DD-finite functions

a computable extension for holonomic functions

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Joint work with: P. Nuspl and V. Pillwein
D-finite functions: the holonomic world
Throughout this talk we consider:

- $\mathbb{K}$: a **computable** field contained in $\mathbb{C}$.
- $\mathbb{K}[[x]]$: ring of formal power series over $\mathbb{K}$.
- $'$ is the standard derivation w.r.t. $x$:

\[
\left( \sum_{n \geq 0} c_n x^n \right)' = \sum_{n \geq 0} (c_n x^n)' = \sum_{n \geq 0} (n+1)c_{n+1}x^n.
\]
All results presented in this talk are included in the SageMath package dd_functions.

- **Repository:**
  https://github.com/Antonio-JP/dd_functions

- **Documentation:**
  https://antonio-jp.github.io/dd_functions/

- **Demo:**
D-finite functions

**Definition**

Let $f(x) \in \mathbb{K}[[x]]$. We say that $f(x)$ is **D-finite** if there exists $d \in \mathbb{N}$ and polynomials $p_0(x), \ldots, p_d(x) \in \mathbb{K}[x]$ (not all zero) such that:

$$p_d(x)f^{(d)}(x) + \ldots + p_0(x)f(x) = 0.$$
Many **special functions** are D-finite:

- **Exponential functions**: $e^x$.
- **Trigonometric functions**: $\sin(x), \cos(x)$.
- **Logarithm function**: $\log(x + 1)$.
- **Bessel functions**: $J_n(x)$.
- **Hypergeometric functions**: $\,_{p}F_{q} \left( \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array}; x \right)$.
- **Airy functions**: $Ai(x), Bi(x)$.
- **Combinatorial generating functions**: $F(x), C(x), \ldots$
Closure properties

\(f(x), g(x)\) D-finite of order \(d_1, d_2\).
\(a(x)\) algebraic over \(\mathbb{K}(x)\) of degree \(p\).

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There are several implementations of D-finite functions:

- `mgfun`: Maple package, by F. Chyzak and B. Salvy
- `HolonomicFunctions`: Mathematica package, by C. Koutschan
- `ore_algebra`: Sage package, by M. Kauers et al.
Differentially definable functions
Non-D-finite examples

There are power series that are not D-finite:

- Double exponential: $f(x) = e^{e^x}$.
- Tangent: $\tan(x) = \frac{\sin(x)}{\cos(x)}$.
- ℘-Weierstrass function.
- Gamma function: $f(x) = \Gamma(x + 1)$.
- Partition Generating Function: $f(x) = \sum_{n \geq 0} p(n)x^n$. 

DD-finite functions
**DD-finite functions**

**Definition**

Let $f(x) \in \mathbb{K}[[x]]$. We say that $f(x)$ is **D-finite** if there exists $d \in \mathbb{N}$ and polynomials $p_0(x), \ldots, p_d(x) \in \mathbb{K}[x]$ (not all zero) such that:

$$p_d(x)f^{(d)}(x) + \ldots + p_0(x)f(x) = 0.$$ 

**D-finite: NO**

- Double exponential: $f(x) = e^{e^x}$.
- Tangent: $\tan(x) = \frac{\sin(x)}{\cos(x)}$. 

**Simple functions**

**Conclusions**
**DD-finite functions**

**Definition**

Let $f(x) \in \mathbb{K}[[x]]$. We say that $f(x)$ is **DD-finite** if there exists $d \in \mathbb{N}$ and **D-finite functions** $r_0(x), \ldots, r_d(x)$ (not all zero) such that:

$$r_d(x)f^{(d)}(x) + \ldots + r_0(x)f(x) = 0.$$ 

**DD-finite: YES**

- **Double exponential**: $f(x) = e^{e^x} \rightarrow f'(x) - e^x f(x) = 0$
- **Tangent**: $\tan(x) = \frac{\sin(x)}{\cos(x)} \rightarrow \cos^2(x) \tan''(x) - 2 \tan(x) = 0$. 
Differentially definable functions

Definition

Let $R \subset \mathbb{K}[x]$ be a differential ring and $f(x) \in \mathbb{K}[x]$. We say that $f(x)$ is differentially definable over $R$ if there exists $d \in \mathbb{N}$ and elements in $R$ $r_0(x), \ldots, r_d(x)$ (not all zero) such that:

$$r_d(x)f^{(d)}(x) + \ldots + r_0(x)f(x) = 0.$$

We denote the set of all diff. definable functions over $R$ by $D(R)$.

- D-finite functions: $D(\mathbb{K}[x])$.
- DD-finite functions: $D(D(\mathbb{K}[x])) = D^2(\mathbb{K}[x])$. 
Characterization via Linear Algebra

Theorem

The following are equivalent:

\[ f(x) \text{ is differentially definable over } R \ (f(x) \in D(R)) \]

\[ \Leftrightarrow \]

The \( F \)-vector space \( \langle f(x), f'(x), f''(x), \ldots \rangle \) has finite dimension.

- \( R \subset K[[x]] \) is a differential subring
- \( F \) is its field of fractions.
Closure properties

\( f(x), g(x) \) D-finite of order \( d_1, d_2 \).
\( a(x) \) algebraic over \( \mathbb{K}(x) \) of degree \( p \).

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\(f(x), g(x)\) in \(D(R)\) of order \(d_1, d_2\).
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Proof for addition:

\[
\langle (f + g)^{(n)} : n \in \mathbb{N} \rangle_F = \langle f^{(n)} + g^{(n)} : n \in \mathbb{N} \rangle_F
\]
\[
\subset \langle f^{(n)} : n \in \mathbb{N} \rangle_F + \langle g^{(n)} : n \in \mathbb{N} \rangle_F
\]
Closure properties

$f(x), g(x)$ in $D(R)$ of order $d_1, d_2$.
$a(x)$ algebraic over $F$ of degree $p$.

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$$\subset \langle f^{(n)} : n \in \mathbb{N} \rangle_F + \langle g^{(n)} : n \in \mathbb{N} \rangle_F$$
$D^n$-finite functions: iterating the process
Remark

\[ R \subset \mathbb{K}[x] \text{ diff. ring } \Rightarrow D(R) \subset \mathbb{K}[x] \text{ diff. ring} \]

\[ \downarrow \]

Iterate the process
**D^n-finite functions**

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Iterate the process

**D^n-finite functions**

D^n-finite functions are the nth iteration over \( \mathbb{K}[x] \), i.e., \( D^n(\mathbb{K}[x]) \).

\[ \mathbb{K}[x] \subset D(\mathbb{K}[x]) \subset D^2(\mathbb{K}[x]) \subset \ldots \subset D^n(\mathbb{K}[x]) \subset \ldots \]
New Properties

\( f(x) \in D^n(\mathbb{K}[x]) \) of order \( d_1 \).
\( g(x) \in D^m(\mathbb{K}[x]) \) of order \( d_2 \).
\( a(x) \) algebraic over \( D^m(\mathbb{K}[x]) \) of degree \( p \).

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New Properties

\[ f(x) \in D^n(K[x]) \text{ of order } d_1. \]
\[ g(x) \in D^m(K[x]) \text{ of order } d_2. \]
\[ a(x) \text{ algebraic over } D^m(K[x]) \text{ of degree } p. \]

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- \( a(x) \) algebraic over \( D^m(K[x]) \) implies \( a(x) \in D^{m+1}(K[x]). \)
New Properties

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- \( a(x) \) algebraic over \( D^m(K[x]) \) implies \( a(x) \in D^{m+1}(\mathbb{K}[x]). \)
- Then \( f(a(x)) \) is in \( D^{n+m+1}(\mathbb{K}[x]). \)
New Properties

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- \(a(x)\) algebraic over \(D^m(K[x])\) implies \(a(x) \in D^{m+1}(K[x]).\)
- Then \(f(a(x))\) is in \(D^{n+m+1}(K[x]).\)
D^n \nsubseteq D^{n+1}: \text{Iterated exponentials}

\[ K[x] \nsubseteq D(K[x]) \subset D^2(K[x]) \subset \ldots \subset D^n(K[x]) \subset \ldots \]

\[ e^x \notin K[x] \]
$D^n \subsetneq D^{n+1}$: Iterated exponentials

\[ K[x] \subsetneq D(K[x]) \subsetneq D^2(K[x]) \subset \ldots \subset D^n(K[x]) \subset \ldots \]

\[ e^x \notin K[x], \quad e^{e^x - 1} \notin D(K[x]). \]
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**Iterated Exponentials**

- $e_0(x) = 1$,
- $\hat{e}_n(x) = \int_0^x e_n(t) dt$,
- $e_{n+1}(x) = \exp(\hat{e}_n(x))$. 
$D^n \subsetneq D^{n+1}$: Iterated exponentials

\[ K[x] \subsetneq D(K[x]) \subsetneq D^2(K[x]) \subsetneq \ldots \subsetneq D^n(K[x]) \subsetneq \ldots \]

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**Iterated Exponentials**

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- $\hat{e}_n(x) = \int_0^x e_n(t) \, dt$,
- $e_{n+1}(x) = \exp(\hat{e}_n(x))$.

\[ \left\{ \begin{array}{l}
    e_{n+1}(x) \in D^{n+1}(K[x]) \\
    e_{n+1}(x) \notin D^n(K[x])
\end{array} \right. \]
Diff. Algebraic functions

Definition

Let $R \subset \mathbb{K}[[x]]$ be a differential ring and $f(x) \in \mathbb{K}[[x]]$. We say that $f(x)$ **differentially algebraic** over $R$ if there is $n \in \mathbb{N}$ and $P(y_0, \ldots, y_n) \in R[y_0, \ldots, y_n]$ such that

$$P(f(x), f'(x), \ldots, f^{(n)}(x)) = 0.$$  

We denote by $DA(R)$ the set of all differentially algebraic functions over $R$. 

**Diff. Algebraic functions**

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We denote by $DA(R)$ the set of all differentially algebraic functions over $R$. 

\[
\begin{array}{ccc}
\text{Diff. definable } & D(R) & \text{Diff. algebraic } \\
\downarrow & & \downarrow \\
\text{Linear diff. equations} & & \text{Polynomial diff. equations}
\end{array}
\]
Inclusion into Diff. Algebraic

- \( D(R) \subset DA(R) \).
- \( R \subset S \Rightarrow DA(R) \subset DA(S) \).
- \( DA(\mathbb{K}[x]) = DA(\mathbb{K}) \).
Inclusion into Diff. Algebraic

- $D(R) \subset DA(R)$.
- $R \subset S \Rightarrow DA(R) \subset DA(S)$.
- $DA(\mathbb{K}[x]) = DA(\mathbb{K})$.

**Proposition**

Let $R \subset \mathbb{K}[[x]]$ be a differential ring. Then $DA(D(R)) = DA(R)$.

The proof is *constructive.*
Inclusion into Diff. Algebraic

- $D(R) \subset DA(R)$.
- $R \subset S \Rightarrow DA(R) \subset DA(S)$.
- $DA(\mathbb{K}[x]) = DA(\mathbb{K})$.
- **Proposition:** $DA(D^n(R)) = DA(D^{n-1}(R))$.

**Theorem**

For all $n \in \mathbb{N}$, if $f(x) \in D^n(\mathbb{K}[x])$, then $f(x) \in DA(\mathbb{K})$. 
Inclusion into Diff. Algebraic

- $D(R) \subset DA(R)$.
- $R \subset S \Rightarrow DA(R) \subset DA(S)$.
- $DA(\mathbb{K}[x]) = DA(\mathbb{K})$.
- **Proposition:** $DA(D^n(R)) = DA(D^{n-1}(R))$.
- **Theorem:** $D^n(\mathbb{K}[x]) \subset DA(\mathbb{K})$.

**Example:** double exponential

\[
\exp(\exp(x) - 1) \quad \longrightarrow \quad f'(x) - \exp(x)f(x) = 0
\]

\[
\downarrow
\]

\[
f''(x)f(x) - f'(x)^2 - f'(x)f(x) = 0
\]
Inclusion into Diff. Algebraic

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- **Proposition:** $DA(D^n(R)) = DA(D^{n-1}(R))$.
- **Theorem:** $D^n(\mathbb{K}[x]) \subset DA(\mathbb{K})$.

**Example:** tangent

$$\tan(x) \rightarrow \cos(x)^2 f''''(x) - 2f(x) = 0$$

$$-2f^{(5)}(x)f''(x)^2f(x) + 12f^{(4)}(x)f'''(x)f''(x)f(x) -$$

$$6f^{(4)}(x)f''(x)^2f'(x) - 12f'''(x)^3f(x) +$$

$$12f'''(x)^2f''(x)f'(x) - 4f''''(x)f''(x)^3 -$$

$$8f''''(x)f''(x)^2f(x) + 8f''(x)^3f'(x) = 0$$
Reverse inclusion

Is the other inclusion true? Can we have $DA(\mathbb{K}[x]) = D^\infty(\mathbb{K}[x])$?
Reverse inclusion

Is the other inclusion true? Can we have $DA(K[x]) = D^\infty(K[x])$?

For some diff. algebraic functions, we can find an $n \in \mathbb{N}$:

**Riccati differential equation**

Let $y(x)$ be a solution to the Riccati differential equation

$$y'(x) = c(x)y(x)^2 + b(x)y(x) + a(x),$$

where $a(x), b(x) \in D^n(K[x])$ and $c(x) \in D^{n-1}(K[x])$.

Then $y(x) \in D^{n+2}(K[x])$. 
Reverse inclusion

Is the other inclusion true? Can we have $DA(\mathbb{K}[x]) = D^\infty(\mathbb{K}[x])$?

But that is not always the case

**Theorem (Noordman, Top, van der Put)**

Let $y(x)$ be a non-constant solution to the differential equation

$$y'(x) = y(x)^3 - y(x)^2.$$  

Then, there is no $n \in \mathbb{N}$ with $y(x) \in D^n(\mathbb{K}[x])$. 

DD-finite functions
Simple functions: handling singularities
Singularities on differential equations

Zero and Singular set

Let $f(x) \in \mathbb{K}[[x]]$:
- Zero set: $Z(f) = \{ \alpha \in \mathbb{C} : f(\alpha) = 0 \}$.
- Singular set: $S(f) = \{ \alpha \in \mathbb{C} : \alpha \text{ singularity of } f \}$.

Theorem

Let $f(x) \in \mathbb{K}[[x]]$ that satisfy the linear diff. equation

$$r_d(x)f^{(d)}(x) + \ldots + r_0(x)f(x) = 0,$$

for some $r_0(x), \ldots, r_d(x) \in \mathbb{K}[[x]]$. Then:

$$S(f) \subseteq Z(r_d) \bigcup_{i=0}^{d} S(r_i).$$
Singularities on differential equations

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$$S(f) \subseteq Z(r_d) \bigcup_{i=0}^{d} S(r_i).$$
Creating new singularities

Closure properties computations may create **new** singularities
Creating new singularities

Closure properties computations may create **new** singularities

Adding two D-finite functions

- $e^x \longrightarrow \partial - 1$.
- $Ai(x) \longrightarrow \partial^2 - x$.
- $e^x + Ai(x)$:

  $$(x - 1)\partial^3 - x\partial^2 - (x^2 - x)\partial + (x^2 - x + 1)$$
Creating new singularities

Closure properties computations may create new singularities

Adding two D-finite functions $I$

- $e^x \rightarrow \partial - 1.$
- $Ai(x) \rightarrow \partial^2 - x.$
- $e^x + Ai(x):$

$$(x - 1)\partial^3 - x\partial^2 - (x^2 - x)\partial + (x^2 - x + 1)$$
Creating new singularities

Closure properties computations may create **new** singularities

**Adding two D-finite functions II**

- \( \log(x + 1) \rightarrow (x + 1)\partial^2 + \partial. \)
- \( \sin(x) \rightarrow \partial^2 + 1. \)
- \( \log(x + 1) + \sin(x): \)

\[
(x + 1)(x^2 + 2x + 3)\partial^4 + (x^2 + 2x + 7)\partial^3 + (x + 1)(x^2 + 2x + 3)\partial^2 + (x^2 + 2x + 7)\partial
\]
Creating new singularities

Closure properties computations may create new singularities

Adding two D-finite functions II

- $\log(x+1) \rightarrow (x+1)\partial^2 + \partial$
- $\sin(x) \rightarrow \partial^2 + 1$
- $\log(x+1) + \sin(x)$:
  
  $$
  (x + 1)(x^2 + 2x + 3)\partial^4 + (x^2 + 2x + 7)\partial^3 + \\
  (x + 1)(x^2 + 2x + 3)\partial^2 + (x^2 + 2x + 7)\partial
  $$

DD-finite functions
Creating new singularities

Closure properties computations may create **new** singularities

**Multiplying two D-finite functions**

- $Ai(x + 1) \rightarrow \partial^2 - x$.
- $J_1(x) \rightarrow x^2 \partial^2 + x\partial + (x - 1)^2$.
- $Ai(x)J_1(x)$:
  \[
  x^2(4x^3 + 4x^2 - 3)\partial^4 + 4x(x - 1)(x^2 + 3x + 3)\partial^3 - \\
  2x^2(4x^4 - 4x^2 + 6x + 9)\partial^2 - \\
  2x^2(6x^3 + 10x^2 - 4x - 27)\partial - \\
  (4x^7 + 12x^6 + 12x^5 - x^4 - 28x^3 - 11x^2 + 6x - 9)
  \]
Desingularization

**INPUT:** a differential equation $\mathcal{L} \cdot f(x) = 0$.

**OUTPUT:** a differential equation $\tilde{\mathcal{L}} \cdot f(x) = 0$ such that:
1. For all $g(x)$ with $\mathcal{L} \cdot g(x) = 0$, $\tilde{\mathcal{L}} \cdot g(x) = 0$.
2. $\tilde{\mathcal{L}}$ has no apparent singularities.

**Previous work on desingularization**
- For differential systems over $\mathbb{Q}(x)$: M. Barkatou et al.
- For Ore Operators: S. Chen, M. Jaroschek, M. Kauers, M. F. Singer

Input $\rightarrow$ Closure property $\rightarrow$ Desing. $\rightarrow$ Output w/o sing.
Desingularization

**INPUT:** a differential equation \( \mathcal{L} \cdot f(x) = 0 \).

**OUTPUT:** a differential equation \( \tilde{\mathcal{L}} \cdot f(x) = 0 \) such that:
1. For all \( g(x) \) with \( \mathcal{L} \cdot g(x) = 0 \), \( \tilde{\mathcal{L}} \cdot g(x) = 0 \).
2. \( \tilde{\mathcal{L}} \) has no apparent singularities.

Our approach

Being able to obtain directly through closure properties an operator that has no new singularities by using only linear algebra.

Input \( \rightarrow \) Closure property \( \rightarrow \) Output w/o new sing.
Two key concepts

Noetherian modules

$M$ is a Noetherian module if all its submodules are finitely generated.

Localization ring

Given a multiplicatively closed set $S \subset R$, the localized ring of $R$ over $S$ (denoted by $R_S$) is the minimal extension to $R$ where we can divide by elements in $S$. 
S-simple diff. definable functions

**Definition**

Let $R \subseteq \mathbb{K}[[x]]$ a differential subring and $S \subseteq R$ multiplicatively closed. We say that $f(x) \in \mathbb{K}[[x]]$ is $S$-simple differentially definable over $R$ if there are $r_0(x), \ldots, r_{d-1}(x) \in R$ and $s(x) \in S$ such that:

$$s(x)f^{(d)}(x) + r_{d-1}(x)f^{(d-1)}(x) + \ldots + r_0(x)f(x) = 0.$$  

We denote the set of all these functions by $D(R, S)$. 

**DD-finite functions**
S-simple diff. definable functions

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We denote the set of all these functions by $D(R, S)$.

**D-finite case**

The set $S$ controls the singularities of the functions in $D(\mathbb{K}[x], S)$!
S-simple diff. definable functions

D-finite case

The set $S$ controls the singularities of the functions in $D(\mathbb{K}[x], S)$!

Consider $f(x) \in D(\mathbb{K}[x], S)$ for the following sets:

1. $S = \mathbb{K}$: $f(x)$ is analytic in $\mathbb{C}$.
2. $S = \{p(x)^n : n \in \mathbb{N}\}$: $f(x)$ can only have singularities in the zeros of $p(x)$.
3. $S = \mathbb{K}[x] \setminus \wp$ where $\wp$ is a prime ideal. Then $f(x)$ does not have singularities in any $\alpha \in V(\wp)$.
Adapting the main characterization

Characterization theorem - diff. def. case

Let $R \subset \mathbb{K}[[x]]$ and $f(x) \in \mathbb{K}[[x]]$. It is equivalent:

1. $f(x) \in D(R)$
2. The following $Fr(R)$-vector space has finite dimension

\[ \langle f, f'(x), f''(x), ... \rangle \]
Adapting the main characterization

Characterization theorem - \( S \)-simple case

Let \( R \subset \mathbb{K}[[x]] \), \( S \subset R \) m. c. and \( f(x) \in \mathbb{K}[[x]] \). It is equivalent:

1. \( f(x) \in D(R, S) \)
2. The following \( R_S \)-module is finitely generated

\[ \langle f, f'(x), f''(x), \ldots \rangle \]
Theorem (addition)

Let $R \subset K[[x]]$ be Noetherian, $S \subset R$ m.c. and $f(x), g(x) \in D(R, S)$. Then $f(x) + g(x)$ is again in $D(R, S)$. 

Proof:

Let $M(f) = \langle f(x), \ldots, f(n)(x) \rangle_{R,S}$. Then $M(f) + M(g) \in R$ Noetherian $\Rightarrow$ $M(f), M(g)$ Noetherian $\Rightarrow$ $M(f) + M(g)$ Noetherian. Hence, $M(f + g)$ is fin. generated and $f(x) + g(x) \in D(R, S)$. 

DD-finite functions
Theorem (addition)

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Addition is closed

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**Proof:**

Let $M(f) = \langle f(x), \ldots, f^{(n)}(x) \rangle_{R_S}$.

$$M(f + g) \subset M(f) + M(g)$$
**Theorem (addition)**

Let $R \subset \mathbb{K}[[x]]$ be Noetherian, $S \subset R$ m.c. and $f(x), g(x) \in D(R, S)$. Then $f(x) + g(x)$ is again in $D(R, S)$.

**Proof:**

Let $M(f) = \langle f(x), \ldots, f^{(n)}(x) \rangle_{R_S}$.

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$R$ Noetherian $\Rightarrow M(f), M(g)$ Noetherian $\Rightarrow M(f) + M(g)$ Noetherian.
Addition is closed

**Theorem (addition)**

Let \( R \subset \mathbb{K}[[x]] \) be Noetherian, \( S \subset R \) m.c. and \( f(x), g(x) \in D(R, S) \). Then \( f(x) + g(x) \) is again in \( D(R, S) \).

**Proof:**

Let \( M(f) = \langle f(x), \ldots, f^{(n)}(x) \rangle_{RS} \).

\[
M(f + g) \subset M(f) + M(g)
\]

\( R \) Noetherian \( \Rightarrow \) \( M(f), M(g) \) Noetherian \( \Rightarrow \)

\[ M(f) + M(g) \] Noetherian

Hence, \( M(f + g) \) is fin. generated and \( f(x) + g(x) \in D(R, S) \).
Closure properties

**Theorem (product)**
Let $R \subset \mathbb{K}[[x]]$ be Noetherian, $S \subset R$ m.c. and $f(x), g(x) \in D(R, S)$. Then $f(x)g(x)$ is again in $D(R, S)$.

**Theorem (derivation)**
Let $R \subset \mathbb{K}[[x]]$ be Noetherian, $S \subset R$ m.c. and $f(x) \in D(R, S)$. Then $f'(x)$ is again in $D(R, S)$.

**Theorem (integration)**
Let $R \subset \mathbb{K}[[x]]$ be Noetherian, $S \subset R$ m.c. and $f(x) \in D(R, S)$. Then for any $F(x)$ with $F'(x) = f(x)$, $F(x) \in D(R, S)$. 
Theorem

Let \( R \subset \mathbb{K}[[x]] \) be Noetherian and \( S \subset R \) multiplicatively closed. Then \( D(R, S) \) is a differential extension of \( R \) closed under integration.

\[ f(x), g(x) \text{ S-simple D-finite of order } d_1, d_2. \]

<table>
<thead>
<tr>
<th>Operation</th>
<th>Result</th>
<th>Order bound</th>
</tr>
</thead>
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<tr>
<td>Addition</td>
<td>((f + g))</td>
<td>Unknown</td>
</tr>
<tr>
<td>Product</td>
<td>((fg))</td>
<td>Unknown</td>
</tr>
<tr>
<td>Differentiation</td>
<td>(f')</td>
<td>Unknown</td>
</tr>
<tr>
<td>Integration</td>
<td>(\int f)</td>
<td>(d_1 + 1)</td>
</tr>
</tbody>
</table>
## Closure properties examples

### Adding two D-finite functions I

- $e^x \rightarrow \partial - 1$.
- $Ai(x) \rightarrow \partial^2 - x$.
- $e^x + Ai(x)$:
  \[
  (x - 1)\partial^3 - x\partial^2 - (x^2 - x)\partial + (x^2 - x + 1)
  \]
Closure properties examples

Adding two D-finite functions I

- \( e^x \longrightarrow \partial - 1 \).
- \( Ai(x) \longrightarrow \partial^2 - x \).
- \( e^x + Ai(x): \)

\[
(x - 1)\partial^3 - x\partial^2 - (x^2 - x)\partial + (x^2 - x + 1) \\
\downarrow \\
\partial^4 - (x^2 + 1)\partial^3 + (x^2 + 1)\partial^2 + (x^3 + x - 2)\partial - (x^3 - x - 1)
\]
Closure properties examples

Adding two D-finite functions II

- $\log(x + 1) \rightarrow (x + 1)\partial^2 + \partial$.
- $\sin(x) \rightarrow \partial^2 + 1$.
- $\log(x + 1) + \sin(x)$:

\[
(x + 1)(x^2 + 2x + 3)\partial^4 + (x^2 + 2x + 7)\partial^3 + (x + 1)(x^2 + 2x + 3)\partial^2 + (x^2 + 2x + 7)\partial
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Closure properties examples

Adding two D-finite functions II

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(x + 1)(x^2 + 2x + 3)\partial^2 + (x^2 + 2x + 7)\partial
\]

\[
\downarrow
\]

\[
4(x + 1)^2\partial^5 + p_4(x)\partial^4 + p_3(x)\partial^3 + p_2(x)\partial^2 + p_1(x)\partial,
\]

\[
\deg(p_i(x)) \leq 5
\]
Closure properties examples

**Multiplying two D-finite functions**

- $Ai(x + 1) \longrightarrow \partial^2 - x$.
- $J_1(x) \longrightarrow x^2 \partial^2 + x \partial + (x - 1)^2$.
- $Ai(x)J_1(x)$:

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x^2(4x^3 + 4x^2 - 3)\partial^4 + 4x(x - 1)(x^2 + 3x + 3)\partial^3 - \\
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Closure properties examples

### Multiplying two D-finite functions

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- \( J_1(x) \rightarrow x^2 \partial^2 + x \partial + (x - 1)^2. \)
- \( Ai(x) J_1(x) : \)

\[
195 x^3 \partial^5 + p_4(x) \partial^4 + p_3(x) \partial^3 + p_2(x) \partial^2 + p_1(x) \partial
\]

\[
\deg(p_i(x)) \leq 12
\]
Simple DD-finite functions

- All results for $S$-simple $D(R)$ require $R$ Noetherian.
- D-finite functions are not Noetherian.
- Can we extend the result to DD-finite functions?
Simple DD-finite functions

- All results for S-simple $D(R)$ require $R$ Noetherian.
- D-finite functions are not Noetherian.
- Can we extend the result to DD-finite functions?

**YES!**

Instead of working with the whole $D(\mathbb{K}[x])$, we restrict to a smaller Noetherian subring.
Simple DD-finite functions

- All results for S-simple D(R) require R Noetherian.
- D-finite functions are not Noetherian.
- Can we extend the result to DD-finite functions?

**Definition**

Let $S \subset \mathbb{K}[[x]]$ be a multiplicatively closed set. We denote the set of **S-simple D^n-finite functions** with $D^n(\mathbb{K}[x], S)$ and we define it recursively by:

- $D^1(\mathbb{K}[x], S) = D(\mathbb{K}[x], \mathbb{K}[x] \cap S)$.
- $D^n(\mathbb{K}[x], S) = D(D^{n-1}(\mathbb{K}[x], S), D^{n-1}(\mathbb{K}[x], S) \cap S)$.

Intuitively, we **control the leading coefficient** of the equations that define the elements in the whole chain of rings.
Conclusions

Achievements
- Extension of the holonomic framework.
- Running implementation of closure properties.
- Relation to differentially algebraic functions.
- Control of singularities throughout closure properties.

Future work
- Fast computation of truncation of $D^n$-finite functions.
- Development of certified numerical evaluations.
- Combinatorial meaning of the induced sequences.
- Multivariate DD-finite functions.
Thank you!

Contact webpage:

- http://www.lix.polytechnique.fr/~jimenezpastor/
- https://www.dk-compmath.jku.at/people/antonio

Code available: