Linear PDE with constant coefficients

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Study the ideal $I = \langle x^2, y^2, xz - yz^2\rangle$

1. Solve the equations $x^2 = y^2 = xz - yz^2 = 0$ Answer: the z-axis.
2. Which polynomials lie in the ideal

$$I = \langle x^2, y^2, x - yz\rangle \cap \langle x^2, y^2, z\rangle?$$

   Answer: A polynomial f lies in $I = \left\langle x^2, y^2, x - yz\right\rangle \cap \left\langle x^2, y^2, z\right\rangle$ if and only if the following conditions hold: Both f and $\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial x}$ vanish on the z-axis, and both $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial f}{\partial x}$ vanish at the origin.
3. Solve the PDE given by I:

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial^2 \varphi}{\partial y^2} = \frac{\partial^2 \varphi}{\partial x \partial z} - \frac{\partial^3 \varphi}{\partial y \partial z^2} = 0.$$

   Answer : $\varphi(x, y, z) = \xi(z) + (y\psi(z) + x\psi'(z)) + \alpha xy + \beta x$.

Setup

$R = \mathbb{C}[\partial_{z_1}, \ldots, \partial_{z_n}]$, elements of R^k acts on functions $\varphi : \mathbb{R}^n \to \mathbb{C}^k$ as follow:

$$R^k \times \mathcal{F}^k \to \mathcal{F}$$

$$p \bullet \varphi \mapsto \sum_{i=1}^{k} p_i \cdot \varphi_i$$

$\mathcal{F}$ is either $\mathcal{D}'(\mathbb{R}^n)$ or $\mathcal{C}_0^\infty(\mathbb{R}^n)$.
An $k \times \ell$ matrix M encodes a linear PDE with constant coefficients.
φ satisfies the PDE given by M if $M_i \bullet \varphi$ vanishes for all column M_i.

$$\mathrm{Sol}(M) := \left\{\varphi \in \mathcal{F}^k : m \bullet \varphi = 0 \text{ for all } m \in M\right\}$$

Example of PDE

$$M = \begin{bmatrix} \partial_1\partial_3 & \partial_1\partial_2 & \partial_1^2\partial_2 \\ \partial_1^2 & \partial_2^2 & \partial_1^2\partial_4 \end{bmatrix}$$

$$\frac{\partial^2\psi_1}{\partial z_1\partial z_3} + \frac{\partial^2\psi_2}{\partial z_1^2} = \frac{\partial^2\psi_1}{\partial z_1\partial z_2} + \frac{\partial^2\psi_2}{\partial z_2^2} = \frac{\partial^3\psi_1}{\partial z_1^2\partial z_2} + \frac{\partial^3\psi_2}{\partial z_1^2\partial z_4} = 0$$

A family of (some) solutions is

$$t^2 \begin{bmatrix} t \\ -s \end{bmatrix} \exp\left(s^2tz_1 + st^2z_2 + s^3z_3 + t^3z_4\right)$$

Exponential solutions

Given M submodule of R^k and $Ass(M) = \{P_1, \ldots, P_s\}$;
$P_i \subseteq R$ is associated to M if there exists $m \in R^k$ such that $(M : m) = P_i$.

$$V(M) = V(P_1) \cup \ldots \cup V(P_s) = V(\langle k \times k \text{ subdeterminants of } M \rangle)$$

is called the characteristic variety of M.
The arithmetic length m_i of M along P_i is the length of the largest submodule of finite length in $(R_{P_i})^k / M_{P_i}$.
If M is a P-primary ideal; its arithmetic length is $\frac{\deg(M)}{\deg(P)}$.

Exponential solutions

Lemma

Fix a $k \times l$ *matrix* $M(\partial)$*. A point* $\mathrm{u} \in \mathbb{C}^n$ *lies in* $V(M)$ *if and only if there exist constants* $c_1, \ldots, c_k \in \mathbb{C}$*, not all zero, such that*

$$\begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} \exp\left(u_1 z_1 + \cdots + u_n z_n\right) \in \mathrm{Sol}(M)$$

Proposition

The solution space $\mathrm{Sol}(M)$ contains an exponential solution $q(\mathrm{z}) \cdot \exp\left(\mathrm{u}^t \mathrm{z}\right)$ if and only if $\mathrm{u} \in V(M)$. Here q is a vector of polynomials.

Finite dimensional case

Theorem ([4])

Consider a submodule $M \subseteq R^k$. *Its solution space* $\mathrm{Sol}(M)$ *is finite-dimensional over* $\mathbb{C}$ *if and only if* $V(M)$ *has dimension* 0. *There is a basis of* $\mathrm{Sol}(M)$ *given by vectors* $q(\mathrm{z})\exp(\mathrm{u}^t\mathrm{z})$, *where* $\mathrm{u} \in V(M)$ *and* $q(\mathrm{z})$ *runs over a finite set of polynomial vectors.*
There exist polynomial solutions if and only if $\mathfrak{m} = \langle x_1, \ldots, x_n \rangle$ *is an associated prime of* M. *The polynomial solutions are found by solving the PDE given by the* $\mathfrak{m}$*-primary component of* M.

$$\dim_{\mathbb{C}} \mathrm{Sol}(M) = \dim_K \left(R^k / M \right) = \mathrm{amult}(M).$$

Theorem (Ehrenpreis–Palamodov Fundamental Principle)

Consider a module $M \subseteq R^k$. *There exist irreducible varieties* $V_1, \ldots, V_s$ *in* $\mathbb{C}^n$ *and finitely many vectors* B_{ij} *of polynomials in* $2n$ *unknowns* (x, z), *such that any solution* $\psi \in \mathcal{F}$ *admits an integral representation*

$$\psi(\mathrm{z}) = \sum_{i=1}^{s} \sum_{j=1}^{m_i} \int_{V_i} B_{ij}(\mathrm{x}, \mathrm{z}) \exp\left(\mathrm{x}^t \mathrm{z}\right) d\mu_{ij}(\mathrm{x})$$

Here μ_{ij} *is a measure supported on the variety* V_i.

Example

The ideal $I = \left\langle \partial_1^2 - \partial_2\partial_3, \partial_3^2 \right\rangle$ represents the PDE

$$\frac{\partial^2 \varphi}{\partial z_1^2} = \frac{\partial^2 \varphi}{\partial z_2 \partial z_3} \quad \text{and} \quad \frac{\partial^2 \varphi}{\partial z_3^2} = 0$$

for a scalar-valued function $\varphi = \varphi(z_1, z_2, z_3)$. Primary and $m_1 = 4$. It reveals the vectors

$$B_1 = 1, B_2 = z_1, B_3 = z_1^2 x_2 + 2z_3, B_4 = z_1^3 x_2 + 6z_1 z_3$$

$$\varphi(\mathrm{z}) = a(z_2) + z_1 b(z_2) + \left(z_1^2 c'(z_2) + 2z_3 c(z_2)\right) + \left(z_1^3 d'(z_2) + 6z_1 z_3 d(z_2)\right)$$

Noetherian operators

Denote $R = \mathbb{C}[\partial_{z_1}, \ldots, \partial_{z_n}] := \mathbb{C}[x_1, \ldots, x_n]$ $(x_i = \partial_{z_i})$.
Find the differential operators $A_{i,j}(\mathrm{x}, \partial_{\mathrm{x}})$ such that

$$m \in M \Longleftrightarrow A_{i,j} \bullet m \in P_i \text{ for all } P_i \in \mathrm{Ass}(M)$$

$A_{i,j}$ are called Noetherian operators and the list $(P_i, A_{i,\cdot})$ is called a differential primary decomposition of M.

Noetherian multiplier

Suppose M is P-primary submodule of R^k.
Consider the differential operator in the Weyl algebra

$$A(\mathrm{x}, \partial_{\mathrm{x}}) = \sum_{\mathrm{r},\mathrm{s} \in \mathbb{N}^n} c_{\mathrm{r},\mathrm{s}} x_1^{r_1} \cdots x_n^{r_n} \partial_{x_1}^{s_1} \cdots \partial_{x_n}^{s_n}, \quad \text{where } c_{\mathrm{r},\mathrm{s}} \in K$$

There is a natural K-linear isomorphism between the Weyl algebra D_n and the polynomial ring $K[\mathrm{x}, \mathrm{z}]$ which takes the operator $A(x, \partial_x)$ to the following polynomial B in $2n$ variables:

$$B(\mathrm{x}, \mathrm{z}) = \sum_{\mathrm{r},\mathrm{s} \in \mathbb{N}^n} c_{\mathrm{r},\mathrm{s}} x_1^{r_1} \cdots x_n^{r_n} z_1^{s_1} \cdots z_n^{s_n}$$

This bijection restricts to a bijection between Noetherian operators and Noetherian multipliers:

$$\mathfrak{B} := \left\{ B \in K[\mathrm{x}, \mathrm{z}] : B(\mathrm{x}, \mathrm{z}) \exp\left(\mathrm{x}^t \mathrm{z}\right) \in \mathrm{Sol}(M) \text{ for all } \mathrm{x} \in V(P) \right\}$$

Differential primary decomposition

Theorem ([2])

Every submodule M of R^k has a differential primary decomposition. We can choose the sets $\mathcal{A}_1, \dots, \mathcal{A}_s$ such that $|\mathcal{A}_i|$ is the arithmetic length of M along the prime P_i.
Moreover If $(P_1, \mathcal{A}_1), \dots, (P_s, \mathcal{A}_s)$ is any differential primary decomposition for M, then $|\mathcal{A}_i| \geq m_i$.

solvePDE algorithm

Algorithm 1 SolvePDE

Input: An arbitrary submodule M of R^k

Output: List of associated primes with corresponding Noetherian multipliers.

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for each associated prime ideal P of M do
    U ← M R_P^k ∩ R^k
    V ← (U : P^∞)
    r ← the smallest number such that V ∩ P^{r+1} R^k is a subset of U
    S ← a maximal set of independent variables modulo P
    K ← Frac(R/P)
    T ← K[y_i : x_i ∉ S]
    γ ← the map defined in (37)
    m ← the irrelevant ideal in T
    Û ← γ(U) + m^{r+1} T^k
    V̂ ← γ(V) + m^{r+1} T^k
    N ← a K-vector space basis of the space of k-tuples of polynomials of degree ≤ r
    Diff(Û) ← the matrix given by the •-product of generators of Û with elements of N
    Diff(V̂) ← the matrix given by the •-product of generators of V̂ with elements of N
    𝒦 ← ker_K(Diff(Û)) / ker_K(Diff(V̂))
    𝒜 ← a K-vector space basis of 𝒦
    ℬ ← lifts of the vectors in 𝒜 ⊂ T^k to vectors in R[dx_1, ..., dx_n]^k
    return the pair (P, ℬ)
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$$\gamma\colon R \to T, \quad x_i \mapsto \begin{cases} y_i + u_i, & \text{if } x_i \notin S, \\ u_i, & \text{if } x_i \in S. \end{cases}$$

solvePDE

Distributed in Macaulay2 with the package "NoehterianOperators" from version 1.18.

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R=QQ[x1,x2,x3];

I=ideal(x1^2-x2*x3,x3^2);

Ideal of R

solvePDE I

{{ideal (x3, x1), {| 1 |, | dx1 |, | x2dx1^2+2dx3 |, | x2dx1^3+6dx1dx3 |}}}

List
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solvePDE

$a(z_2), \;\; z_1 b(z_2),$

$$\int (x_2 z_1^2 + z_3) \exp(x_2 z_2) dx_2 = z_1^2 \partial_{z_2} \left(\int \exp(x_2 z_2) dx_2 \right) + z_3 \int \exp(z_2 x_2) dx_2$$
$$= z_1^2 c_1'(z_2) + z_3 c_2(z_2),$$

$$\int (x_2 z_1^3 + 6 z_1 z_3) \exp(x_2 z_2) dx_2 = z_1^3 \partial_{z_2} \left(\int \exp(x_2 z_2) dx_2 \right) + 6 z_1 z_3 \int \exp(z_2 x_2) dx_2$$
$$= z_1^3 d_1'(z_2) + 6 z_1 z_3 d_2(z_2),$$

Linear PDE with polynomial coefficients

Consider the Weyl algebra $D = \mathbb{C}\langle z_1, \ldots, z_n, \partial_1, \ldots, \partial_n \rangle$, and I a D-ideal.
$\text{in}_{(-w,w)}(I)$ is fixed under the action of the n-dimensional algebraic torus $(\mathbb{C}^*)^n$: $t_i \bullet x_i = \frac{1}{t_i} x_i$ and $t_i \bullet \partial_i = t_i \partial_{t_i}$.
$\text{in}_{(-w,w)}(I)$ is generated by operators $x^a p(\theta) \partial^b$ where $a, b \in \mathbb{N}^n$ where $\theta_i = z_i \partial_i$ [5, Theorem 2.3.3].
Consider

$$[\theta_b] := \prod_{i=1}^{n} \prod_{j=0}^{b_i - 1} (\theta_i - j)$$

The distraction J of $\text{in}_{(-w,w)}(I)$ is the ideal in $\mathbb{C}[\theta]$ generated by all polynomials $[\theta_b]\, p(\theta - b) = x^b p(\theta) \partial^b$.
$Sol(J) = Sol(\text{in}_{(-w,w)}(I))$ which can often be lift to $Sol(I)$.

J. Chen and Y. Cid-Ruiz.
Primary decomposition of modules: a computational differential approach, 2021.

Y. Cid-Ruiz and B. Sturmfels.
Primary decomposition with differential operators, 2021.

R. A. E. Manssour, M. Härkönen, and B. Sturmfels.
Linear pde with constant coefficients, 2021.

U. Oberst.
Finite dimensional systems of partial differential or difference equations.

M. S. S. Takayama.
Gröbner deformations of hypergeometric differential equations.