Linear PDE with constant coefficients

Joint work with Marc Härkönen and Bernd Sturmfels

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Motivation

Study the ideal \( I = \langle x^2, y^2, xz - yz^2 \rangle \)

1. Solve the equations \( x^2 = y^2 = xz - yz^2 = 0 \) \textbf{Answer:} the \( z \)-axis.

2. Which polynomials lie in the ideal

\[
I = \langle x^2, y^2, x - yz \rangle \cap \langle x^2, y^2, z \rangle ?
\]

\textbf{Answer:} A polynomial \( f \) lies in \( I = \langle x^2, y^2, x - yz \rangle \cap \langle x^2, y^2, z \rangle \) if and only if the following conditions hold: Both \( f \) and \( \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial x} \) vanish on the \( z \)-axis, and both \( \frac{\partial^2 f}{\partial x \partial y} \) and \( \frac{\partial f}{\partial x} \) vanish at the origin.

3. Solve the PDE given by \( I \):

\[
\frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial^2 \varphi}{\partial y^2} = \frac{\partial^2 \varphi}{\partial x \partial z} - \frac{\partial^3 \varphi}{\partial y \partial z^2} = 0.
\]

\textbf{Answer :} \( \varphi(x, y, z) = \xi(z) + (y \psi(z) + x \psi'(z)) + \alpha xy + \beta x \).
$R = \mathbb{C}[\partial z_1, \ldots, \partial z_n]$, elements of $R^k$ acts on functions $\varphi : \mathbb{R}^n \to \mathbb{C}^k$ as follow:

$$R^k \times \mathcal{F}^k \to \mathcal{F}$$

$$p \cdot \varphi \mapsto \sum_{i=1}^{k} p_i \cdot \varphi_i$$

$\mathcal{F}$ is either $\mathcal{D}'(\mathbb{R}^n)$ or $C^\infty_0(\mathbb{R}^n)$.

An $k \times \ell$ matrix $M$ encodes a linear PDE with constant coefficients. $\varphi$ satisfies the PDE given by $M$ if $M_i \cdot \varphi$ vanishes for all column $M_i$.

$$\text{Sol}(M) := \left\{ \varphi \in \mathcal{F}^k : m \cdot \varphi = 0 \text{ for all } m \in M \right\}$$
Example of PDE

\[ M = \left[ \begin{array}{ccc} \frac{\partial}{\partial z_1} & \frac{\partial}{\partial z_2} & \frac{\partial^2}{\partial z_1 \partial z_2} \\ \frac{\partial^2}{\partial z_1} & \frac{\partial^2}{\partial z_2} & \frac{\partial^2}{\partial z_1 \partial z_4} \end{array} \right] \]

\[ \frac{\partial^2 \psi_1}{\partial z_1 \partial z_3} + \frac{\partial^2 \psi_2}{\partial z_1^2} = \frac{\partial^2 \psi_1}{\partial z_1 \partial z_2} + \frac{\partial^2 \psi_2}{\partial z_2^2} = \frac{\partial^3 \psi_1}{\partial z_1 \partial z_2} + \frac{\partial^3 \psi_2}{\partial z_1^2 \partial z_4} = 0 \]

A family of (some) solutions is

\[ t^2 \left[ \begin{array}{c} t \\ -s \end{array} \right] \exp \left( s^2 tz_1 + st^2 z_2 + s^3 z_3 + t^3 z_4 \right) \]
Exponential solutions

Given $M$ submodule of $R^k$ and $\text{Ass}(M) = \{P_1, \ldots, P_s\}$; $P_i \subseteq R$ is associated to $M$ if there exists $m \in R^k$ such that $(M : m) = P_i$.

$$V(M) = V(P_1) \cup \ldots \cup V(P_s) = V(\langle k \times k \text{ subdeterminants of } M \rangle)$$

is called the characteristic variety of $M$.

The arithmetic length $m_i$ of $M$ along $P_i$ is the length of the largest submodule of finite length in $(R_{P_i})^k / M_{P_i}$.

If $M$ is a $P$-primary ideal; its arithmetic length is $\frac{\text{deg}(M)}{\text{deg}(P)}$. 
Exponential solutions

Lemma

Fix a $k \times l$ matrix $M(\partial)$. A point $u \in \mathbb{C}^n$ lies in $V(M)$ if and only if there exist constants $c_1, \ldots, c_k \in \mathbb{C}$, not all zero, such that
\[
\begin{pmatrix}
c_1 \\
\vdots \\
c_k
\end{pmatrix}
\exp(u_1 z_1 + \cdots + u_n z_n) \in \text{Sol}(M)
\]

Proposition

The solution space $\text{Sol}(M)$ contains an exponential solution $q(z) \cdot \exp(u^t z)$ if and only if $u \in V(M)$. Here $q$ is a vector of polynomials.
Finite dimensional case

Theorem ([4])

Consider a submodule $M \subseteq \mathbb{R}^k$. Its solution space $\text{Sol}(M)$ is finite-dimensional over $\mathbb{C}$ if and only if $V(M)$ has dimension 0. There is a basis of $\text{Sol}(M)$ given by vectors $q(z) \exp(u^t z)$, where $u \in V(M)$ and $q(z)$ runs over a finite set of polynomial vectors. There exist polynomial solutions if and only if $m = \langle x_1, \ldots, x_n \rangle$ is an associated prime of $M$. The polynomial solutions are found by solving the PDE given by the $m$-primary component of $M$.

$$\dim_{\mathbb{C}} \text{Sol}(M) = \dim_{K} \left( \mathbb{R}^k / M \right) = \text{amult}(M).$$
Theorem (Ehrenpreis–Palamodov Fundamental Principle)

Consider a module \( M \subseteq \mathbb{R}^k \). There exist irreducible varieties \( V_1, \ldots, V_s \) in \( \mathbb{C}^n \) and finitely many vectors \( B_{ij} \) of polynomials in \( 2n \) unknowns \((x, z)\), such that any solution \( \psi \in \mathcal{F} \) admits an integral representation

\[
\psi(z) = \sum_{i=1}^{s} \sum_{j=1}^{m_i} \int_{V_i} B_{ij}(x, z) \exp \left( x^t z \right) d\mu_{ij}(x)
\]

Here \( \mu_{ij} \) is a measure supported on the variety \( V_i \).
The ideal $I = \langle \partial^2_1 - \partial_2 \partial_3, \partial^2_3 \rangle$ represents the PDE

$$
\frac{\partial^2 \varphi}{\partial z_1^2} = \frac{\partial^2 \varphi}{\partial z_2 \partial z_3} \quad \text{and} \quad \frac{\partial^2 \varphi}{\partial z_3^2} = 0
$$

for a scalar-valued function $\varphi = \varphi(z_1, z_2, z_3)$. Primary and $m_1 = 4$. It reveals the vectors

$$B_1 = 1, B_2 = z_1, B_3 = z_1^2 x_2 + 2z_3, B_4 = z_1^3 x_2 + 6z_1 z_3$$

$$\varphi(z) = a(z_2) + z_1 b(z_2) + (z_1^2 c'(z_2) + 2z_3 c(z_2)) + (z_1^3 d'(z_2) + 6z_1 z_3 d(z_2))$$
Denote $R = \mathbb{C}[\partial z_1, \ldots, \partial z_n] := \mathbb{C}[x_1, \ldots, x_n]$ ($x_i = \partial z_i$).

Find the differential operators $A_{i,j}(x, \partial x)$ such that

$$m \in M \iff A_{i,j} \cdot m \in P_i \text{ for all } P_i \in \text{Ass}(M)$$

$A_{i,j}$ are called Noetherian operators and the list $(P_i, A_{i, \cdot})$ is called a differential primary decomposition of $M$. 
Suppose $M$ is $P$-primary submodule of $R^k$.
Consider the differential operator in the Weyl algebra

$$A(x, \partial_x) = \sum_{r,s \in \mathbb{N}^n} c_{r,s} x_1^{r_1} \cdots x_n^{r_n} \partial_{x_1}^{s_1} \cdots \partial_{x_n}^{s_n}, \quad \text{where } c_{r,s} \in K$$

There is a natural $K$-linear isomorphism between the Weyl algebra $D_n$ and the polynomial ring $K[x, z]$ which takes the operator $A(x, \partial_x)$ to the following polynomial $B$ in $2n$ variables:

$$B(x, z) = \sum_{r,s \in \mathbb{N}^n} c_{r,s} x_1^{r_1} \cdots x_n^{r_n} z_1^{s_1} \cdots z_n^{s_n}$$

This bijection restricts to a bijection between Noetherian operators and Noetherian multipliers:

$$\mathcal{B} := \{ B \in K[x, z] : B(x, z) \exp (x^t z) \in \text{Sol}(M) \text{ for all } x \in V(P) \}$$
Theorem ([2])

Every submodule $M$ of $\mathbb{R}^k$ has a differential primary decomposition. We can choose the sets $A_1, \ldots, A_s$ such that $|A_i|$ is the arithmetic length of $M$ along the prime $P_i$. Moreover if $(P_1, A_1), \ldots, (P_s, A_s)$ is any differential primary decomposition for $M$, then $|A_i| \geq m_i$. 
solvePDE algorithm

Algorithm 1 SolvePDE

\textbf{Input:} An arbitrary submodule $M$ of $R^k$

\textbf{Output:} List of associated primes with corresponding Noetherian multipliers.

1: \textbf{for} each associated prime ideal $P$ of $M$ \textbf{do}
2: \hspace{1em} $U \leftarrow MR_P^k \cap R^k$
3: \hspace{1em} $V \leftarrow (U : P^{\infty})$
4: \hspace{1em} $r \leftarrow$ the smallest number such that $V \cap P^{r+1} R^k$ is a subset of $U$
5: \hspace{1em} $S \leftarrow$ a maximal set of independent variables modulo $P$
6: \hspace{1em} $\mathbb{K} \leftarrow \text{Frac}(R/P)$
7: \hspace{1em} $T \leftarrow \mathbb{K}[y_i : x_i \notin S]$
8: \hspace{1em} $\gamma \leftarrow$ the map defined in (37)
9: \hspace{1em} $m \leftarrow$ the irrelevant ideal in $T$
10: \hspace{1em} $\hat{U} \leftarrow \gamma(U) + m^{r+1}T^k$
11: \hspace{1em} $\hat{V} \leftarrow \gamma(V) + m^{r+1}T^k$
12: \hspace{1em} $N \leftarrow$ a $\mathbb{K}$-vector space basis of the space of $k$-tuples of polynomials of degree $\leq r$
13: \hspace{1em} $\text{Diff}(\hat{U}) \leftarrow$ the matrix given by the $\bullet$-product of generators of $\hat{U}$ with elements of $N$
14: \hspace{1em} $\text{Diff}(\hat{V}) \leftarrow$ the matrix given by the $\bullet$-product of generators of $\hat{V}$ with elements of $N$
15: \hspace{1em} $\mathcal{K} \leftarrow \ker_{\mathbb{K}}(\text{Diff}(\hat{U}))/\ker_{\mathbb{K}}(\text{Diff}(\hat{V}))$
16: \hspace{1em} $\mathcal{A} \leftarrow$ a $\mathbb{K}$-vector space basis of $\mathcal{K}$
17: \hspace{1em} $\mathcal{B} \leftarrow$ lifts of the vectors in $\mathcal{A} \subset T^k$ to vectors in $R[dx_1, \ldots, dx_n]^k$
18: \hspace{1em} \textbf{return} the pair $(P, \mathcal{B})$

\hspace{1em} $\gamma : R \rightarrow T, \hspace{1em} x_i \mapsto \begin{cases} y_i + u_i, & \text{if } x_i \notin S, \\ u_i, & \text{if } x_i \in S. \end{cases}$
Distributed in Macaulay2 with the package "NoehterianOperators" from version 1.18.

```plaintext
R=QQ[x1,x2,x3];

I=ideal(x1^2-x2*x3,x3^2);

Ideal of R

solvePDE I

{"ideal (x3, x1), { | 1 |, | dx1 |, | x2dx1^2+2dx3 |, | x2dx1^3+6dx1dx3 |}}

List
```
solvePDE

\[ a(z_2), \; z_1 b(z_2), \]

\[
\int (x_2 z_1^2 + z_3) \exp(x_2 z_2) dx_2 = z_1^2 \partial_{z_2} \left( \int \exp(x_2 z_2) dx_2 \right) + z_3 \int \exp(z_2 x_2) dx_2 = z_1^2 c_1'(z_2) + z_3 c_2(z_2),
\]

\[
\int (x_2 z_1^3 + 6z_1 z_3) \exp(x_2 z_2) dx_2 = z_1^3 \partial_{z_2} \left( \int \exp(x_2 z_2) dx_2 \right) + 6z_1 z_3 \int \exp(z_2 x_2) dx_2 = z_1^3 d_1'(z_2) + 6z_1 z_3 d_2(z_2),
\]
Consider the Weyl algebra $D = \mathbb{C}\langle z_1, \ldots, z_n, \partial_1, \ldots, \partial_n \rangle$, and $I$ a $D$-ideal.

$\text{in}_{(-w,w)}(I)$ is fixed under the action of the n-dimensional algebraic torus $(\mathbb{C}^*)^n$: $t_i \cdot x_i = \frac{1}{t_i} x_i$ and $t_i \cdot \partial_i = t_i \partial_i$.

$\text{in}_{(-w,w)}(I)$ is generated by operators $x^a p(\theta) \partial^b$ where $a, b \in \mathbb{N}^n$ where $\theta_i = z_i \partial_i$ [5, Theorem 2.3.3].

Consider

$$[\theta_b] := \prod_{i=1}^{n} \prod_{j=0}^{b_i-1} (\theta_i - j)$$

The distraction $J$ of $\text{in}_{(-w,w)}(I)$ is the ideal in $\mathbb{C}[\theta]$ generated by all polynomials $[\theta_b] p(\theta - b) = x^b p(\theta) \partial^b$.

$\text{Sol}(J) = \text{Sol}(\text{in}_{(-w,w)}(I))$ which can often be lift to $\text{Sol}(I)$. 

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