DD-finite functions

a computable extension for holonomic functions

Antonio Jiménez-Pastor

JOHANNES KEPLER UNIVERSITY LINZ

Doctoral Program
Computational Mathematics
Numerical Analysis and Symbolic Computation

ÉCOLE POLYTECHNIQUE
IP PARIS

MAX Seminar (Sep. 2021)

Joint work with: P. Nuspl and V. Pillwein

Outline

D-finite functions: the holonomic world

Basic notation

Throughout this talk we consider:

- $\mathbb{K}$: a **computable** field contained in $\mathbb{C}$.
- $\mathbb{K}[[x]]$: ring of formal power series over $\mathbb{K}$.
- $'$ is the standard derivation w.r.t. x:

$$\left(\sum_{n\geq 0} c_n x^n\right)' = \sum_{n\geq 0} (c_n x^n)' = \sum_{n\geq 0} (n+1)c_{n+1}x^n.$$

Links to package

Package dd_functions

All results presented in this talk are included in the SageMath package dd_functions.

- **Repository:** https://github.com/Antonio-JP/dd_functions
- **Documentation:** https://antonio-jp.github.io/dd_functions/
- **Demo:** https://mybinder.org/v2/gh/Antonio-JP/dd_functions.git/master?filepath=dd_functions_demo.ipynb

D-finite functions

Definition

Let $f(x) \in \mathbb{K}[[x]]$. We say that $f(x)$ is D-finite if there exists $d \in \mathbb{N}$ and polynomials $p_0(x), \ldots, p_d(x) \in \mathbb{K}[x]$ (not all zero) such that:

$$p_d(x)f^{(d)}(x) + \ldots + p_0(x)f(x) = 0.$$

Examples

Many special functions are D-finite:

- Exponential functions: e^x.
- Trigonometric functions: $\sin(x)$, $\cos(x)$.
- Logarithm function: $\log(x+1)$.
- Bessel functions: $J_n(x)$.
- Hypergeometric functions: ${}_pF_q\left(\begin{array}{c} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{array};x\right)$.
- Airy functions: $Ai(x), Bi(x)$.
- Combinatorial generating functions: $F(x), C(x), \ldots$

Closure properties

$f(x), g(x)$ D-finite of order d_1, d_2.
$a(x)$ algebraic over $\mathbb{K}(x)$ of degree p.

Property	**Function**	**Order bound**
Addition	$f(x) + g(x)$	$d_1 + d_2$
Product	$f(x)g(x)$	$d_1 d_2$
Differentiation	$f'(x)$	d_1
Integration	$\int f(x)$	$d_1 + 1$
Be Algebraic	$a(x)$	p

Working with D-finite functions

There are several implementations of D-finite functions:

- *mgfun*: Maple package, by F. Chyzak and B. Salvy
- *HolonomicFunctions*: Mathematica package, by C. Koutschan
- *ore_algebra*: Sage package, by M. Kauers et al.

Differentially definable functions

Non-D-finite examples

There are power series that **are not** D-finite:

- Double exponential: $f(x) = e^{e^x}$.
- Tangent: $\tan(x) = \frac{\sin(x)}{\cos(x)}$.

- $\wp$-Weierstrass function.
- Gamma function: $f(x) = \Gamma(x+1)$.
- Partition Generating Function: $f(x) = \sum_{n\geq 0} p(n)x^n$.

DD-finite functions

Definition

Let $f(x) \in \mathbb{K}[[x]]$. We say that $f(x)$ is D-finite if there exists $d \in \mathbb{N}$ and polynomials $p_0(x), \ldots, p_d(x) \in \mathbb{K}[x]$ (not all zero) such that:

$$p_d(x)f^{(d)}(x) + \ldots + p_0(x)f(x) = 0.$$

D-finite: **NO**

- Double exponential: $f(x) = e^{e^x}$.
- Tangent: $\tan(x) = \frac{\sin(x)}{\cos(x)}$.

DD-finite functions

Definition

Let $f(x) \in \mathbb{K}[[x]]$. We say that $f(x)$ is DD-finite if there exists $d \in \mathbb{N}$ and D-finite functions $r_0(x), \ldots, r_d(x)$ (not all zero) such that:

$$r_d(x)f^{(d)}(x) + \ldots + r_0(x)f(x) = 0.$$

DD-finite: **YES**

- Double exponential: $f(x) = e^{e^x} \rightarrow f'(x) - e^x f(x) = 0$
- Tangent: $\tan(x) = \frac{\sin(x)}{\cos(x)} \rightarrow \cos^2(x)\tan''(x) - 2\tan(x) = 0.$

Differentially definable functions

Definition

Let $R \subset \mathbb{K}[[x]]$ be a differential ring and $f(x) \in \mathbb{K}[[x]]$. We say that $f(x)$ is differentially definable over R if there exists $d \in \mathbb{N}$ and elements in R $r_0(x), \ldots, r_d(x)$ (not all zero) such that:

$$r_d(x) f^{(d)}(x) + \ldots + r_0(x) f(x) = 0.$$

We denote the set of all diff. definable functions over R by $\mathrm{D}(R)$.

- D-finite functions: $\mathrm{D}(\mathbb{K}[x])$.
- DD-finite functions: $\mathrm{D}(\mathrm{D}(\mathbb{K}[x])) = \mathrm{D}^2(\mathbb{K}[x])$.

Characterization via Linear Algebra

Theorem

The following are equivalent:

$f(x)$ is differentially definable over R ($f(x) \in \mathrm{D}(R)$)

$$\Updownarrow$$

The F-vector space $\langle f(x), f'(x), f''(x), \dots \rangle$ has finite dimension.

- $R \subset K[[x]]$ is a differential subring
- F is its field of fractions.

Closure properties

$f(x), g(x)$ D-finite of order d_1, d_2.
$a(x)$ algebraic over $\mathbb{K}(x)$ of degree p.

Property	**Function**	**Order bound**
Addition	$f(x)+g(x)$	d_1+d_2
Product	$f(x)g(x)$	d_1d_2
Differentiation	$f'(x)$	d_1
Integration	$\int f(x)$	d_1+1
Be Algebraic	$a(x)$	p

Closure properties

$f(x), g(x)$ in $D(R)$ of order d_1, d_2.
$a(x)$ algebraic over F of degree p.

Property	**Function**	**Order bound**
Addition	$f(x)+g(x)$	d_1+d_2
Product	$f(x)g(x)$	$d_1 d_2$
Differentiation	$f'(x)$	d_1
Integration	$\int f(x)$	d_1+1
Be Algebraic	$a(x)$	p

Closure properties

$f(x), g(x)$ in D(R) of order d_1, d_2.
$a(x)$ algebraic over F of degree p.

Property	**Function**	**Order bound**
Addition	$f(x) + g(x)$	$d_1 + d_2$
Product	$f(x)g(x)$	$d_1 d_2$
Differentiation	$f'(x)$	d_1
Integration	$\int f(x)$	$d_1 + 1$
Be Algebraic	$a(x)$	p

Proof for addition:

$$\begin{aligned}\langle (f+g)^{(n)} \ : \ n \in \mathbb{N}\rangle_F &= \langle f^{(n)} + g^{(n)} \ : \ n \in \mathbb{N}\rangle_F \\ &\subset \langle f^{(n)} \ : \ n \in \mathbb{N}\rangle_F + \langle g^{(n)} \ : \ n \in \mathbb{N}\rangle_F\end{aligned}$$

Closure properties

$f(x), g(x)$ in $D(R)$ of order d_1, d_2.
$a(x)$ algebraic over F of degree p.

Property	**Function**	**Order bound**
Addition	$f(x)+g(x)$	d_1+d_2
Product	$f(x)g(x)$	$d_1 d_2$
Differentiation	$f'(x)$	d_1
Integration	$\int f(x)$	d_1+1
Be Algebraic	$a(x)$	p

Proof for addition:

$$\langle (f+g)^{(n)} \ : \ n \in \mathbb{N}\rangle_F = \langle f^{(n)} + g^{(n)} \ : \ n \in \mathbb{N}\rangle_F$$
$$\subset \langle f^{(n)} \ : \ n \in \mathbb{N}\rangle_F + \langle g^{(n)} \ : \ n \in \mathbb{N}\rangle_F$$

D^n-finite functions: iterating the process

D^n-finite functions

Remark

$$R \subset \mathbb{K}[[x]] \text{ diff. ring} \Rightarrow \mathrm{D}(R) \subset \mathbb{K}[[x]] \text{ diff. ring}$$

$$\downarrow$$

Iterate the process

D^n-finite functions

Remark

$$R \subset \mathbb{K}[[x]] \text{ diff. ring} \Rightarrow \mathrm{D}(R) \subset \mathbb{K}[[x]] \text{ diff. ring}$$

$$\downarrow$$

Iterate the process

D^n-finite functions

D^n-finite functions are the *n*th iteration over $\mathbb{K}[x]$, i.e., $\mathrm{D}^n(\mathbb{K}[x])$.

$$\mathbb{K}[x] \subset \mathrm{D}(\mathbb{K}[x]) \subset \mathrm{D}^2(\mathbb{K}[x]) \subset \ldots \subset \mathrm{D}^n(\mathbb{K}[x]) \subset \ldots$$

New Properties

$f(x) \in D^n(\mathbb{K}[x])$ of order d_1.
$g(x) \in D^m(\mathbb{K}[x])$ of order d_2.
$a(x)$ algebraic over $D^m(\mathbb{K}[x])$ of degree p.

Property	**Function**	**Is in**	**Order bound**
Composition	$f \circ g$	$D^{n+m}(\mathbb{K}[x])$	d_1
Alg. subs.	$f \circ a$	$D^{n+m}(\mathbb{K}[x])$	pd_1

New Properties

$f(x) \in D^n(\mathbb{K}[x])$ of order d_1.
$g(x) \in D^m(\mathbb{K}[x])$ of order d_2.
$a(x)$ algebraic over $D^m(\mathbb{K}[x])$ of degree p.

Property	**Function**	**Is in**	**Order bound**
Composition	$f \circ g$	$D^{n+m}(\mathbb{K}[x])$	d_1
Alg. subs.	$f \circ a$	$D^{n+m}(\mathbb{K}[x])$	pd_1

- $a(x)$ algebraic over $D^m(K[x])$ implies $a(x) \in D^{m+1}(\mathbb{K}[x])$.

New Properties

$f(x) \in D^n(\mathbb{K}[x])$ of order d_1.
$g(x) \in D^m(\mathbb{K}[x])$ of order d_2.
$a(x)$ algebraic over $D^m(\mathbb{K}[x])$ of degree p.

Property	**Function**	**Is in**	**Order bound**
Composition	$f \circ g$	$D^{n+m}(\mathbb{K}[x])$	d_1
Alg. subs.	$f \circ a$	$D^{n+m}(\mathbb{K}[x])$	pd_1

- $a(x)$ algebraic over $D^m(K[x])$ implies $a(x) \in D^{m+1}(\mathbb{K}[x])$.
- Then $f(a(x))$ is in $D^{n+m+1}(\mathbb{K}[x])$.

New Properties

$f(x) \in D^n(\mathbb{K}[x])$ of order d_1.
$g(x) \in D^m(\mathbb{K}[x])$ of order d_2.
$a(x)$ algebraic over $D^m(\mathbb{K}[x])$ of degree p.

Property	**Function**	**Is in**	**Order bound**
Composition	$f \circ g$	$D^{n+m}(\mathbb{K}[x])$	d_1
Alg. subs.	$f \circ a$	$D^{n+m}(\mathbb{K}[x])$	pd_1

- $a(x)$ algebraic over $D^m(K[x])$ implies $a(x) \in D^{m+1}(\mathbb{K}[x])$.
- Then $f(a(x))$ is in $D^{n+m+1}(\mathbb{K}[x])$.

$D^n \subsetneq D^{n+1}$: Iterated exponentials

$$K[x] \subsetneq D(K[x]) \subset D^2(K[x]) \subset \ldots \subset D^n(K[x]) \subset \ldots$$

$$e^x \notin K[x]$$

$D^n \subsetneq D^{n+1}$: Iterated exponentials

$$K[x] \subsetneq D(K[x]) \subsetneq D^2(K[x]) \subset \ldots \subset D^n(K[x]) \subset \ldots$$

$$e^x \notin K[x], \qquad e^{e^x-1} \notin D(K[x]).$$

$\mathrm{D}^n \subsetneq \mathrm{D}^{n+1}$: Iterated exponentials

$$K[x] \subsetneq \mathrm{D}(K[x]) \subsetneq \mathrm{D}^2(K[x]) \subset \ldots \subset \mathrm{D}^n(K[x]) \subset \ldots$$

$$e^x \notin K[x], \qquad e^{e^x - 1} \notin \mathrm{D}(K[x]).$$

$D^n \subsetneq D^{n+1}$: Iterated exponentials

$$K[x] \subsetneq D(K[x]) \subsetneq D^2(K[x]) \subset \ldots \subset D^n(K[x]) \subset \ldots$$

$$e^x \notin K[x], \qquad e^{e^x-1} \notin D(K[x]).$$

Iterated Exponentials

- $e_0(x) = 1$,
- $\hat{e}_n(x) = \int_0^x e_n(t)dt$,
- $e_{n+1}(x) = \exp(\hat{e}_n(x))$.

$D^n \subsetneq D^{n+1}$: Iterated exponentials

$$K[x] \subsetneq D(K[x]) \subsetneq D^2(K[x]) \subsetneq \ldots \subsetneq D^n(K[x]) \subsetneq \ldots$$

$$e^x \notin K[x], \qquad e^{e^x - 1} \notin D(K[x]).$$

Iterated Exponentials

- $e_0(x) = 1$,
- $\hat{e}_n(x) = \int_0^x e_n(t)dt$,
- $e_{n+1}(x) = \exp(\hat{e}_n(x))$.

$$\begin{cases} e_{n+1}(x) \in D^{n+1}(K[x]) \\ e_{n+1}(x) \notin D^n(K[x]) \end{cases}$$

Diff. Algebraic functions

Definition

Let $R \subset \mathbb{K}[[x]]$ be a differential ring and $f(x) \in \mathbb{K}[[x]]$. We say that $f(x)$ differentially algebraic over R if there is $n \in \mathbb{N}$ and $P(y_0, \ldots, y_n) \in R[y_0, \ldots, y_n]$ such that

$$P(f(x), f'(x), \ldots, f^{(n)}(x)) = 0.$$

We denote by $\mathrm{DA}(R)$ the set of all differentially algebraic functions over R.

Diff. Algebraic functions

Definition

Let $R \subset \mathbb{K}[[x]]$ be a differential ring and $f(x) \in \mathbb{K}[[x]]$. We say that $f(x)$ differentially algebraic over R if there is $n \in \mathbb{N}$ and $P(y_0, \ldots, y_n) \in R[y_0, \ldots, y_n]$ such that

$$P(f(x), f'(x), \ldots, f^{(n)}(x)) = 0.$$

We denote by $\mathrm{DA}(R)$ the set of all differentially algebraic functions over R.

Diff. definable $\mathrm{D}(R)$	Diff. algebraic $\mathrm{DA}(R)$
$\downarrow$	$\downarrow$
Linear diff. equations	Polynomial diff. equations

Inclusion into Diff. Algebraic

- $\mathrm{D}(R) \subset \mathrm{DA}(R)$.
- $R \subset S \Rightarrow \mathrm{DA}(R) \subset \mathrm{DA}(S)$.
- $\mathrm{DA}(\mathbb{K}[x]) = \mathrm{DA}(\mathbb{K})$.

Inclusion into Diff. Algebraic

- $\mathrm{D}(R) \subset \mathrm{DA}(R)$.
- $R \subset S \Rightarrow \mathrm{DA}(R) \subset \mathrm{DA}(S)$.
- $\mathrm{DA}(\mathbb{K}[x]) = \mathrm{DA}(\mathbb{K})$.

Proposition

Let $R \subset \mathbb{K}[[x]]$ be a differential ring. Then $\mathrm{DA}(\mathrm{D}(R)) = \mathrm{DA}(R)$.

The proof is constructive.

Inclusion into Diff. Algebraic

- $D(R) \subset DA(R)$.
- $R \subset S \Rightarrow DA(R) \subset DA(S)$.
- $DA(\mathbb{K}[x]) = DA(\mathbb{K})$.
- **Proposition:** $DA(D^n(R)) = DA(D^{n-1}(R))$.

Theorem

For all $n \in \mathbb{N}$, if $f(x) \in D^n(\mathbb{K}[x])$, then $f(x) \in DA(\mathbb{K})$.

Inclusion into Diff. Algebraic

- $D(R) \subset DA(R)$.
- $R \subset S \Rightarrow DA(R) \subset DA(S)$.
- $DA(\mathbb{K}[x]) = DA(\mathbb{K})$.
- **Proposition:** $DA(D^n(R)) = DA(D^{n-1}(R))$.
- **Theorem:** $D^n(\mathbb{K}[x]) \subset DA(\mathbb{K})$.

Example: double exponential

$$\exp(\exp(x) - 1) \longrightarrow f'(x) - \exp(x) f(x) = 0$$

$$\downarrow$$

$$f''(x) f(x) - f'(x)^2 - f'(x) f(x) = 0$$

Inclusion into Diff. Algebraic

- $\mathrm{D}(R) \subset \mathrm{DA}(R)$.
- $R \subset S \Rightarrow \mathrm{DA}(R) \subset \mathrm{DA}(S)$.
- $\mathrm{DA}(\mathbb{K}[x]) = \mathrm{DA}(\mathbb{K})$.
- **Proposition:** $\mathrm{DA}(\mathrm{D}^n(R)) = \mathrm{DA}(\mathrm{D}^{n-1}(R))$.
- **Theorem:** $\mathrm{D}^n(\mathbb{K}[x]) \subset \mathrm{DA}(\mathbb{K})$.

Example: tangent

$$\tan(x) \longrightarrow \cos(x)^2 f''(x) - 2f(x) = 0$$

$$\downarrow$$

$$-2f^{(5)}(x)f''(x)^2f(x) + 12f^{(4)}(x)f'''(x)f''(x)f(x) - 6f^{(4)}(x)f''(x)^2f'(x) - 12f'''(x)^3f(x) + 12f'''(x)^2f''(x)f'(x) - 4f'''(x)f''(x)^3 - 8f'''(x)f''(x)^2f(x) + 8f''(x)^3f'(x) = 0$$

Reverse inclusion

Is the other inclusion true? Can we have $DA(\mathbb{K}[x]) = D^{\infty}(\mathbb{K}[x])$?

Reverse inclusion

Is the other inclusion true? Can we have $DA(\mathbb{K}[x]) = D^\infty(\mathbb{K}[x])$?

For some diff. algebraic functions, we can find an $n \in \mathbb{N}$:

Riccati differential equation

Let $y(x)$ be a solution to the Riccati differential equation

$$y'(x) = c(x)y(x)^2 + b(x)y(x) + a(x),$$

where $a(x), b(x) \in D^n(\mathbb{K}[x])$ and $c(x) \in D^{n-1}(\mathbb{K}[x])$.
Then $y(x) \in D^{n+2}(\mathbb{K}[x])$.

Reverse inclusion

Is the other inclusion true? Can we have $\mathrm{DA}(\mathbb{K}[x]) = \mathrm{D}^\infty(\mathbb{K}[x])$?

But that is not always the case

Theorem (Noordman, Top, van der Put)

Let $y(x)$ be a non-constant solution to the differential equation

$$y'(x) = y(x)^3 - y(x)^2.$$

Then, there is no $n \in \mathbb{N}$ with $y(x) \in \mathrm{D}^n(\mathbb{K}[x])$.

Simple functions: handling singularities

Singularities on differential equations

Zero and Singular set

Let $f(x) \in \mathbb{K}[[x]]$:

- Zero set: $Z(f) = \{\alpha \in \mathbb{C} \ : \ f(\alpha) = 0\}$.
- Singular set: $S(f) = \{\alpha \in \mathbb{C} \ : \ \alpha \text{ singularity of } f\}$.

Theorem

Let $f(x) \in \mathbb{K}[[x]]$ that satisfy the linear diff. equation

$$r_d(x)f^{(d)}(x) + \ldots + r_0(x)f(x) = 0,$$

for some $r_0(x), \ldots, r_d(x) \in \mathbb{K}[[x]]$. Then:

$$S(f) \subseteq Z(r_d) \bigcup_{i=0}^{d} S(r_i).$$

Singularities on differential equations

Zero and Singular set

Let $f(x) \in \mathbb{K}[[x]]$:

- Zero set: $Z(f) = \{\alpha \in \mathbb{C} \;:\; f(\alpha) = 0\}$.
- Singular set: $S(f) = \{\alpha \in \mathbb{C} \;:\; \alpha \text{ singularity of } f\}$.

Theorem

Let $f(x) \in \mathbb{K}[[x]]$ that satisfy the linear diff. equation

$$r_d(x)f^{(d)}(x) + \ldots + r_0(x)f(x) = 0,$$

for some $r_0(x), \ldots, r_d(x) \in \mathbb{K}[[x]]$. Then:

$$S(f) \subseteq Z(r_d) \bigcup_{i=0}^{d} S(r_i).$$

Creating new singularities

Closure properties computations may create **new** singularities

Creating new singularities

Closure properties computations may create **new** singularities

Adding two D-finite functions I

- $e^x \longrightarrow \partial - 1$.
- $Ai(x) \longrightarrow \partial^2 - x$.
- $e^x + Ai(x)$:

$$(x-1)\partial^3 - x\partial^2 - (x^2 - x)\partial + (x^2 - x + 1)$$

Creating new singularities

Closure properties computations may create **new** singularities

Adding two D-finite functions I

- $e^x \longrightarrow \partial - 1$.
- $Ai(x) \longrightarrow \partial^2 - x$.
- $e^x + Ai(x)$:

$$(x-1)\partial^3 - x\partial^2 - (x^2 - x)\partial + (x^2 - x + 1)$$

Creating new singularities

Closure properties computations may create **new** singularities

Adding two D-finite functions II

- $\log(x+1) \longrightarrow (x+1)\partial^2 + \partial$.
- $\sin(x) \longrightarrow \partial^2 + 1$.
- $\log(x+1) + \sin(x)$:

$$(x+1)(x^2+2x+3)\partial^4 + (x^2+2x+7)\partial^3 + \\ (x+1)(x^2+2x+3)\partial^2 + (x^2+2x+7)\partial$$

Creating new singularities

Closure properties computations may create **new** singularities

Adding two D-finite functions II

- $\log(x+1) \longrightarrow (x+1)\partial^2 + \partial.$
- $\sin(x) \longrightarrow \partial^2 + 1.$
- $\log(x+1) + \sin(x)$:

$$(x+1)(x^2+2x+3)\partial^4 + (x^2+2x+7)\partial^3 + \\ (x+1)(x^2+2x+3)\partial^2 + (x^2+2x+7)\partial$$

Creating new singularities

Closure properties computations may create **new** singularities

Multiplying two D-finite functions

- $Ai(x+1) \longrightarrow \partial^2 - x$.
- $J_1(x) \longrightarrow x^2\partial^2 + x\partial + (x-1)^2$.
- $Ai(x)J_1(x)$:

$$x^2(4x^3+4x^2-3)\partial^4 + 4x(x-1)(x^2+3x+3)\partial^3 - \\ 2x^2(4x^4-4x^2+6x+9)\partial^2 - \\ 2x^2(6x^3+10x^2-4x-27)\partial - \\ (4x^7+12x^6+12x^5-x^4-28x^3-11x^2+6x-9)$$

Desingularization

Desingularization

- **INPUT:** a differential equation $\mathcal{L} \cdot f(x) = 0$.
- **OUTPUT:** a differential equation $\tilde{\mathcal{L}} \cdot f(x) = 0$ such that:
  1. For all $g(x)$ with $\mathcal{L} \cdot g(x) = 0$, $\tilde{\mathcal{L}} \cdot g(x) = 0$.
  2. $\tilde{\mathcal{L}}$ has no apparent singularities.

Previous work on desingularization

- For differential systems over $\mathbb{Q}(x)$: M. Barkatou et al.
- For Ore Operators: S. Chen, M. Jaroschek, M. Kauers, M. F. Singer

Input $\rightarrow$ Closure property $\rightarrow$ Desing. $\rightarrow$ Output w/o sing.

Desingularization

Desingularization

- **INPUT:** a differential equation $\mathcal{L} \cdot f(x) = 0$.
- **OUTPUT:** a differential equation $\tilde{\mathcal{L}} \cdot f(x) = 0$ such that:
  1. For all $g(x)$ with $\mathcal{L} \cdot g(x) = 0$, $\tilde{\mathcal{L}} \cdot g(x) = 0$.
  2. $\tilde{\mathcal{L}}$ has no apparent singularities.

Our approach

Being able to obtain **directly** through closure properties an operator that has **no new singularities** by using *only* linear algebra.

Input $\rightarrow$ Closure property $\rightarrow$ Output w/o new sing.

Two key concepts

Noetherian modules

M is a Noetherian module if all its submodules are finitely generated.

Localization ring

Given a multiplicatively closed set $S \subset R$, the localized ring of R over S (denoted by R_S) is the minimal extension to R where we can divide by elements in S.

S-simple diff. definable functions

Definition

Let $R \subset \mathbb{K}[[x]]$ a differential subring and $S \subset R$ multiplicatively closed. We say that $f(x) \in \mathbb{K}[[x]]$ is S-simple differentially definable over R if there are $r_0(x), \ldots, r_{d-1}(x) \in R$ and $s(x) \in S$ such that:

$$s(x)f^{(d)}(x) + r_{d-1}(x)f^{(d-1)}(x) + \ldots + r_0(x)f(x) = 0.$$

We denote the set of all these functions by $\mathrm{D}(R, S)$.

S-simple diff. definable functions

Definition

Let $R \subset \mathbb{K}[[x]]$ a differential subring and $S \subset R$ multiplicatively closed. We say that $f(x) \in \mathbb{K}[[x]]$ is S-simple differentially definable over R if there are $r_0(x), \ldots, r_{d-1}(x) \in R$ and $s(x) \in S$ such that:

$$s(x)f^{(d)}(x) + r_{d-1}(x)f^{(d-1)}(x) + \ldots + r_0(x)f(x) = 0.$$

We denote the set of all these functions by $\mathrm{D}(R, S)$.

D-finite case

The set S controls the singularities of the functions in $\mathrm{D}(\mathbb{K}[x], S)$!!

S-simple diff. definable functions

D-finite case

The set S controls the singularities of the functions in $D(\mathbb{K}[x], S)$!!

Consider $f(x) \in D(\mathbb{K}[x], S)$ for the following sets:

1. $S = \mathbb{K}$: $f(x)$ is analytic in $\mathbb{C}$.
2. $S = \{p(x)^n \ : \ n \in \mathbb{N}\}$: $f(x)$ can only have singularities in the zeros of $p(x)$.
3. $S = \mathbb{K}[x] \setminus \wp$ where $\wp$ is a prime ideal. Then $f(x)$ **does not** have singularities in any $\alpha \in V(\wp)$.

Adapting the main characterization

Characterization theorem - diff. def. case

Let $R \subset \mathbb{K}[[x]]$ and $f(x) \in \mathbb{K}[[x]]$. It is equivalent:

1. $f(x) \in \mathrm{D}(R)$
2. The following $\mathrm{Fr}(R)$-vector space has finite dimension

$$\langle f, f'(x), f''(x), \ldots \rangle$$

Adapting the main characterization

Characterization theorem - S-simple case

Let $R \subset \mathbb{K}[[x]]$, $S \subset R$ m. c. and $f(x) \in \mathbb{K}[[x]]$. It is equivalent:

1. $f(x) \in \mathrm{D}(R, S)$
2. The following R_S-module is finitely generated

$$\langle f, f'(x), f''(x), \ldots \rangle$$

Addition is closed

Theorem (addition)

Let $R \subset \mathbb{K}[[x]]$ be Noetherian, $S \subset R$ m.c. and $f(x), g(x) \in \mathrm{D}(R,S)$. Then $f(x) + g(x)$ is again in $\mathrm{D}(R,S)$

Addition is closed

Theorem (addition)

Let $R \subset \mathbb{K}[[x]]$ be Noetherian, $S \subset R$ m.c. and $f(x), g(x) \in \mathrm{D}(R, S)$. Then $f(x) + g(x)$ is again in $\mathrm{D}(R, S)$

Proof:
Let $M(f) = \langle f(x), \ldots, f^{(n)}(x) \rangle_{R_S}$.

Addition is closed

Theorem (addition)

Let $R \subset \mathbb{K}[[x]]$ be Noetherian, $S \subset R$ m.c. and $f(x), g(x) \in \mathrm{D}(R, S)$. Then $f(x) + g(x)$ is again in $\mathrm{D}(R, S)$

Proof:

Let $M(f) = \langle f(x), \dots, f^{(n)}(x) \rangle_{R_S}$.

$$M(f+g) \subset M(f) + M(g)$$

Addition is closed

Theorem (addition)

Let $R \subset \mathbb{K}[[x]]$ be Noetherian, $S \subset R$ m.c. and $f(x), g(x) \in \mathrm{D}(R, S)$. Then $f(x) + g(x)$ is again in $\mathrm{D}(R, S)$

Proof:

Let $M(f) = \langle f(x), \ldots, f^{(n)}(x) \rangle_{R_S}$.

$$M(f+g) \subset M(f) + M(g)$$

$$R \text{ Noetherian} \Rightarrow M(f), M(g) \text{ Noetherian} \Rightarrow M(f) + M(g) \text{ Noetherian}$$

Addition is closed

Theorem (addition)

Let $R \subset \mathbb{K}[[x]]$ be Noetherian, $S \subset R$ m.c. and $f(x), g(x) \in D(R,S)$. Then $f(x) + g(x)$ is again in $D(R,S)$

Proof:
Let $M(f) = \langle f(x), \ldots, f^{(n)}(x) \rangle_{R_S}$.

$$M(f+g) \subset M(f) + M(g)$$

$$R \text{ Noetherian} \Rightarrow M(f), M(g) \text{ Noetherian} \Rightarrow M(f) + M(g) \text{ Noetherian}$$

Hence, $M(f+g)$ is fin. generated and $f(x) + g(x) \in D(R,S)$.

Closure properties

Theorem (product)

Let $R \subset \mathbb{K}[[x]]$ be Noetherian, $S \subset R$ m.c. and $f(x), g(x) \in \mathrm{D}(R, S)$. Then $f(x)g(x)$ is again in $\mathrm{D}(R, S)$

Theorem (derivation)

Let $R \subset \mathbb{K}[[x]]$ be Noetherian, $S \subset R$ m.c. and $f(x) \in \mathrm{D}(R, S)$. Then $f'(x)$ is again in $\mathrm{D}(R, S)$

Theorem (integration)

Let $R \subset \mathbb{K}[[x]]$ be Noetherian, $S \subset R$ m.c. and $f(x) \in \mathrm{D}(R, S)$. Then for any $F(x)$ with $F'(x) = f(x)$, $F(x) \in \mathrm{D}(R, S)$.

Closure properties

Theorem

Let $R \subset \mathbb{K}[[x]]$ be Noetherian and $S \subset R$ multiplicatively closed. Then $D(R, S)$ is a differential extension of R closed under integration.

$f(x), g(x)$ S-simple D-finite of order d_1, d_2.

	in $D(R, S)$	**Order bound**
Addition	$(f+g)$	Unknown
Product	(fg)	Unknown
Differentiation	f'	Unknown
Integration	$\int f$	$d_1 + 1$

Closure properties examples

Adding two D-finite functions I

- $e^x \longrightarrow \partial - 1$.
- $Ai(x) \longrightarrow \partial^2 - x$.
- $e^x + Ai(x)$:

$$(x-1)\partial^3 - x\partial^2 - (x^2 - x)\partial + (x^2 - x + 1)$$

Closure properties examples

Adding two D-finite functions I

- $e^x \longrightarrow \partial - 1$.
- $Ai(x) \longrightarrow \partial^2 - x$.
- $e^x + Ai(x)$:

$$(x-1)\partial^3 - x\partial^2 - (x^2 - x)\partial + (x^2 - x + 1)$$
$$\downarrow$$
$$\partial^4 - (x^2+1)\partial^3 + (x^2+1)\partial^2 + (x^3 + x - 2)\partial - (x^3 - x - 1)$$

Closure properties examples

Adding two D-finite functions II

- $\log(x+1) \longrightarrow (x+1)\partial^2 + \partial$.
- $\sin(x) \longrightarrow \partial^2 + 1$.
- $\log(x+1) + \sin(x)$:

$$(x+1)(x^2+2x+3)\partial^4 + (x^2+2x+7)\partial^3 + \\ (x+1)(x^2+2x+3)\partial^2 + (x^2+2x+7)\partial$$

Closure properties examples

Adding two D-finite functions II

- $\log(x+1) \longrightarrow (x+1)\partial^2 + \partial$.
- $\sin(x) \longrightarrow \partial^2 + 1$.
- $\log(x+1) + \sin(x)$:

$$(x+1)(x^2+2x+3)\partial^4 + (x^2+2x+7)\partial^3 + (x+1)(x^2+2x+3)\partial^2 + (x^2+2x+7)\partial$$
$$\downarrow$$
$$4(x+1)^2\partial^5 + p_4(x)\partial^4 + p_3(x)\partial^3 + p_2(x)\partial^2 + p_1(x)\partial,$$
$$\deg(p_i(x)) \leq 5$$

Closure properties examples

Multiplying two D-finite functions

- $Ai(x+1) \longrightarrow \partial^2 - x$.
- $J_1(x) \longrightarrow x^2\partial^2 + x\partial + (x-1)^2$.
- $Ai(x)J_1(x)$:

$$x^2(4x^3+4x^2-3)\partial^4 + 4x(x-1)(x^2+3x+3)\partial^3 - \\ 2x^2(4x^4-4x^2+6x+9)\partial^2 - \\ 2x^2(6x^3+10x^2-4x-27)\partial - \\ (4x^7+12x^6+12x^5-x^4-28x^3-11x^2+6x-9)$$

Closure properties examples

Multiplying two D-finite functions

- $Ai(x+1) \longrightarrow \partial^2 - x$.
- $J_1(x) \longrightarrow x^2\partial^2 + x\partial + (x-1)^2$.
- $Ai(x)J_1(x)$:

$$195x^3\partial^5 + p_4(x)\partial^4 + p_3(x)\partial^3 + p_2(x)\partial^2 + p_1(x)\partial$$
$$\deg(p_i(x)) \leq 12$$

Simple DD-finite functions

- All results for S-simple $\mathrm{D}(R)$ require R Noetherian.
- D-finite functions are not Noetherian.
- Can we extend the result to DD-finite functions?

Simple DD-finite functions

- All results for S-simple $D(R)$ require R Noetherian.
- D-finite functions are not Noetherian.
- Can we extend the result to DD-finite functions?

YES!

Instead of working with the whole $D(\mathbb{K}[x])$, we restrict to a smaller Noetherian subring.

Simple DD-finite functions

- All results for S-simple $D(R)$ require R Noetherian.
- D-finite functions are not Noetherian.
- Can we extend the result to DD-finite functions?

Definition

Let $S \subset \mathbb{K}[[x]]$ be a multiplicatively closed set. We denote the set of S-simple D^n-finite functions with $D^n(\mathbb{K}[x], S)$ and we define it recursively by:

- $D^1(\mathbb{K}[x], S) = D(\mathbb{K}[x], \mathbb{K}[x] \cap S)$.
- $D^n(\mathbb{K}[x], S) = D(D^{n-1}(\mathbb{K}[x], S), D^{n-1}(\mathbb{K}[x], S) \cap S)$.

Intuitively, we control the leading coefficient of the equations that define the elements in the whole chain of rings.

Conclusions

Achievements

- Extension of the holonomic framework.
- Running implementation of closure properties.
- Relation to differentially algebraic functions.
- Control of singularities throughout closure properties.

Future work

- Fast computation of truncation of D^n-finite functions.
- Development of certified numerical evaluations.
- Combinatorial meaning of the induced sequences.
- Multivariate DD-finite functions.

Thank you!

Contact webpage:

- http://www.lix.polytechnique.fr/~jimenezpastor/
- https://www.dk-compmath.jku.at/people/antonio

Code available:

- https://github.com/Antonio-JP/dd_functions