Differential Flatness for Fractional Order Dynamic Systems

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December 14, 2021

Outline

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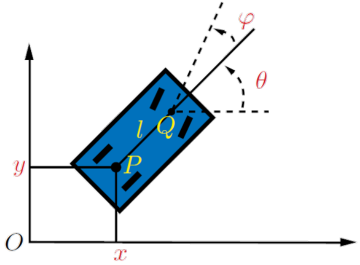


Figure: *Non-holonomic vehicle*

$(x, y, \theta) \in X \subset \mathbb{R}^3$: state vector

$(u, \varphi) \in U \subset \mathbb{R}^2$: input vector

$$\begin{cases} \dot{x} = u \ \cos(\theta) \\ \dot{y} = u \ \sin(\theta) \\ \dot{\theta} = \frac{u}{l} \ \tan(\varphi) \end{cases} \quad (1)$$

Let $z = (x, y) = (z_1, z_2)$:

$$\theta = \tan^{-1}\left(\frac{\dot{z}_2}{\dot{z}_1}\right)$$
$$u = \sqrt{\dot{z}_1^2 + \dot{z}_2^2} \quad (2)$$
$$\varphi = \tan^{-1}\left(l \frac{\ddot{z}_2 \dot{z}_1 - \dot{z}_2 \ddot{z}_1}{(\dot{z}_1^2 + \dot{z}_1^2)^{3/2}}\right)$$

System (1) is said to be differentially flat and z is called flat output.

Differential flatness

Definition

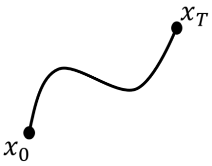
A nonlinear system of the form $\dot{x} = f(x, u)$ with $x = (x_1, \ldots, x_n)$ is the state vector, $u = (u_1, \ldots, u_m)$ is the input vector and $m \leq n$, is said to be differentially flat if and only if there exists $z = (z_1, \ldots, z_m)$, called flat output such that:

- $z = \psi(x, u, \dot{u}, \ldots, u^{(v)})$
- $z_1, \ldots, z_m$ and their successive derivatives are linearly independent
- $x = \varphi_0(z, \dot{z}, \ldots, z^{(\rho)}) \triangleq \varphi_0(\overline{z}^{(\rho)})$ and $u = \varphi_1(z, \dot{z}, \ldots, z^{(\rho+1)}) \triangleq \varphi_1(\overline{z}^{(\rho+1)})$
- The differential equation $\dot{\varphi}_0(\overline{z}) = f(\varphi_0(\overline{z}), \varphi_1(\overline{z}))$ is identically satisfied.

M.Fliess, J.Lévine, P.Martin, P.Rouchon (1991)

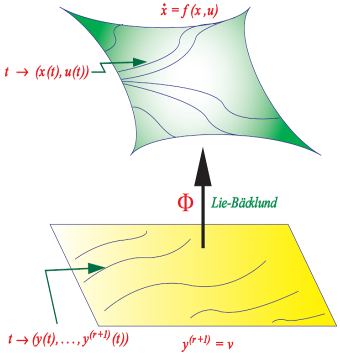
What for?

Generating reference trajectories



Differential flatness:
To every curve $t \longrightarrow z(t)$ differentiable at least $\rho + 1$ times it corresponds a trajectory

$$t \longrightarrow \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} \varphi_0(\overline{z}^{(\rho)}(t)) \\ \varphi_1(\overline{z}^{(\rho+1)}(t)) \end{pmatrix}$$



- The differential flatness has been discovered in 1991 by M. Fliess, J.Lévine, P.Martin and P.Rouchon for the class of nonlinear systems of integer order.
- Several computation methods of the flat output have been developed:
  - based on the diagonal Smith decomposition
  - based on the unimodular completion algorithm.
- The differential flatness has been extended to the class of linear systems of fractional order, as well as the computation methods of the flat output.
- Attempts have been made to extend the differential flatness to nonlinear systems of fractional order (open problem).

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Definition

Let $\gamma \in \mathbb{R}_+$, $n = \min\{k \in \mathbb{N} \mid k > \gamma\}$, $\nu = n - \gamma \in [0, 1[$ and $f \in \mathcal{C}^\infty([0, +\infty[)$. The fractional derivative of order γ, or Riemann-Liouville is defined by:

$$\mathbf{D}^\gamma f(t) = \mathbf{D}^n \big(\mathbf{I}^\nu f(t)\big) \triangleq \left(\frac{d}{dt}\right)^n \Bigg(\underbrace{\frac{1}{\Gamma(\nu)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\nu}} d\tau}_{\text{Cauchy integral}}\Bigg) \quad (3)$$

where

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \quad \forall x \in \mathbb{R}^* \setminus \mathbb{N}^-.$$

is the generalized factorial ($\Gamma(n+1) = n!$) and $\Gamma(\nu+1) = \nu\Gamma(\nu)$.

Theorem (Commutativity of fractional integral)

For all $\gamma, \kappa > 0$ *and all* $f \in \mathcal{C}^{\infty}([0, +\infty[)$, *we have:*

$$I^{\gamma} I^{\kappa} f(t) = I^{\kappa} I^{\gamma} f(t) = I^{\gamma+\kappa} f(t) \tag{4}$$

Lemma

The fractional derivative is not commutative and we have:

$$\begin{aligned} \mathbf{D}^{\gamma} f(t) &= \mathbf{D}^{n} \mathbf{I}^{\nu} f(t) \\ &= \mathbf{I}^{\nu} \mathbf{D}^{n} f(t) + \sum_{j=0}^{n-1} \frac{t^{\nu-n+j}}{\Gamma(n-\nu+j+1)} \delta^{(j)} f \end{aligned} \tag{5}$$

where $\delta^{(j)} f = (-1)^{j} f^{(j)}(0)$ *is the* j^{th} *derivative of the Dirac measure at* $t = 0$.

- In systems theory, the natural signal space generally considered is the space of causal signals:

$$\mathfrak{H}_0 \triangleq \{f : \mathbb{R} \mapsto \mathbb{R} \mid f \in \mathcal{C}^\infty([0, +\infty[), f(t) = 0, \forall t \leq 0\}. \tag{6}$$

- If $f \in \mathfrak{H}_0$, for all $\gamma, \kappa > 0$:

$$\mathbf{D}^\gamma(\mathbf{I}^\gamma f(t)) = \mathbf{I}^\gamma(\mathbf{D}^\gamma f(t)) = f(t) \tag{7}$$

$$\mathbf{I}^{-\gamma} f(t) = \mathbf{D}^\gamma f(t), \quad \mathbf{D}^{-\gamma} f(t) = \mathbf{I}^\gamma f(t) \tag{8}$$

$$\mathbf{D}^\kappa(\mathbf{D}^\gamma f(t)) = \mathbf{D}^\gamma(\mathbf{D}^\kappa f(t)) = \mathbf{D}^{\gamma+\kappa} f(t) \tag{9}$$

- Factional polynomial:

$$\mathbb{R}[\mathbf{D}^\gamma] \triangleq \left\{P = \sum_{k=0}^{K} c_k \mathbf{D}^{k\gamma}, c_k \in \mathbb{R}\right\}$$

The set $(\mathbb{R}[\mathbf{D}^\gamma], +, \times)$ is a (commutative) principal ideal domain.

- Fractional polynomial matrix:

$$\mathbb{R}[\mathbf{D}^\gamma]^{p\times q} : \text{ the set of } \mathbb{R}[\mathbf{D}^\gamma] - \text{matrices of size } p \times q$$

- Unimodular matrix:

$$GL_p(\mathbb{R}[\mathbf{D}^\gamma]) \triangleq \{M \in \mathbb{R}[\mathbf{D}^\gamma]^{p\times p} \mid \exists N \in \mathbb{R}[\mathbf{D}^\gamma]^{p\times p}; MN = I_p\}$$

- **Remark:** In $\mathbb{R}[\mathbf{D}^\gamma]$, properties of polynomial division, Bézout identity, definitions of G.C.D, L.C.M. and prime polynomials are the same as in the principal ideal domain of real polynomials.

Theorem (Diagonal Smith decomposition)

Let $M \in \mathbb{R}[\mathbf{D}^\gamma]^{p\times q}$ with $p \leq q$ (resp. $p \geq q$), then there exist two matrices $S \in GL_p(\mathbb{R}[\mathbf{D}^\gamma])$ and $T \in GL_q(\mathbb{R}[\mathbf{D}^\gamma])$ and a matrix $\Delta = \mathrm{diag}\{\delta_1, \ldots, \delta_\sigma, 0, \ldots, 0\} \in \mathbb{R}[\mathbf{D}^\gamma]^{p\times p}$ (resp. $\mathbb{R}[\mathbf{D}^\gamma]^{q\times q}$) such that:

$$SMT = \begin{pmatrix} \Delta & 0_{p\times(q-p)} \end{pmatrix} \quad \left(resp.\ \ SMT = \begin{pmatrix} \Delta \\ 0_{(p-q)\times q} \end{pmatrix} \right). \tag{10}$$

In Δ, δ_i, for $i = 1, \ldots, \sigma$, is a $\mathbf{D}^\gamma$-polynomial such that δ_i divides δ_j for all $i \leq j \leq \sigma$ and $\sigma = \mathrm{rank}(M)$.

Theorem

A $\mathbf{D}^\gamma$-polynomial matrix of size $p \times q$ is hyper-regular if, and only if $\Delta = I_p$ (resp. $\Delta = I_q$).

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- Consider the following representation of a linear fractional system:

$$x^{(\gamma)} = Ax + Bu \tag{11}$$

  where $x \in (\mathfrak{H}_0)^n$ is the state vector, $u \in (\mathfrak{H}_0)^m$ is the input vector, $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ with $\mathrm{rank}(B) = m$ and $m \leq n$.

- The system (11) can be written in the form:

$$F \begin{pmatrix} x \\ u \end{pmatrix} = 0 \tag{12}$$

  where $F \triangleq \begin{pmatrix} A & -B \end{pmatrix} \in \mathbb{R}[\mathbf{D}^{\gamma}]^{n \times (n+m)}$ with $\mathrm{rank}(F) = n$.

Definition[1]: Fractional flat system

The system $F \begin{pmatrix} x \\ u \end{pmatrix} = 0$ is a fractional flat system if, and only if

1. $\exists\ P \in \mathbb{R}[\mathbf{D}^\gamma]^{m \times (n+m)}$ and $Q \in \mathbb{R}[\mathbf{D}^\gamma]^{(n+m) \times m}$ s.t. $PQ = I_m$;
2. $\exists\ z \in (\mathfrak{H}_0)^m$ s.t. $z = P \begin{pmatrix} x \\ u \end{pmatrix}$ and conversely $\begin{pmatrix} x \\ u \end{pmatrix} = Qz$.

z is called fractional flat output and the matrices P and Q are called defining matrices.

Theorem

The system $F \begin{pmatrix} x \\ u \end{pmatrix} = 0$ *is a fractional flat system if, and only if* F *is hyper-regular over* $\mathbb{R}[\mathbf{D}^\gamma]$.

[1] Victor, S., Melchior, P., Lévine, J., and Oustaloup, A. (2015)

Proposition:

If $B \in \mathbb{R}[\mathbf{D}^\gamma]^{n\times m}$ is hyper-regular, i.e. if there exists $M \in GL_n(\mathbb{R}[\mathbf{D}^\gamma])$ such that $MB = \begin{pmatrix} I_m \\ 0_{(n-m)\times m} \end{pmatrix}$, then there exist two matrices $\widetilde{F} \in \mathbb{R}[\mathbf{D}^\gamma]^{(n-m)\times n}$ and $R \in \mathbb{R}[\mathbf{D}^\gamma]^{m\times n}$ such that the system $F\begin{pmatrix} x \\ u \end{pmatrix} = 0$ is equivalent to

$$\begin{cases} Rx = u \\ \widetilde{F}x = 0 \end{cases} \quad . \tag{13}$$

Theorem

Let the matrix B be hyper-regular, then the system (13) is fractionally flat if, and only if the matrix $\widetilde{F}$ of the implicit form is hyper-regular over $\mathbb{R}[\mathbf{D}^\gamma]$.

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- Consider the fractional linear system $F\begin{pmatrix} x \\ u \end{pmatrix} = 0$
- Algorithm:
  - **Input:** the matrix $F \in \mathbb{R}[\mathbf{D}^\gamma]^{n\times(n+m)}$
  - **Output:** defining matrices $P \in \mathbb{R}[\mathbf{D}^\gamma]^{m\times(n+m)}$ and $Q \in \mathbb{R}[\mathbf{D}^\gamma]^{(n+m)\times m}$
  - **Procedure:**
    1. Check if F is hyper-regular. If not, return "the system is not flat"
    2. Else, find $U \in GL_{n+m}(\mathbb{R}[\mathbf{D}^\gamma])$ s.t.:
$$FU = \begin{pmatrix} I_n & 0_{n\times m} \end{pmatrix} \tag{14}$$
    3. The defining matrices P and Q are given by:
$$Q = U\begin{pmatrix} 0_{n\times m} \\ I_m \end{pmatrix} \text{ and } P = \begin{pmatrix} 0_{m\times n} & I_m \end{pmatrix} U^{-1}$$
    4. Return P and Q and the fractional flat output is given by $z = P\begin{pmatrix} x \\ u \end{pmatrix}$.

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Definition (Unimodular completion)

Given a hyper-regular matrix $M \in \mathbb{R}[\mathbf{D}^\gamma]^{p \times q}$ with $p \leq q$, we say that $N \in \mathbb{R}[\mathbf{D}^\gamma]^{(q-p) \times q}$ is a unimodular completion of M if, and only if,

$$\begin{pmatrix} M \\ N \end{pmatrix} \in GL_q(\mathbb{R}[\mathbf{D}^\gamma]).$$

Proposition

Let $F \in \mathbb{R}[\mathbf{D}^\gamma]^{n \times (n+m)}$ be hyper-regular. Then, the vector z is a fractional flat output if, and only if, the defining matrix $P \in \mathbb{R}[\mathbf{D}^\gamma]^{m \times (n+m)}$ such that

$$z = P \begin{pmatrix} x \\ u \end{pmatrix} \tag{15}$$

is a unimodular completion of F.

Algorithm

This algorithm is iterative and has 3 steps: reduction, zero-space decomposition and elimination.

- **Input:** the matrix $F \in \mathbb{R}[\mathbf{D}^\gamma]^{n\times(n+m)}$
- **Output:** the defining matrices $P \in \mathbb{R}[\mathbf{D}^\gamma]^{m\times(n+m)}$ and $Q \in \mathbb{R}[\mathbf{D}^\gamma]^{(n+m)\times m}$
- **Procedure:**
  - **Starting point:** Decompose F into the form

$$\left(F_{0,[0]} + F_{1,[0]}\ \mathbf{D}^\gamma\right) v_{[0]}{}^2 = 0 \tag{16}$$

where $F_{0,[0]}$ and $F_{1,[0]}$ in $\mathbb{R}^{n\times(n+m)}$ are two coefficient matrices and $v_{[0]} = (x, u)^T$.

[2] The index in brackets indicates the iteration number.

Algorithm

– **Step 1: Reduction**

- For iteration i, we start from:

$$\left(F_{0,[i]} + F_{1,[i]}\ \mathbf{D}^{\gamma}\right) v_{[i]} = 0 \tag{17}$$

- we consider the change of coordinates

$$v_{[i]} = F_{1,[i]}^{\dagger R}\, v_{[i+1]} + F_{1,[i]}^{\perp R}\, w_{[i+1]} \tag{18}$$

where $F_{1,[i]}^{\dagger R}$ is the right pseudo-inverse:

$$F_{1,[i]} F_{1,[i]}^{\dagger R} = I_n \tag{19}$$

and $F_{1,[i]}^{\perp R}$ is the right-orthonormal:

$$F_{1,[i]} F_{1,[i]}^{\perp R} = 0. \tag{20}$$

Algorithm

– **Step 1: Reduction**

- By injecting equation (18) in (17) we get

$$v_{[i+1]}^{(\gamma)} + \mathfrak{A}_{[i]} v_{[i+1]} + \mathfrak{B}_{[i]} w_{[i+1]} = 0 \tag{21}$$

with

$$\mathfrak{A}_{[i]} = F_{0,[i]} F_{1,[i]}^{\dagger R} \quad \text{and} \quad \mathfrak{B}_{[i]} = F_{0,[i]} F_{1,[i]}^{\perp R}. \tag{22}$$

- Check the matrix $\mathfrak{B}_{[i]} \in \mathbb{R}^{n_i \times m_i}$:
  - If $\mathfrak{B}_{[i]} \equiv 0$ return: *the system is not flat.*
  - If $\text{rank}(\mathfrak{B}_{[i]}) = r_i < m_i$ then a zero-space decomposition is needed to reduce the dimension.
  - If $\text{rank}(\mathfrak{B}_{[i]}) = m_i$, *i.e.* $\mathfrak{B}_{[i]}$ is of full column rank, we move on to the elimination step.

Algorithm

– **Step 2: Zero-space decomposition:** $\text{rank}(\mathfrak{B}_{[i]}) = r_i < m_i$

- decompose the matrix $F^{\perp R}_{1,[i]} = \left(\widetilde{F}^{\perp R}_{1,[i]} \;\; Z_{[i]}\right)$ such that

$$\mathfrak{B}_{[i]} = F_{0,[i]}\left(\widetilde{F}^{\perp R}_{1,[i]} \;\; Z_{[i]}\right) = \left(\widetilde{\mathfrak{B}}_{[i]} \;\; 0\right) \tag{23}$$

  with $\text{rank}(\widetilde{\mathfrak{B}}_{[i]}) = r_i$.
- For this purpose, we introduce:

$$Z_{[i]} := F^{\perp R}_{1,[i]}\mathfrak{B}^{\perp R}_{[i]} \quad \text{and} \quad \widetilde{F}^{\perp R}_{1,[i]} := F^{\perp R}_{1,[i]}\left((\mathfrak{B}^{\perp R}_{[i]})^{\perp L}\right)^T \tag{24}$$

- The change of coordinates (18) is replaced by

$$v_{[i]} = F^{\dagger R}_{1,[i]}\, v_{[i+1]} + \widetilde{F}^{\perp R}_{1,[i]}\, w_{[i+1]} + Z_{[i]}\, z_{[i+1]} \tag{25}$$

  and equation (21) becomes:

$$v^{(\gamma)}_{[i+1]} + \mathfrak{A}_{[i]} v_{[i+1]} + \widetilde{\mathfrak{B}}_{[i]} w_{[i+1]} = 0. \tag{26}$$

Algorithm

- **Step 3: Elimination:** $\text{rank}(\mathfrak{B}_{[i]}) < n_i$:
  - multiply (21) by $\mathfrak{B}_{[i]}^{\perp L}$:

$$\left(F_{0,[i+1]} + \mathbf{D}^{\gamma}\ F_{1,[i+1]}\right) v_{[i+1]} = 0, \tag{27}$$

    with $F_{0,[i+1]} = \mathfrak{B}_{[i]}^{\perp L}\mathfrak{A}_{[i]}$ and $F_{1,[i+1]} = \mathfrak{B}_{[i]}^{\perp L}$.
  - In this case, we eliminate the variable $w_{[i+1]}$ from equation (21) and the dimension of the system is reduced.
  - We repeat the procedure for the iteration $i+1$.
  - The calculations stop at iteration k when a full row rank of $\mathfrak{B}_{[k]}$ is reached.

Algorithm

- **Construction of the defining matrices:**
  - In each iteration i, a relation between $v_{[i]}$ and $v_{[i+1]}$ is established:

$$v_{[i+1]} = F_{1,[i]} v_{[i]}. \tag{28}$$

  - Conversely, we have:

$$\begin{aligned} v_{[i]} &= \left(F_{1,[i]}^{\dagger R} - F_{1,[i]}^{\perp R}(\mathfrak{B}_{[i]}^{\dagger L}\mathbf{D}^{(\gamma)} + \mathfrak{B}_{[i]}^{\dagger L}\mathfrak{A}_{[i]})\right) v_{[i+1]} \\ &= G_{[i]}(\mathbf{D}^{(\gamma)}) v_{[i+1]}. \end{aligned} \tag{29}$$

  - After a finite number $k+1$ of iterations:

$$v_{[k+1]} = F_{1,[k]} F_{1,[k-1]} \ldots F_{1,[0]} v_{[0]} := \mathbf{P}\, v_{[0]} \tag{30}$$

and

$$\begin{aligned} v_{[0]} &= G_{[0]}(\mathbf{D}^{(\gamma)}) \ldots G_{[k-1]}(\mathbf{D}^{(\gamma)}) G_{[k]}(\mathbf{D}^{(\gamma)}) v_{[k+1]} \\ &:= \mathbf{G}\, v_{[k+1]}. \end{aligned} \tag{31}$$

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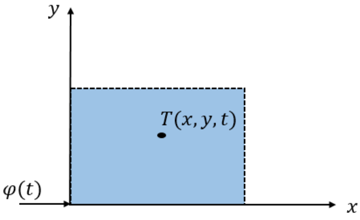

$T(x,y,t)$ is the temperature at (x,y)
$\varphi(t)$ is the heat flux applied at $(0,0)$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-\frac{1}{\alpha}\frac{\partial}{\partial t}\right)T(x,y,t)=0 \tag{32}$$

where α is the coefficient of diffusivity.

- In polar coordinates: $x = \rho\cos(\theta)$, $y = \rho\sin(\theta)$ the heat equation becomes:

$$\frac{1}{\alpha}\frac{\partial T(\rho,\theta,t)}{\partial t} = \frac{\partial^2 T(\rho,\theta,t)}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial T(\rho,\theta,t)}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 T(\rho,\theta,t)}{\partial^2 \theta}. \tag{33}$$

- the boundary conditions:

$$-\lambda \lim_{\rho\to 0}\frac{\pi}{2}\rho\frac{\partial T(\rho,\theta,t)}{\partial \rho} = \varphi(t), \qquad \forall t > 0 \tag{34}$$

and

$$\frac{\partial T(\rho,0,t)}{\partial \rho} = 0 \quad \text{and} \quad \frac{\partial T(\rho,\frac{\pi}{2},t)}{\partial \rho} = 0, \qquad \forall \rho > 0 \tag{35}$$

where λ is the coefficient of conductivity.

- the limit condition:

$$\lim_{\rho\to\infty} T(\rho,\theta,t) = 0, \qquad \forall\, \theta \in \left[0, \frac{\pi}{2}\right], \quad \forall t > 0 \tag{36}$$

- the initial condition known as Cauchy condition:

$$T(\rho,\theta,0) = 0, \qquad \forall \rho > 0, \quad \forall \theta \in \left[0, \frac{\pi}{2}\right]. \tag{37}$$

- The Laplace transformation of the equation (33) is given by:

$$\frac{s}{\alpha}\widehat{T}(\rho,\theta,s) = \frac{\partial^2 \widehat{T}(\rho,\theta,s)}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \widehat{T}(\rho,\theta,s)}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 \widehat{T}(\rho,\theta,s)}{\partial^2 \theta} \tag{38}$$

where

$$\widehat{T}(\rho,\theta,s) = \int_0^{+\infty} T(\rho,\theta,t)e^{-st}dt$$

is the Laplace transformation of $T(\rho,\theta,t)$.

- Using the separation of variables method: $\widehat{T}(\rho,\theta,s) = \widehat{T}_\rho(\rho,s)\widehat{T}_\theta(\theta,s)$

$$\frac{\rho^2}{\widehat{T}_\rho(\rho,s)}\frac{\partial^2 \widehat{T}_\rho(\rho,s)}{\partial \rho^2} + \frac{\rho}{\widehat{T}_\rho(\rho,s)}\frac{\partial \widehat{T}_\rho(\rho,s)}{\partial \rho} - \frac{s\rho^2}{\alpha} + \frac{1}{\widehat{T}_\theta(\theta,s)}\frac{\partial^2 \widehat{T}_\theta(\theta,s)}{\partial \theta^2} = 0. \tag{39}$$

- Because of the symmetry of the metallic sheet, $\widehat{T}_\theta(\theta,s) = A_1(s)$, then equation (39) becomes:

$$\rho^2\frac{\partial^2 \widehat{T}_\rho(\rho,s)}{\partial \rho^2} + \rho\frac{\partial \widehat{T}_\rho(\rho,s)}{\partial \rho} - \frac{s\rho^2}{\alpha}\widehat{T}_\rho(\rho,s) = 0 \tag{40}$$

- Equation (40) is a modified Bessel equation. Its solution is a modified Bessel function of the form:

$$\widehat{T}_\rho(\rho, s) = B_1(s) I_0\Big(\rho\sqrt{\frac{s}{\alpha}}\Big) + B_2(s) K_0\Big(\rho\sqrt{\frac{s}{\alpha}}\Big). \tag{41}$$

- According to the limit condition (36), $B_1(s) = 0$, and the solution of the thermal bi-dimensional system is given by:

$$\widehat{T}(\rho, \theta, s) = A_1(s)\, B_2(s) K_0\Big(\rho\sqrt{\frac{s}{\alpha}}\Big). \tag{42}$$

- According to the boundary condition, the heat flux is given by:

$$\widehat{\varphi}(s) = \lambda \frac{\pi}{2} A_1(s) B_2(s) \tag{43}$$

- The thermal impedance is defined by:

$$\widehat{H}(\rho, \theta, s) = \frac{\widehat{T}(\rho, \theta, s)}{\widehat{\varphi}(s)} = \frac{2\sqrt{2\pi}}{\lambda\pi} \frac{1}{\sqrt{\rho\sqrt{\frac{s}{\alpha}}}} e^{-\rho\sqrt{\frac{s}{\alpha}}} \tag{44}$$

- Applying the Padé approximation at the order $\mathbf{K}$ at a point (ρ_0, θ_0):

$$\widehat{H}_{\mathbf{K}}(\rho_0, \theta_0, s) \approx \frac{\sqrt{2\pi\sqrt{\alpha}}}{\lambda\pi\sqrt{\rho_0}} \frac{\displaystyle\sum_{k=0}^{\mathbf{K}} (-1)^k C'_k s^{\frac{k}{2}}}{\displaystyle\sum_{k=0}^{\mathbf{K}} C'_k s^{\frac{2k+1}{4}}} \tag{45}$$

with $C'_k = \dfrac{C_k}{|C_{\mathbf{K}}|}$ and $C_k = \dfrac{(2\mathbf{K}-k)!\mathbf{K}!}{(2\mathbf{K}!)k!(\mathbf{K}-k)!}\left(\dfrac{\rho_0}{\sqrt{\alpha}}\right)^k$.

- The system is of fractional order $\nu = \frac{1}{4}$.

- The linear system of fractional order $\nu = \frac{1}{4}$:

$$\begin{cases} X^{(\nu)} = AX + BU, \\ T_{\mathbf{K}}(x_0, y_0, t) = CX \end{cases} \tag{46}$$

with A is a real square matrix of dimension $(2\mathbf{K}+1)$,

$$A = \begin{pmatrix} 0 & -C'_{\mathbf{K}-1} & 0 & -C'_{\mathbf{K}-2} & \cdots & 0 & -C'_0 & 0 \\ & & & I_{2\mathbf{K}} & & & & 0_{2\mathbf{K}\times 1} \end{pmatrix}$$

$$B = \begin{pmatrix} 1 \\ 0_{2\mathbf{K}\times 1} \end{pmatrix} \in \mathbb{R}^{(2\mathbf{K}+1)\times 1},$$

and

$$C = \frac{\sqrt{2\pi\sqrt{\alpha}}}{\lambda\pi\sqrt{\rho_0}} \begin{pmatrix} (-1)^{\mathbf{K}} C'_{\mathbf{K}} & 0 & \cdots & 0 & -C'_1 & 0 & C'_0 \end{pmatrix} \in \mathbb{R}^{1\times(2\mathbf{K}+1)}.$$

- $X = \begin{pmatrix} X_0 & \ldots & X_{2\mathbf{K}} \end{pmatrix}$ is the state vector and $U = \varphi(t)$ is the input.

- The system (46) can also be written into the form

$$\mathcal{A}X = BU, \qquad T_{\mathbf{K}}(x_0, y_0, t) = CX \tag{47}$$

where $\mathcal{A}$ is of size $(2\mathbf{K}+1) \times (2\mathbf{K}+1)$ and given by

$$\mathcal{A} = \begin{pmatrix} \mathbf{D}^{\frac{1}{4}} & C'_{\mathbf{K}-1} & 0 & \cdots & 0 & C'_0 & 0 \\ -1 & \mathbf{D}^{\frac{1}{4}} & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & -1 & \ddots & \ddots & & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & 0 \\ 0 & & \cdots & & 0 & -1 & \mathbf{D}^{\frac{1}{4}} \end{pmatrix}.$$

- The matrix B is hyper-regular, system (47) becomes:

$$RX = U, \quad \widetilde{F}X = 0 \tag{48}$$

with $\widetilde{F} = \begin{pmatrix} -1 & \mathbf{D}^{\frac{1}{4}} & 0 & \cdots & 0 \\ 0 & -1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & \mathbf{D}^{\frac{1}{4}} \end{pmatrix}$ and

$R = \begin{pmatrix} \mathbf{D}^{\frac{1}{4}} & C'_{\mathbf{K}-1} & 0 & \cdots & 0 & C'_0 & 0 \end{pmatrix}.$

Computation of the flat output:

- For iteration $i = 0$ the matrix $\widetilde{F}$ can be written in the form

$$\widetilde{F} = \widetilde{F}_{0,[0]} + \widetilde{F}_{1,[0]}\mathbf{D}^{\frac{1}{4}} \tag{49}$$

where

$$\widetilde{F}_{0,[0]} = \begin{pmatrix} -1 & 0 & \cdots & \cdots & 0 \\ 0 & -1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & -1 & 0 \end{pmatrix}, \quad \widetilde{F}_{1,[0]} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \cdots & & 0 & 1 \end{pmatrix}$$

- We compute

$$\mathfrak{A}_{[0]} = \widetilde{F}_{0,[0]}\widetilde{F}_{1,[0]}^{\dagger R} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & -1 & 0 \end{pmatrix},$$

$$\mathfrak{B}_{[0]} = \widetilde{F}_{0,[0]}\widetilde{F}_{1,[0]}^{\perp R} = \begin{pmatrix} -1 \\ 0_{(2\mathbf{K}-1)\times 1} \end{pmatrix}$$

- The matrix $\mathfrak{B}_{[0]} = \begin{pmatrix} -1 \\ 0_{(2\mathbf{K}-1)\times 1} \end{pmatrix}$ is of full column rank but may not be of full row rank.
- If $\operatorname{rank} \mathfrak{B}_{[0]} = r < 2\mathbf{K}$, then we move to the Elimination step and we compute

$$\widetilde{F}_{0,[1]} = \mathfrak{B}_{[0]}^{\perp L} \mathfrak{A}_{[0]} \quad \text{and} \quad \widetilde{F}_{1,[1]} = \mathfrak{B}_{[0]}^{\perp L}. \tag{50}$$

- we apply the Reduction step for iteration $i = 1$.
- Finally, the computations end at iteration $i = 2\mathbf{K} - 1$ where $\mathfrak{B}_{[2\mathbf{K}-1]}$ reaches full row rank.

- The unimodular completion matrix of $\widetilde{F}$ is then given by:

$$P = \widetilde{F}_{1,[2\mathbf{K}-1]} \ldots \widetilde{F}_{1,[1]} \widetilde{F}_{1,[0]} = \begin{pmatrix} 0_{1\times 2\mathbf{K}} & 1 \end{pmatrix}, \tag{51}$$

- The fractional flat output is given by

$$Z = PX = \begin{pmatrix} 0_{1\times 2\mathbf{K}} & 1 \end{pmatrix} \begin{pmatrix} X_0 \\ \vdots \\ X_{2\mathbf{K}} \end{pmatrix} = X_{2\mathbf{K}}. \tag{52}$$

- Conversely, the state X can be computed by

$$X = G(\mathbf{D}^{\frac{1}{4}})Z = G_{[0]}(\mathbf{D}^{\frac{1}{4}})G_{[1]}(\mathbf{D}^{\frac{1}{4}}) \ldots G_{[2\mathbf{K}-1]}(\mathbf{D}^{\frac{1}{4}})Z \tag{53}$$

where $G(\mathbf{D}^{\frac{1}{4}})$ is given by

$$G(\mathbf{D}^{\frac{1}{4}}) = \begin{pmatrix} \mathbf{D}^{\frac{\mathbf{K}}{2}} & \mathbf{D}^{\frac{\mathbf{K}}{2}-\frac{1}{4}} & \ldots & \mathbf{D}^{\frac{1}{4}} & 1 \end{pmatrix}^T. \tag{54}$$

- The input U is computed by

$$U = RX = \sum_{k=0}^{\mathbf{K}} C'_k \mathbf{D}^{\frac{2k+1}{4}} Z \tag{55}$$

- The output $T_{\mathbf{K}}$ is computed by

$$T_{\mathbf{K}} = CX = \frac{\sqrt{2\pi\sqrt{\alpha}}}{\lambda\pi\sqrt{\rho_0}} \sum_{k=0}^{\mathbf{K}} (-1)^k C'_k \mathbf{D}^{\frac{k}{2}} Z. \tag{56}$$

Application to trajectory planning

- We want the temperature to go from a resting state of temperature $T_0 = 0°C$ to a resting state of temperature $T_f = 30°C$ in a period of $2500s$, at the point $x_0 = 0.005m$ and $y_0 = 0.002m$.
- The initial and final conditions of the temperature are the following:

$$\begin{aligned} &T(x_0, y_0, 0) \triangleq T_0 = 0, \\ &T(x_0, y_0, t_f) \triangleq T_f = 30, \quad t_f = 2500s, \\ &T^{(l)}(x_0, y_0, 0) = 0, \quad l = 1, 2, \\ &T^{(l)}(x_0, y_0, t_f) = 0, \quad l = 1, 2. \end{aligned} \tag{57}$$

- Translate these conditions to conditions on the flat output Z:

$$Z_{ref}(t) = \sum_{j=0}^{r} \eta_j t^j$$

In these simulations we took $\mathbf{K} = 20$:

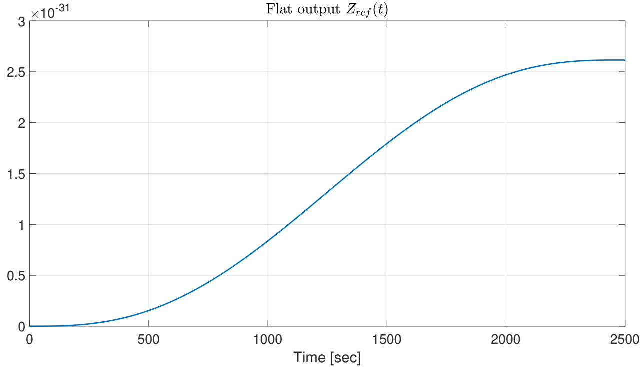

Figure: *Reference trajectory of the flat output* $Z_{ref}(t)$

Reference trajectory of the input $\varphi_{ref}(t)$

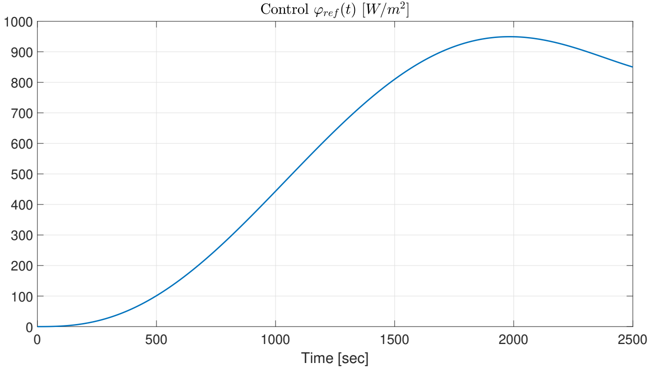

Figure: *Reference trajectory of the heat flux* $\varphi(t)$

Reference trajectory of the temperature $T_{ref}(x_0, y_0, t)$

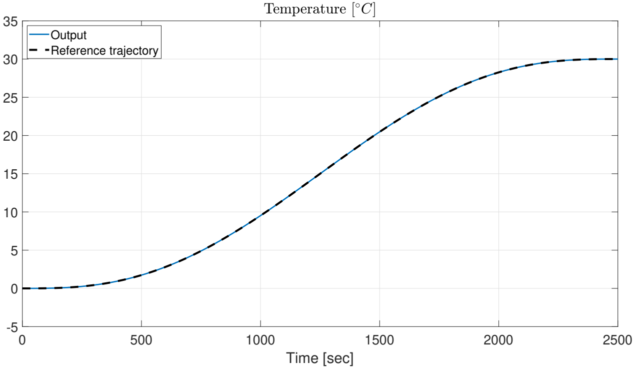

Figure: *Reference trajectory vs. exact solution*

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- The nonlinear fractional order systems are of the form:

$$x^{(\nu)} = f(x, u) \tag{58}$$

where $x = (x_1, \ldots, x_n)$ is the state vector, $u = (u_1, \ldots, u_m)$ is the input vector and ν the fractional order derivative.

- In the literature, there exists two approaches for the nonlinear flatness: **differential algebra** and **differential geometry** of jets of infinite order.

- Studies were done via two ways:
  - using the standard differential operator d
  - using the fractional differential operator d_ν.

- In both ways, the extension to the fractional order is not consistent with the integer case.

- This case remains an open problem.

Thank you!