)\}$. 

$\text{card}(W) = \text{card}(W_{\lambda}) = n_1^{\ell_1} \ldots n_r^{\ell_r}$ and $\text{card}(W'_{\lambda}) = \ell_1! \ldots \ell_r!$. 


For a partition \( \lambda = (n_1^{\ell_1} \dotsc n_r^{\ell_r}) \) and \( a = (a_{i,j})_{1 \leq i \leq r, 1 \leq j \leq \ell_i} \), the compression mapping:

\[
E_\lambda(a) = (E_{i,1}(a_{i,1}, \dotsc, a_{i,\ell_i}), \dotsc, E_{i,\ell_i}(a_{i,1}, \dotsc, a_{i,\ell_i}))_{1 \leq i \leq r} \in \mathbb{K}^\ell,
\]

where \( E_{i,j} \)'s the \( j \)-th elementary symmetric function of \( a_{i,1}, \dotsc, a_{i,\ell_i} \).

**Example:** \( \lambda = (1, 2, 2) \) of \( n = 5 \), then

\[
E_{(1,2,2)}(3, 4, 4, 5, 5) = (E_{1,1}(3), E_{2,1}(4, 5), E_{2,2}(4, 5)) = (3, 9, 20)
\]
For a partition \( \lambda = (n_1^{\ell_1} \ldots n_r^{\ell_r}) \) and \( a = (a_{i,j})_{1 \leq i \leq r, 1 \leq j \leq \ell_i} \), the compression mapping:

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\( W \subset \overline{K}^n \) a \( S_n \)-invariant set \( W_\lambda := S_n(W \cap C_\lambda) \), \( W'_\lambda := E_\lambda(W \cap C_\lambda) \subset \overline{K}^\ell \)
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$W_{(1,2)} = \{(3, 4, 4), (4, 3, 4), (4, 4, 3)\}$, $W_{(3)} = \{(5, 5, 5)\}$, and $W_{(1,1,1)} = \emptyset$

and $W'_{(1,2)} = \{(3, 4)\}$ and $W_{(3)} = \{(5)\}$
For a partition \( \lambda = (n_1^{\ell_1} \ldots n_r^{\ell_r}) \) and \( a = (a_{i,j})_{1 \leq i \leq r, 1 \leq j \leq \ell_i} \), the compression mapping :

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E_\lambda(a) = (E_{i,1}(a_{i,1}, \ldots, a_{i,\ell_i}), \ldots, E_{i,\ell_i}(a_{i,1}, \ldots, a_{i,\ell_i}))_{1 \leq i \leq r} \in \overline{\mathbb{K}}^\ell,
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**Example :** Suppose \( n = 3 \) and \( W = \{(5, 5, 5), (3, 4, 4), (3, 4, 4), (4, 4, 3)\} \). Then

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W_{(1,2)} = \{(3, 4, 4), (4, 3, 4), (4, 4, 3)\}, \ W_{(3)} = \{(5, 5, 5)\}, \text{ and } W_{(1,1,1)} = \emptyset
\]

and \( W'_{(1,2)} = \{(3, 4)\} \) and \( W_{(3)} = \{(5)\}\)

\[
\frac{\text{card}(W)}{\text{card}(W_\lambda)} = \binom{n}{n_1, \ldots, n_1, \ldots, n_r, \ldots, n_r} = \frac{n!}{n_1!^{\ell_1} \cdots n_r!^{\ell_r}} \quad \text{and} \quad \frac{\text{card}(W_\lambda)}{\text{card}(W'_\lambda)} = \ell_1! \cdots \ell_r!\]
Let $\lambda = (n_1^{\ell_1} \ldots n_r^{\ell_r})$ a partition of $n$ and $z_i = (z_{i,1}, \ldots, z_{i,\ell_i})$ sequence of $\ell_i$ variables
Let \( \lambda = (n_1^{\ell_1} \ldots n_r^{\ell_r}) \) a partition of \( n \) and \( z_i = (z_{i,1}, \ldots, z_{i,\ell_i}) \) sequence of \( \ell_i \) variables Define the \( \mathbb{K} \)-algebra homomorphism \( T_\lambda : \mathbb{K}[x_1, \ldots, x_n] \to \mathbb{K}[z_1, \ldots, z_r] \)

\[
(x_1, \ldots, x_n) \mapsto (\underbrace{z_{1,1}, \ldots, z_{1,1}}_{n_1}, \ldots, \underbrace{z_{1,\ell_1}, \ldots, z_{1,\ell_1}}_{n_1}, \ldots, \underbrace{z_{r,1}, \ldots, z_{r,1}}_{n_r}, \ldots, \underbrace{z_{r,\ell_r}, \ldots, z_{r,\ell_r}}_{n_r})
\]
Let $\lambda = (n_1^{\ell_1} \ldots n_r^{\ell_r})$ a partition of $n$ and $z_i = (z_{i,1}, \ldots, z_{i,\ell_i})$ sequence of $\ell_i$ variables.

Define the $\mathbb{K}$-algebra homomorphism $T_\lambda : \mathbb{K}[x_1, \ldots, x_n] \rightarrow \mathbb{K}[z_1, \ldots, z_r]$

$$(x_1, \ldots, x_n) \mapsto (z_{1,1}, \ldots, z_{1,1}, \ldots, z_{1,\ell_1}, \ldots, z_{1,\ell_1}, \ldots, z_{r,1}, \ldots, z_{r,1}, \ldots, z_{r,\ell_r}, \ldots, z_{r,\ell_r})$$

**Example:** Let $\lambda = (1, 2, 2)$ of $n = 5$, then

$$T_{(1,2,2)}(x_1, x_2, x_3, x_4, x_5) = (z_{1,1}, z_{2,1}, z_{2,1}, z_{2,2}, z_{2,2})$$
Let $\lambda = (n_1^{\ell_1} \ldots n_r^{\ell_r})$ a partition of $n$ and $z_i = (z_{i,1}, \ldots, z_{i,\ell_i})$ sequence of $\ell_i$ variables

Define the $K$-algebra homomorphism $T_\lambda : K[x_1, \ldots, x_n] \rightarrow K[z_1, \ldots, z_r]$

$$(x_1, \ldots, x_n) \mapsto (\underbrace{z_{1,1}, \ldots, z_{1,1}}_{n_1}, \ldots, \underbrace{z_{1,\ell_1}, \ldots, z_{1,\ell_1}}_{n_1}, \ldots, \underbrace{z_{r,1}, \ldots, z_{r,1}}_{n_r}, \ldots, \underbrace{z_{r,\ell_r}, \ldots, z_{r,\ell_r}}_{n_r})$$

**Example:** $\lambda = (1, 2, 2)$ of $n = 5$, then

$$T_{(1,2,2)}(x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3) = z_{1,1}^3 + 2z_{2,1}^3 + 2z_{2,2}^3$$
Let $\lambda = (n_1^{\ell_1} \ldots n_r^{\ell_r})$ a partition of $n$ and $z_i = (z_{i,1}, \ldots, z_{i,\ell_i})$ sequence of $\ell_i$ variables

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**Example**: $\lambda = (1, 2, 2)$ of $n = 5$, then

$$T_{(1,2,2)}(G) = (T_\lambda(g_{i,j}))_{i,j} \text{ for } G = (g_{i,j}) \in K[x_1, \ldots, x_n]^{p \times q}$$
Let $\lambda = (n_1^{\ell_1} \ldots n_r^{\ell_r})$ a partition of $n$ and $z_i = (z_{i,1}, \ldots, z_{i,\ell_i})$ sequence of $\ell_i$ variables

Define the $\mathbb{K}$-algebra homomorphism $T_\lambda : \mathbb{K}[x_1, \ldots, x_n] \rightarrow \mathbb{K}[z_1, \ldots, z_r]$

$$(x_1, \ldots, x_n) \mapsto (z_{1,1}, \ldots, z_{1,1}, \ldots, z_{1,\ell_1}, \ldots, z_{1,\ell_1}, \ldots, z_{1,1}, \ldots, z_{1,\ell_1}, \ldots, z_{r,1}, \ldots, z_{r,1}, \ldots, z_{r,\ell_r}, \ldots, z_{r,\ell_r})$$

**Example**: $\lambda = (1, 2, 2)$ of $n = 5$, then

$$T_{(1,2,2)}(x_1, x_2, x_3, x_4, x_5) = (z_{1,1}, z_{2,1}, z_{2,1}, z_{2,2}, z_{2,2})$$

**Properties**: Denote $S_\lambda := S_{\ell_1} \times \cdots \times S_{\ell_r}$ and let $f$ be a $S_n$-invariant. Then

- $T_\lambda(f)$ is $S_\lambda$-invariant if $f$ is $S_n$-invariant
Back to polynomial systems

Let $\lambda = (n_1^{\ell_1} \ldots n_r^{\ell_r})$ a partition of $n$ and $z_i = (z_{i,1}, \ldots, z_{i,\ell_i})$ sequence of $\ell_i$ variables

Define the $K$-algebra homomorphism $T_\lambda : K[x_1, \ldots, x_n] \to K[z_1, \ldots, z_r]$

$$(x_1, \ldots, x_n) \mapsto (z_{1,1}, \ldots, z_{1,1}, \ldots, z_{1,\ell_1}, \ldots, z_{1,\ell_1}, \ldots, z_{r,1}, \ldots, z_{r,1}, \ldots, z_{r,\ell_r}, \ldots, z_{r,\ell_r})$$

**Example:** $\lambda = (1, 2, 2)$ of $n = 5$, then

$$T_{(1,2,2)}(x_1, x_2, x_3, x_4, x_5) = (z_{1,1}, z_{2,1}, z_{2,1}, z_{2,2}, z_{2,2})$$

**Properties:** Denote $S_\lambda := S_{\ell_1} \times \cdots \times S_{\ell_r}$ and let $f$ be a $S_n$-invariant. Then

- $T_{(1,2,2)}(x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3) = z_{1,1}^3 + 2z_{2,1}^3 + 2z_{2,2}^3$ is $S_1 \times S_2$-invariant
Let $\lambda = (n_1^{\ell_1} \ldots n_r^{\ell_r})$ a partition of $n$ and $z_i = (z_{i,1}, \ldots, z_{i,\ell_i})$ sequence of $\ell_i$ variables.

Define the $\mathbb{K}$-algebra homomorphism $T_\lambda : \mathbb{K}[x_1, \ldots, x_n] \to \mathbb{K}[z_1, \ldots, z_r]$

$$(x_1, \ldots, x_n) \mapsto (z_{1,1}, \ldots, z_{1,1}, \ldots, z_{1,\ell_1}, \ldots, z_{1,\ell_1}, \ldots, z_{r,1}, \ldots, z_{r,1}, \ldots, z_{r,\ell_r}, \ldots, z_{r,\ell_r})$$

**Example :** $\lambda = (1, 2, 2)$ of $n = 5$, then

$$T_{(1,2,2)}(x_1, x_2, x_3, x_4, x_5) = (z_{1,1}, z_{2,1}, z_{2,1}, z_{2,2}, z_{2,2})$$

**Properties :** Denote $S_\lambda := S_{\ell_1} \times \cdots \times S_{\ell_r}$ and let $f$ be a $S_n$-invariant. Then

- $T_\lambda(f)$ is $S_\lambda$-invariant if $f$ is $S_n$-invariant

- discarding some duplicated columns from $T_\lambda(\nabla f)$ gives a $S_\lambda$-equivariant system,

$$\nabla f = \left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right)$$
Let $\lambda = (n_1^{\ell_1} \ldots n_r^{\ell_r})$ a partition of $n$ and $z_i = (z_{i,1}, \ldots, z_{i,\ell_i})$ sequence of $\ell_i$ variables

Define the $\mathbb{K}$-algebra homomorphism $T_\lambda : \mathbb{K}[x_1, \ldots, x_n] \to \mathbb{K}[z_1, \ldots, z_r]$

$$(x_1, \ldots, x_n) \mapsto \left( \underbrace{z_{1,1}, \ldots, z_{1,1}}_{n_1}, \ldots, \underbrace{z_{1,\ell_1}, \ldots, z_{1,\ell_1}}_{n_1}, \ldots, \underbrace{z_{r,1}, \ldots, z_{r,1}}_{n_r}, \ldots, \underbrace{z_{r,\ell_r}, \ldots, z_{r,\ell_r}}_{n_r} \right)$$

**Example:** $\lambda = (1, 2, 2)$ of $n = 5$, then

$$T_{(1,2,2)}(x_1, x_2, x_3, x_4, x_5) = (z_{1,1}, z_{2,1}, z_{2,1}, z_{2,2}, z_{2,2})$$

**Properties:** Denote $S_\lambda := S_{\ell_1} \times \cdots \times S_{\ell_r}$ and let $f$ be a $S_n$-invariant. Then

- $T_{(1,2,2)}(x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3) = z_{1,1}^3 + 2z_{2,1}^3 + 2z_{2,2}^3$ is $S_1 \times S_2$-invariant

- discarding some **duplicated** columns from $T_\lambda(\nabla f)$ gives a $S_\lambda$-**equivariant** system,

$$T_{(1,2,2)}(\nabla x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3) = 3T_{(1,2,2)}(x_1^2, x_2^2, x_3^2, x_4^2, x_5^2) = 3(z_{1,1}^2, z_{2,1}^2, z_{2,1}^2, z_{2,2}^2, z_{2,2}^2)$$

the sequence $(z_{1,1}^2, z_{2,1}^2, z_{2,2}^2)$ is $S_1 \times S_2$-equivariant but **NOT** $S_1 \times S_2$-invariant
From $S_\lambda$-equivariant to $S_\lambda$-invariant

Recall

- $\lambda = (n_1^{\ell_1}, \ldots, n_r^{\ell_r})$ a partition of $n$ of length $\ell$ and $S_\lambda := S_{\ell_1} \times \cdots \times S_{\ell_r}$
- $z_i = (z_{i,1}, \ldots, z_{i,\ell_i})$ for $i = 1, \ldots, r$

We index $(z_1, \ldots, z_r) = (z_1, \ldots, z_{\ell})$. 
From $S_\lambda$-equivariant to $S_\lambda$-invariant

Recall

- $\lambda = (n_1^{\ell_1}, \ldots, n_r^{\ell_r})$ a partition of $n$ of length $\ell$ and $S_\lambda := S_{\ell_1} \times \cdots \times S_{\ell_r}$
- $z_i = (z_i, 1, \ldots, z_i, \ell_i)$ for $i = 1, \ldots, r$

We index $(z_1, \ldots, z_r) = (z_1, \ldots, z_\ell)$.

A sequence of polynomials $q = (q_1, \ldots, q_\ell)$ in $\mathbb{K}[z_1, \ldots, z_r]$ is $S_\lambda$-equivariant if

$$q_i(z_{\sigma(1)}, \ldots, z_{\sigma(\ell)}) = q_{\sigma(i)}(z_1, \ldots, z_\ell)$$

for all $i = 1, \ldots, \ell$ and $\sigma \in S_\lambda$
**From $S_\lambda$-equivariant to $S_\lambda$-invariant**

Recall

- $\lambda = (n_1^{\ell_1}, \ldots, n_r^{\ell_r})$ a partition of $n$ of length $\ell$ and $S_\lambda := S_{\ell_1} \times \cdots \times S_{\ell_r}$
- $z_i = (z_{i,1}, \ldots, z_{i,\ell_i})$ for $i = 1, \ldots, r$

We index $(z_1, \ldots, z_r) = (z_1, \ldots, z_\ell)$.

**Proposition**: Suppose $q$ is $S_\lambda$-equivariant and $z_i - z_j$ divides $q_i - q_j$. Then there exists an algorithm $\text{Symmetrize}(\lambda, q)$ which returns $p = (p_1, \ldots, p_\ell)$ s.t.

- $p$ is $S_\lambda$-invariant
- $p$ and $q$ generate the same ideal in a suitable localization of $\mathbb{K}[z_1, \ldots, z_r]$, that is, $pU = q$, where $U$ has a determinant unit in $\mathbb{K}[z_1, \ldots, z_r, 1/\Delta]$ with $\Delta = \prod_{1 \leq i < j \leq \ell} (z_i - z_j)$
- $\deg(p_i) \leq \delta - \ell + i$ with $\delta = \deg(q)$  \hspace{1em} $p_i = 0$ if $\ell \geq \delta + i$
- the runtime is $O^\sim(\ell^3 \binom{\ell + \delta}{\delta})$ operations in $\mathbb{K}$
From $S_\lambda$-equivariant to $S_\lambda$-invariant

Recall

• $\lambda = (n_1^{\ell_1}, \ldots, n_r^{\ell_r})$ a partition of $n$ of length $\ell$ and $S_\lambda := S_{\ell_1} \times \cdots \times S_{\ell_r}$

• $z_i = (z_{i,1}, \ldots, z_{i,\ell_i})$ for $i = 1, \ldots, r$

We index $(z_1, \ldots, z_r) = (z_1, \ldots, z_\ell)$.

Proposition: Suppose $q$ is $S_\lambda$-equivariant and $z_i - z_j$ divides $q_i - q_j$. Then there exists an algorithm $\text{Symmetrize}(\lambda, q)$ which returns $p = (p_1, \ldots, p_\ell)$ s.t.

• $p$ is $S_\lambda$-invariant

• $p$ and $q$ generate the same ideal in a suitable localization of $\mathbb{K}[z_1, \ldots, z_r]$, that is, $pU = q$, where $U$ has a determinant unit in $\mathbb{K}[z_1, \ldots, z_r, 1/\Delta]$ with $\Delta = \prod_{1 \leq i < j \leq \ell} (z_i - z_j)$

• $\deg(p_i) \leq \delta - \ell + i$ with $\delta = \deg(q)$ \quad $p_i = 0$ if $\ell \geq \delta + i$

• the runtime is $O^*(\ell^3 \binom{\ell + \delta}{\delta})$ operations in $\mathbb{K}$

Note [Hubert, 2009] has an algorithm which symmetrizes polynomials constructed via a generating set of rational invariants; but we wish to avoid rational functions
Sketch of the main algorithm

Input: symmetric polynomials $\phi$ and $(f_1, \ldots, f_s)$ in $\mathbb{K}[x_1, \ldots, x_n]$ 

Condition: $f = (f_1, \ldots, f_s)$ satisfies (A) and $W(\phi, f)$ is finite 

Output: a representation for $W(\phi, f)$ 

Assumption (A): the Jacobian matrix of $f$ has full rank at any solution of $f$
Sketch of the main algorithm

**Input:** symmetric polynomials $\phi$ and $(f_1, \ldots, f_s)$ in $\mathbb{K}[x_1, \ldots, x_n]$

**Condition:** $f = (f_1, \ldots, f_s)$ satisfies (A) and $W(\phi, f)$ is finite

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for all partitions $\lambda$ of $n$

1. compute $g = \mathcal{T}_\lambda(f)$ and $\mathcal{T}_\lambda(\text{jac}(f, \phi))$
Sketch of the main algorithm

**Input**: symmetric polynomials $\phi$ and $(f_1, \ldots, f_s)$ in $\mathbb{K}[x_1, \ldots, x_n]$

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**Output**: a representation for $W(\phi, f)$

**Assumption (A)**: the Jacobian matrix of $f$ has full rank at any solution of $f$

for all partitions $\lambda$ of $n$

1. compute $g = \mathbb{T}_\lambda(f)$ and $\mathbb{T}_\lambda(\text{jac}(f, \phi))$
2. discard duplicated columns of $\mathbb{T}_\lambda(\text{jac}(f, \phi))$ to obtain $L \in \mathbb{K}[z_1, \ldots, z_r]^{(s+1) \times \ell}$
Sketch of the main algorithm

**Input**: symmetric polynomials $\phi$ and $(f_1, \ldots, f_s)$ in $\mathbb{K}[x_1, \ldots, x_n]$

**Condition**: $f = (f_1, \ldots, f_s)$ satisfies (A) and $W(\phi, f)$ is finite

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3. apply Symmetrize algorithm on row vectors of $L$ to obtain matrix $H$
Sketch of the main algorithm

**Input:** symmetric polynomials \( \phi \) and \((f_1, \ldots, f_s)\) in \(\mathbb{K}[x_1, \ldots, x_n]\)

**Condition:** \(f = (f_1, \ldots, f_s)\) satisfies (A) and \(W(\phi, f)\) is finite

**Output:** a representation for \(W(\phi, f)\)

**Assumption (A):** the Jacobian matrix of \(f\) has full rank at any solution of \(f\)

for all partitions \(\lambda\) of \(n\)

1. compute \(g = \mathbb{T}_\lambda(f)\) and \(\mathbb{T}_\lambda(\text{jac}(f, \phi))\)
2. discard duplicated columns of \(\mathbb{T}_\lambda(\text{jac}(f, \phi))\) to obtain \(L \in \mathbb{K}[z_1, \ldots, z_r]^{(s+1) \times \ell}\)
3. apply Symmetrize algorithm on row vectors of \(L\) to obtain matrix \(H\)
4. find \(\zeta_g\) and \(\zeta_H\) with entries in \(\mathbb{K}[e_1, \ldots, e_r]\) s.t.

\[
\zeta_g(E_{1,1}(z_1), \ldots, E_{r,\ell}(z_r)) = g \quad \text{and} \quad \zeta_H(E_{1,1}(z_1), \ldots, E_{r,\ell}(z_r)) = H
\]

with \(\text{deg}(E_{i,k}) = k\); so \(\text{wdeg}(e_{i,k}) = k\)
Sketch of the main algorithm

Input: symmetric polynomials $\phi$ and $(f_1, \ldots, f_s)$ in $K[x_1, \ldots, x_n]$  

Condition: $f = (f_1, \ldots, f_s)$ satisfies (A) and $W(\phi, f)$ is finite  

Output: a representation for $W(\phi, f)$  

Assumption (A): the Jacobian matrix of $f$ has full rank at any solution of $f$ for all partitions $\lambda$ of $n$

1. compute $g = T_\lambda(f)$ and $T_\lambda(\text{jac}(f, \phi))$
2. discard duplicated columns of $T_\lambda(\text{jac}(f, \phi))$ to obtain $L \in K[z_1, \ldots, z_r]^{(s+1) \times \ell}$
3. apply Symmetrize algorithm on row vectors of $L$ to obtain matrix $H$
4. find $\zeta_g$ and $\zeta_H$ with entries in $K[e_1, \ldots, e_r]$ s. t.
   \[ \zeta_g(E_{1,1}(z_1), \ldots, E_{r,\ell_r}(z_r)) = g \quad \text{and} \quad \zeta_H(E_{1,1}(z_1), \ldots, E_{r,\ell_r}(z_r)) = H \]
   with $\text{deg}(E_{i,k}) = k$; so $\text{wdeg}(e_{i,k}) = k$
5. find $R_\lambda = \text{Homotopy\_weighted}(\zeta_g, \zeta_H)$
Part II:
Let $\mathbb{Q}$ be a field and $f_1, \ldots, f_s$ be polynomials in $\mathbb{Q}[x_1, \ldots, x_n]$

**Input:** $f_1 = \cdots = f_s = 0$ that defines $S \subset \mathbb{R}^n$

**Output:** true iff $S \neq \emptyset$ else false

This is a decision problem.
Let \( \mathbb{Q} \) be a field and \( f_1, \ldots, f_s \) be polynomials in \( \mathbb{Q}[x_1, \ldots, x_n] \).

**Input:** \( f_1 = \cdots = f_s = 0 \) that defines \( S \subset \mathbb{R}^n \)

**Output:** true iff \( S \neq \emptyset \) else false

This is a decision problem.

**Input:** \( f_1 = \cdots = f_s = 0 \) that defines \( S \subset \mathbb{R}^n \)

**Output:** Some points in \( S \) whenever they exist

- how to encode them? What to do if \( |S| = \infty \)?
- representative points in all the connected components of \( S \)
- quantitative results on the number of connected components of \( S \)?

Exact/Symbolic computation.
Collins’ Cylindrical Algebraic Decomposition algorithm

- complexity **doubly exponential** in $n$
- implementations are limited to small $n$

[Hong, McCallum, Arnon, Brown, Strzebonski, Anai, Sturm, Weispfenning]
State-of-the-art

Collins’ Cylindrical Algebraic Decomposition algorithm

- complexity **doubly exponential** in \( n \)
- implementations are limited to small \( n \)

[Hong, McCallum, Arnon, Brown, Strzebonski, Anai, Sturm, Weispfenning]

\[\Rightarrow\] Quest for algorithms **singly exponential** in \( n \)

The critical point method

[Grigoriev-Vorobjov], [Canny] [Renegar], [Heintz-Roy-Solerno], [Basu-Pollack-Roy],
[Bank-Giusti-Heintz-Mbakop], [Aubry-Rouillier-Safey El Din], [Rouillier-Roy-Safey El Din] [Safey El Din-Schost]
Main idea: studying a map that

• reaches an extremum on each connected component of $S$
• whose critical locus is zero-dimensional
Critical point method

Reduction of the dimension through Global Optimization

**Main idea**: studying a map that
- reaches an extremum on each connected component of $S$
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**Representation of finite sets**: using univariate polynomials $(v, v_1, \ldots, v_n)$
Critical point method

Reduction of the dimension through Global Optimization

**Main idea:** studying a map that
- reaches an extremum on each connected component of $S$
- whose critical locus is zero-dimensional

**Representation of finite sets:** using univariate polynomials $(v, v_1, \ldots, v_n)$

**Existence:** from $n$-variate to univariate problems
Our goal with symmetry

\( f = (f_1, \ldots, f_s) \) are symmetric polynomials in \( \mathbb{Q}[x_1, \ldots, x_n] \)

**Input:** \( f_1 = \cdots = f_s = 0 \) that defines \( S \subset \mathbb{R}^n \)

**Output:** true iff \( S \neq \emptyset \) else false

**Assumption (A):** the Jacobian matrix of \( f \) has full rank at any point of \( V(f) \)
Our goal with symmetry

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\( \leadsto \) exploit the **symmetry** to reduce the cost of computations

**Theorem** [Labahn-Riener-Safey El Din-Schost-Vu, preprint 2023]

There exists a randomized algorithm that takes \( f \) as input and decides the existence of real points in \( V(f) \). The runtime is polynomial in \( d^s, \binom{n+d}{d}, \text{ and } \binom{n}{s+1} \).
Our goal with symmetry

\( f = (f_1, \ldots, f_s) \) are symmetric polynomials in \( \mathbb{Q}[x_1, \ldots, x_n] \)

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There exists a randomized algorithm that takes \( f \) as input and decides the existence of real points in \( V(f) \). The runtime is polynomial in \( d^s, \binom{n+d}{d}, \) and \( \binom{n}{s+1} \).

**Observe:** The runtime is

- polynomial in \( n \) when \( n \) and \( d \) are fixed
- equal to \( n^{O(1)} 2^n \) when \( d = n \)
- subexponential in \( n \) when \( d \approx n^\alpha \) with \( \alpha < 1 \)
Assumption (A) with symmetry

Recall, for \( \lambda = (n_1^{\ell_1} \ldots n_r^{\ell_r}) \) a partition of \( n \) of length \( \ell = \ell_1 + \cdots + \ell_r \)

- \( S_\lambda := S_{\ell_1} \times \cdots \times S_{\ell_r} \) and \( E_{i,j} : j\)-th elementary symmetric function in \( z_i = (z_{i,1}, \ldots, z_{i,\ell_i}) \)
- the \( \mathbb{K} \)-algebra homomorphism \( \mathbb{T}_\lambda : \mathbb{K}[x_1, \ldots, x_n] \to \mathbb{K}[z_1, \ldots, z_r] \)
  \[
  (x_1, \ldots, x_n) \mapsto (z_{1,1}, \ldots, z_{1,1}, \ldots, z_{1,\ell_1}, \ldots, z_{1,\ell_1}, \ldots, z_{r,1}, \ldots, z_{r,1}, \ldots, z_{r,\ell_r}, \ldots, z_{r,\ell_r})
  \]

Given \( f = (f_1, \ldots, f_s) \) in \( \mathbb{Q}[x_1, \ldots, x_n] \); all are symmetric

Assumption (A) : the Jacobian matrix of \( f \) has rank \( s \) at any point of \( V(f) \)
Assumption (A) with symmetry

Recall, for $\lambda = (n_1^{\ell_1} \ldots n_r^{\ell_r})$ a partition of $n$ of length $\ell = \ell_1 + \cdots + \ell_r$

- $S_\lambda := S_{\ell_1} \times \cdots \times S_{\ell_r}$ and $E_{i,j} : j$-th elementary symmetric function in $z_i = (z_{i,1}, \ldots, z_{i,\ell_i})$
- the $\mathbb{K}$-algebra homomorphism $\mathbb{T}_\lambda : \mathbb{K}[x_1, \ldots, x_n] \to \mathbb{K}[z_1, \ldots, z_r]$

$(x_1, \ldots, x_n) \mapsto \left( z_{1,1}, \ldots, \underbrace{z_{1,\ell_1}}_{n_1}, \ldots, \underbrace{z_{1,\ell_1}}_{n_1}, \ldots, \underbrace{z_{1,\ell_1}}_{n_1}, \ldots, \underbrace{z_{r,1}}_{n_r}, \ldots, \underbrace{z_{r,1}}_{n_r}, \ldots, \underbrace{z_{r,\ell_r}}_{n_r}, \ldots, \underbrace{z_{r,\ell_r}}_{n_r} \right)$

Given $f = (f_1, \ldots, f_s)$ in $\mathbb{Q}[x_1, \ldots, x_n]$; all are symmetric

Assumption (A) : the Jacobian matrix of $f$ has rank $s$ at any point of $V(f)$

Then, $g := \mathbb{T}_\lambda(f)$ is $S_\lambda$-invariant and also satisfies (A)

$$\mathbb{T}_\lambda(\text{jac}(f)) = \text{jac}(g) \cdot M,$$ where $M = \text{diag}(M_1, \ldots, M_r) \in \mathbb{K}^{\ell \times n}$

$$M_i = \begin{pmatrix} \frac{1}{n_i} & \cdots & \frac{1}{n_i} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{n_i} & \cdots & \frac{1}{n_i} \end{pmatrix} \in \mathbb{K}^{\ell_i \times n_i \ell_i} \text{ of rank } \ell_i; \text{ so } \text{rank}(M) = \ell$$
Assumption (A) with symmetry

Recall, for $\lambda = (n_1^{\ell_1} \ldots n_r^{\ell_r})$ a partition of $n$ of length $\ell = \ell_1 + \cdots + \ell_r$
- $S_\lambda := S_{\ell_1} \times \cdots \times S_{\ell_r}$ and $E_{i,j}: j$-th elementary symmetric function in $z_i = (z_{i,1}, \ldots, z_{i,\ell_i})$
- the $\mathbb{K}$-algebra homomorphism $\mathbb{T}_\lambda: \mathbb{K}[x_1, \ldots, x_n] \to \mathbb{K}[z_1, \ldots, z_r]$

$$(x_1, \ldots, x_n) \mapsto (z_{1,1}, \ldots, z_{1,\ell_1}, \ldots, z_{1,1}, \ldots, z_{1,\ell_1}, \ldots, z_{r,1}, \ldots, z_{r,\ell_r}, \ldots, z_{r,1}, \ldots, z_{r,\ell_r})$$

Given $f = (f_1, \ldots, f_s)$ in $\mathbb{Q}[x_1, \ldots, x_n]$; all are symmetric

**Assumption (A):** the Jacobian matrix of $f$ has rank $s$ at any point of $V(f)$

Then, $g := \mathbb{T}_\lambda(f)$ is $S_\lambda$-invariant and also satisfies (A)

$$\mathbb{T}_\lambda(\text{jac}(f)) = \text{jac}(g) \cdot M,$$

where $M \in \mathbb{K}^{\ell \times n}$ of rank $\ell$

**Example:** $n = 7$ and $\lambda = (2, 2, 3)$. Then $M = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$ of rank $2 + 1$
Assumption (A) with symmetry

Recall, for \( \lambda = (n_1^{\ell_1} \ldots n_r^{\ell_r}) \) a partition of \( n \) of length \( \ell = \ell_1 + \cdots + \ell_r \)

- \( S_\lambda := S_{\ell_1} \times \cdots \times S_{\ell_r} \) and \( E_{i,j} : j \)-th elementary symmetric function in \( z_i = (z_i,1, \ldots, z_i,\ell_i) \)
- the \( K \)-algebra homomorphism \( \mathbb{T}_\lambda : K[x_1, \ldots, x_n] \to K[z_1, \ldots, z_r] \)
  \[
  (x_1, \ldots, x_n) \mapsto \left( z_{1,1}, \ldots, z_{1,1}, \ldots, z_{1,\ell_1}, \ldots, z_{1,\ell_1}, \ldots, z_{r,1}, \ldots, z_{r,1}, \ldots, z_{r,\ell_r}, \ldots, z_{r,\ell_r} \right)
  \]

Given \( f = (f_1, \ldots, f_s) \) in \( \mathbb{Q}[x_1, \ldots, x_n] \); all are symmetric

**Assumption (A)**: the Jacobian matrix of \( f \) has rank \( s \) at any point of \( V(f) \)

Then, \( g := \mathbb{T}_\lambda(f) \) is \( S_\lambda \)-invariant and also satisfies (A)

\[
\mathbb{T}_\lambda(\text{jac}(f)) = \text{jac}(g) \cdot M,
\]
where \( M \in K^{\ell \times n} \) of rank \( \ell \)

and for \( c \in V(\mathbb{T}_\lambda(f)) \cap \mathbb{C}^\ell \), there exists \( u \in V(f) \cap \mathbb{C}^n \) s.t. \( \mathbb{T}_\lambda(\text{jac}(f))(c) = \text{jac}(f)(u) \). Thus

\[
\text{jac}(f)(u) = \text{jac}(g)(c) \cdot M
\]

The left kernel of \( \text{jac}(f)(u) \) is trivial by (A), so is \( \text{jac}(g)(c) \).
Assumption (A) with symmetry

Recall, for $\lambda = (n_1^{\ell_1} \ldots n_r^{\ell_r})$ a partition of $n$ of length $\ell = \ell_1 + \cdots + \ell_r$

- $S_\lambda := S_{\ell_1} \times \cdots \times S_{\ell_r}$ and $E_{i,j}$: $j$-th elementary symmetric function in $z_i = (z_{i,1}, \ldots, z_{i,\ell_i})$
- the $K$-algebra homomorphism $\mathbb{T}_\lambda : K[x_1, \ldots, x_n] \to K[z_1, \ldots, z_r]$\[
(x_1, \ldots, x_n) \mapsto \left(\underbrace{z_{1,1}, \ldots, z_{1,1}}_{n_1}, \ldots, \underbrace{z_{1,\ell_1}, \ldots, z_{1,\ell_1}}_{n_1}, \ldots, \underbrace{z_{r,1}, \ldots, z_{r,1}}_{n_r}, \ldots, \underbrace{z_{r,\ell_r}, \ldots, z_{r,\ell_r}}_{n_r}\right)
\]

Given $f = (f_1, \ldots, f_s)$ in $Q[x_1, \ldots, x_n]$; all are symmetric

Assumption (A) : the Jacobian matrix of $f$ has rank $s$ at any point of $V(f)$

Then, $\zeta_g$ in $K[e_1, \ldots, e_r]$ is also satisfies (A), where $\zeta_g(E_{i,j}) = g$

$$\text{jac}(g) = \text{jac}(\zeta_g)(E_{i,j}) \cdot V, \text{ where } V = \text{diag}(V_1, \ldots, V_r)$$

with $V_i$ the Vandermonde matrix of $(E_{i,1}, \ldots, E_{i,\ell_i})$
Assumption (A) with symmetry

Recall, for $\lambda = (n_1^{\ell_1} \ldots n_r^{\ell_r})$ a partition of $n$ of length $\ell = \ell_1 + \cdots + \ell_r$

- $S_{\lambda} := S_{\ell_1} \times \cdots \times S_{\ell_r}$ and $E_{i,j} : j$-th elementary symmetric function in $z_i = (z_{i,1}, \ldots, z_{i,\ell_i})$
- the $K$-algebra homomorphism $T_{\lambda} : K[x_1, \ldots, x_n] \rightarrow K[z_1, \ldots, z_r]$

$$(x_1, \ldots, x_n) \mapsto (z_{1,1}, \ldots, z_{1,1}, \ldots, z_{1,\ell_1}, \ldots, z_{1,\ell_1}, \ldots, z_{r,1}, \ldots, z_{r,1}, \ldots, z_{r,\ell_r}, \ldots, z_{r,\ell_r})$$

Given $f = (f_1, \ldots, f_s)$ in $Q[x_1, \ldots, x_n]$; all are symmetric

**Assumption (A)**: the Jacobian matrix of $f$ has rank $s$ at any point of $V(f)$

Then,

- $g := T_{\lambda}(f)$ is $S_{\lambda}$-invariant and also satisfies (A) and
- $\zeta_g$ in $K[e_1, \ldots, e_r]$ also satisfies (A), where $\zeta_g(E_{i,j}) = g$
Sketch of the main algorithm

Input: \( f_1 = \cdots = f_s = 0 \) that defines \( S \subset \mathbb{R}^n \); all are symmetric

Output: true iff \( S \neq \emptyset \) else false

Assumption (A): the Jacobian matrix of \( f \) has full rank at any point of \( V(f) \)

for a partition \( \lambda \) of \( n \) of length at least \( s \)

1. compute \( g = T_\lambda(f) \in \mathbb{Q}[z_1, \ldots, z_r] \)
Sketch of the main algorithm

**Input:** \( f_1 = \cdots = f_s = 0 \) that defines \( S \subset \mathbb{R}^n \); all are symmetric

**Output:** true iff \( S \neq \emptyset \) else false

**Assumption (A):** the Jacobian matrix of \( f \) has full rank at any point of \( V(f) \)

for a partition \( \lambda \) of \( n \) of length at least \( s \)

1. compute \( g = T_\lambda(f) \in \mathbb{Q}[z_1, \ldots, z_r] \)
2. construct a good \( S_\lambda \)-invariant map \( \phi \) in \( \mathbb{Q}[z_{i,k}] \) s.t
Sketch of the main algorithm

Input: \( f_1 = \cdots = f_s = 0 \) that defines \( S \subset \mathbb{R}^n \); all are symmetric

Output: true iff \( S \neq \emptyset \) else false

Assumption (A): the Jacobian matrix of \( f \) has full rank at any point of \( V(f) \)

for a partition \( \lambda \) of \( n \) of length at least \( s \)

1. compute \( g = T_\lambda(f) \in \mathbb{Q}[z_1, \ldots, z_r] \)

2. construct a good \( S_\lambda \)-invariant map \( \phi \) in \( \mathbb{Q}[z_{i,k}] \) s.t

3. find \( \zeta_g \) and \( \zeta_\phi \) in \( \mathbb{K}[e_1, \ldots, e_r] \) s.t \( \zeta_g(E_{i,j}) = g \) and \( \zeta_\phi(E_{i,j}) = \phi \)
Sketch of the main algorithm

**Input:** \( f_1 = \cdots = f_s = 0 \) that defines \( S \subset \mathbb{R}^n \); all are symmetric

**Output:** true iff \( S \neq \emptyset \) else false

**Assumption (A):** the Jacobian matrix of \( \mathbf{f} \) has full rank at any point of \( V(\mathbf{f}) \)

for a partition \( \lambda \) of \( n \) of length at least \( s \)

1. compute \( \mathbf{g} = \mathbf{T}_\lambda(\mathbf{f}) \in \mathbb{Q}[\mathbf{z}_1, \ldots, \mathbf{z}_r] \)

2. construct a **good** \( S_\lambda \)-invariant map \( \phi \) in \( \mathbb{Q}[\mathbf{z}_i,k] \) s.t

3. find \( \mathbf{\zeta}_g \) and \( \mathbf{\zeta}_\phi \) in \( \mathbb{K}[\mathbf{e}_1, \ldots, \mathbf{e}_r] \) s.t \( \mathbf{\zeta}_g(E_{i,j}) = \mathbf{g} \) and \( \mathbf{\zeta}_\phi(E_{i,j}) = \phi \)

4. compute critical point set \( W \) of \( \mathbf{\zeta}_\phi \) restricted to \( V(\mathbf{\zeta}_g) \)
   - \( W = \text{Homotopy weighted}(\mathbf{\zeta}_g, \text{jac}(\mathbf{\zeta}_g, \mathbf{\zeta}_\phi)) \)
   - \( W \) is encoded by univariate polynomials \((v, v_{1,1}, \ldots, v_{r,\ell_r})\)
Sketch of the main algorithm

**Input:** \( f_1 = \cdots = f_s = 0 \) that defines \( S \subset \mathbb{R}^n \); all are symmetric

**Output:** true iff \( S \neq \emptyset \) else false

**Assumption (A):** the Jacobian matrix of \( f \) has full rank at any point of \( V(f) \)

for a partition \( \lambda \) of \( n \) of length at least \( s \)

1. compute \( g = T_\lambda (f) \in \mathbb{Q}[z_1, \ldots, z_r] \)

2. construct a good \( S_\lambda \)-invariant map \( \phi \) in \( \mathbb{Q}[z_{i,k}] \) s.t.

3. find \( \zeta_g \) and \( \zeta_\phi \) in \( \mathbb{K}[e_1, \ldots, e_r] \) s.t. \( \zeta_g (E_{i,j}) = g \) and \( \zeta_\phi (E_{i,j}) = \phi \)

4. compute critical point set \( W \) of \( \zeta_\phi \) restricted to \( V(\zeta_g) \)
   - \( W = \text{Homotopy weighted}(\zeta_g, \text{jac}(\zeta_g, \zeta_\phi)) \)
   - \( W \) is encoded by univariate polynomials \( (v, v_{1,1}, \ldots, v_{r,\ell_r}) \)

5. existence of real roots of bi-variate polynomial systems \( (v, v_{i,1}, \ldots, v_{i,\ell_i}) \)
   - from \( e_{i,j} \) coordinates back to \( (z_1, \ldots, z_r) \) then to \( (x_1, \ldots, x_n) \)
   - use Vieta polynomials \( \rho_i \):= \( u^{\ell_i} - v_{i,1}(t)u^{\ell_i-1} + \cdots + (-1)^{\ell_i} e_{i,\ell_i}(t) \in \mathbb{C}[t][u] \)
Given $S_{\lambda}$-invariant polynomials $g$ in $\mathbb{Q}[z_1, \ldots, z_r]$; $\lambda = (n_1^{\ell_1} \ldots, n_r^{\ell_r})$
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Construct a $S_\lambda$-invariant map $\phi$ in $\mathbb{Q}[z_1, \ldots, z_r]$ s.t. $\phi$

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* $\phi$ is a proper map
* $\phi = \phi_1 + \cdots + \phi_m$, where $\phi_k$ the homogeneous component of degree $k$
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$\sim$ the leading form of $\phi$ is Newton power sums of even degree

\[\sum_{i=1}^{r} c_{i} P_{i,\ell_i+1} + \sum_{i=1}^{r} \ell_i \sum_{k=0}^{\infty} c_{i,k} P_{i,k}\]

with $c_{i,k}$ are random numbers in $\mathbb{Q}$ and $c_{i,k} = 1$ if $\ell_i$ is odd and $c_{i,k} = 0$ if $\ell_i$ is even
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- $\phi = \phi_1 + \cdots + \phi_m$, where $\phi_k$ the homogeneous component of degree $k$
  if $\phi_m$ is positive definite; then, $\phi$ is proper
  $\rightsquigarrow$ the leading form of $\phi$ is Newton power sums of even degree

(ii.) has finite number of critical points on $V(g)$
  $\rightsquigarrow$ random linear combination of Newton power sums upto high enough degrees

$$\phi := \sum_{i=1}^{r} c_i P_{i,\ell_i+1} + \sum_{i=1}^{r} \sum_{k=0}^{\ell_i} c_{i,j} P_{i,k} \quad \text{where} \quad P_{i,k} := z_{i,1}^k + \cdots + z_{i,\ell_i}^k$$

with $c_{i,j}$ are random numbers in $\mathbb{Q}$ and $c_i = 1$ if $\ell_i$ is odd and $c_i = 0$ if $\ell_i$ is even
Consider \( f = x_1^2 + x_2^2 + x_3^2 + x_4^2 - 6x_1x_2x_3x_4 - 1 \), then \( S = V(f) \cap \mathbb{R}^4 \) is non-empty.
An illustrative example

Consider \( f = x_1^2 + x_2^2 + x_3^2 + x_4^2 - 6x_1x_2x_3x_4 - 1 \), then \( S = V(f) \cap \mathbb{R}^4 \) is non-empty

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- compute $g = \mathbb{T}_{(4)}(f) = -6z_{1,1}^4 + 4z_{1,1}^2 - 1 = -2z_{1,1}^4 - (2z_{1,1}^2 - 1)^2 < 0$ for all $z_{1,1} \in \mathbb{R}$
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- take \( \phi = 5(z_{1,1}^2 + z_{1,2}^2) - 9(z_{1,2} + z_{1,2}) - 3 \).

- find \( \zeta_\phi = 5e_{1,1}^2 - 9e_{1,1} - 10e_{1,2} - 3 \) and \( \zeta_g = 2e_{1,1}^2 - 6e_{1,2}^2 - 4e_{1,2} - 1 \)
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- critical point set $W$ of $\zeta_{\phi}$ restricted to $\zeta_g$ are solutions to $\zeta_g = \det(\text{jac}(\zeta_g, \zeta_{\phi})) = 0$; $W$ is encoded by $v, v_{1,1}, v_{1,2}$

\[ v = 200t^4 - 360t^3 + 62t^2 + 60t - 27, v_{1,1} = t, \text{ and } v_{1,2} = -1/6t^3 + 9/20t^2 - 31/600t - 1/20 \]
Consider $f = x_1^2 + x_2^2 + x_3^2 + x_4^2 - 6x_1x_2x_3x_4 - 1$, then $S = V(f) \cap \mathbb{R}^4$ is non-empty.

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- critical point set $W$ of $\zeta_\phi$ restricted to $\zeta_g$ are solutions to $\zeta_g = \det(\text{jac}(\zeta_g, \zeta_\phi)) = 0$; $W$ is encoded by $\nu, \nu_{1,1}, \nu_{1,2}$

\[\nu = 200t^4 - 360t^3 + 62t^2 + 60t - 27, \nu_{1,1} = t, \text{ and } \nu_{1,2} = -1/6t^3 + 9/20t^2 - 31/600t - 1/20\]

- check the system

\[\rho_1 = \nu = 0, \text{ with } \rho_1 = \nu'u^2 - \nu_{1,1}u + \nu_{2,1} \in \mathbb{Q}[t, u],\]

has real solutions.
See you in Tromsø this summer!