Fast algorithm and sharp degree bounds for one block quantifier elimination over the reals

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MAX Seminar
May 10, 2021

One block quantifier elimination (QE)

A *basic semi-algebraic* set: the set of real solutions of

$$f_1 = \cdots = f_s = 0,\ g_1 \bullet 0, \ldots,\ g_v \bullet 0,$$

with $\bullet \in \{>, \geq\}$, $f_i, g_j \in \mathbb{R}[x_1, \ldots, x_n]$.

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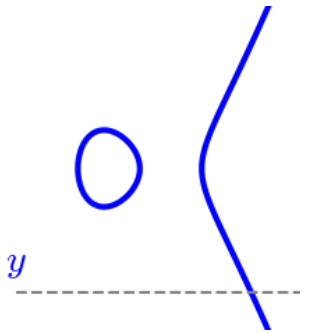

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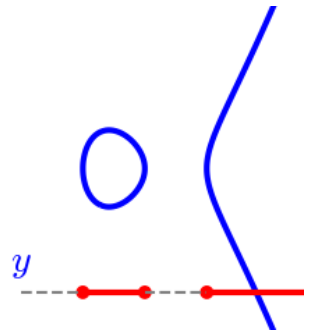

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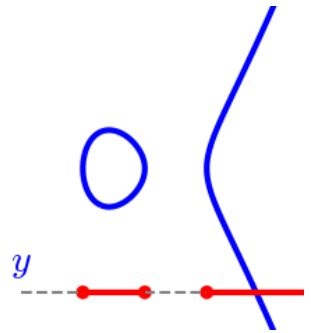

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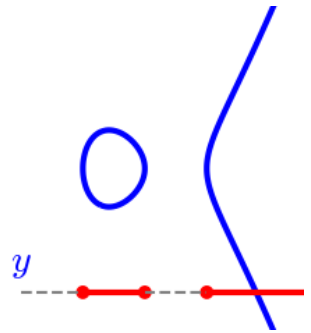

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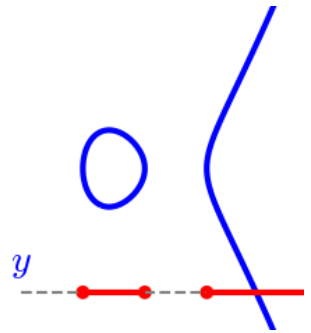

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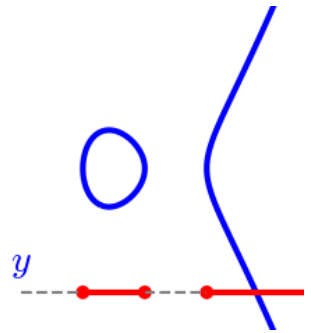

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Algorithms, Complexity, Implementation

Motivations

Many applications

- Perspective-Three-Point Problem
  [Faugère, Moroz, Rouillier, Safey El Din '08]
- Kuramoto model of synchronization
  [Kuramoto '75; Harris, Hauenstein, Szanto '20]
- Dynamic of epidemics
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- Program verification [Kapur '06]
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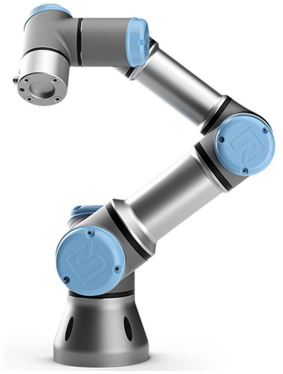

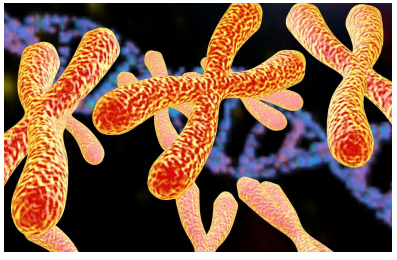

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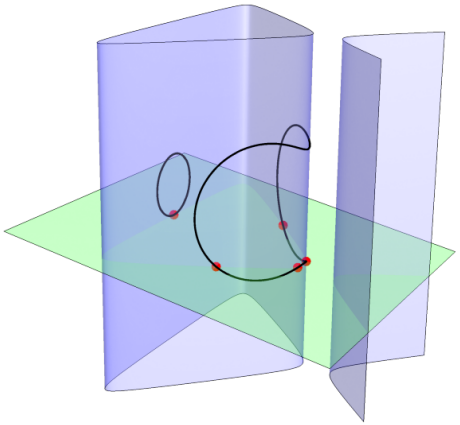
Experimental mathematics

Counter-examples for a conjecture in [Huisman '03]

⇝ Compute a hyperplane that intersects a smooth curve at only real points.

(on-going work with Plaumann, Manevich and Safey El Din)



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We need a practical algorithm and better complexity analysis!

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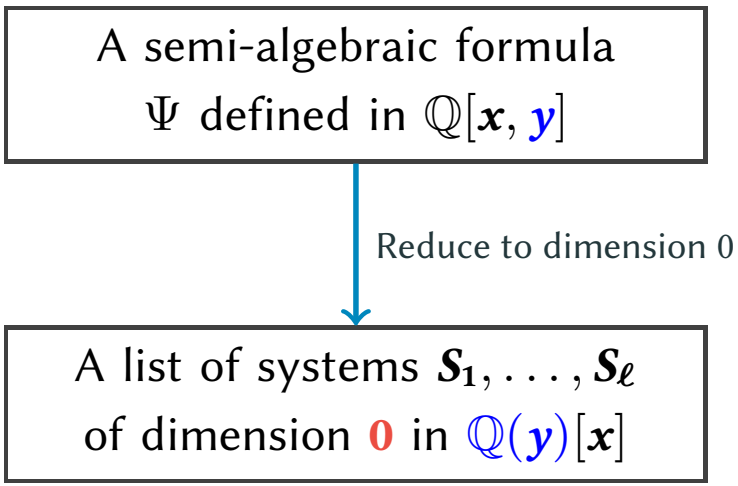

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A list of systems $\boldsymbol{S_1}, \ldots, \boldsymbol{S_\ell}$
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For almost every $\eta \in \mathbb{C}^t$:

- The sets of complex solutions of $\boldsymbol{S_i}(\boldsymbol{x}, \eta)$ are finite.
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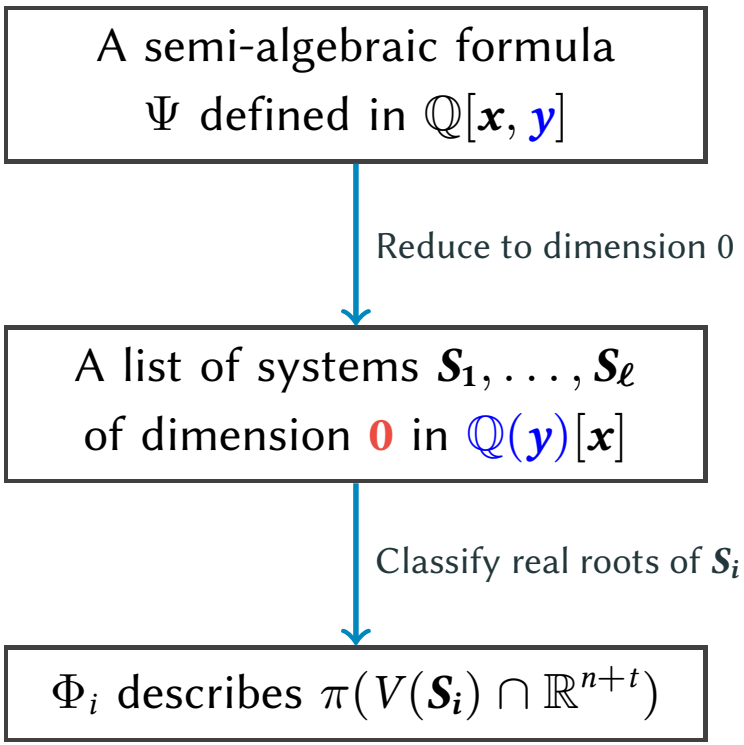

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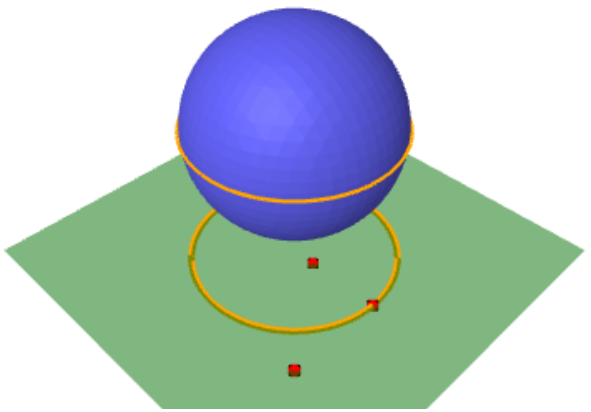

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Return $\Phi = \vee_{i=1}^{\ell} \Phi_i$

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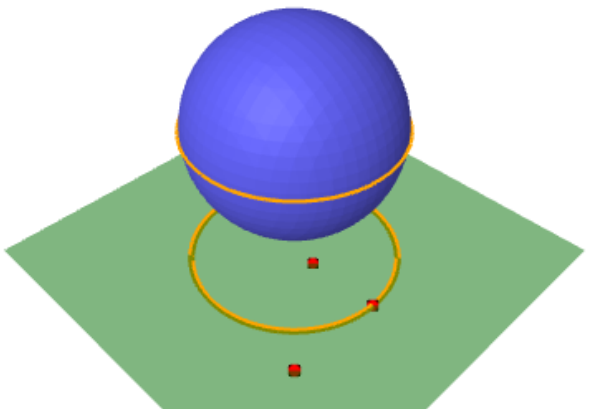

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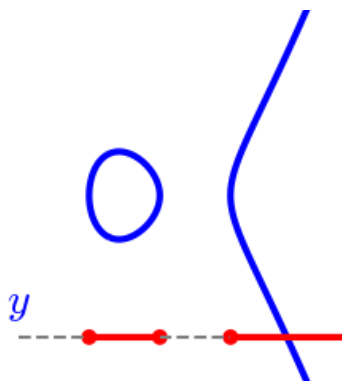

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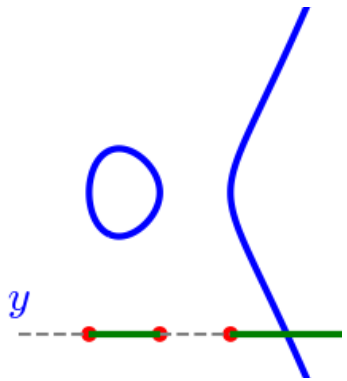
Weaker output: $(y^3 - y > 0)$
(*almost* equivalent)



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- In: f satisfies *regularity assumptions*
- Out: A formula Φ defining a *dense* subset of the *interior* of $\pi(\mathcal{V} \cap \mathbb{R}^{n+t})$

Solve examples that are out of reach of the state-of-the-art software.

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Solve examples that are out of reach of the state-of-the-art software.

t	n	s	NEW QE	MAPLE	MATHEMATICA
2	3	2	**7 s**	$>$ 120 h	$>$ 120 h
2	4	2	**2 m**	$>$ 120 h	$>$ 120 h
2	5	2	**20 m**	$>$ 120 h	$>$ 120 h
2	6	2	**3 h**	$>$ 120 h	$>$ 120 h
2	7	2	**8 h**	$>$ 120 h	$>$ 120 h
3	3	2	**2 m**	$>$ 120 h	$>$ 120 h
3	4	2	**25 m**	$>$ 120 h	$>$ 120 h
3	5	2	**10 h**	$>$ 120 h	$>$ 120 h
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Random dense systems with $D = 2$

Main results

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t	D	NEW RRC	MAPLE[RF]	MAPLE[RC]
2	[3, 2]	**5 s**	1 m	12 s
2	[2, 2, 2]	**34 s**	17m	2 m
2	[3, 3]	**3 m**	2 h	5 m
3	[2, 2]	**27 s**	36 s	12 m
3	[3, 2]	**3 h**	86 h	37 h
3	[2, 2, 2]	**32 h**	$>$ 120 h	$>$ 120 h
3	[4, 2]	**90 h**	$>$ 120 h	$>$ 120 h
4	[2, 2]	**8 m**	$>$ 120 h	$>$ 120 h

Random dense systems

RF: RootFinding RC: RegularChains

Complexity results

Generic input $f = (f_1, \ldots, f_s) \subset \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_t]$ with $\deg(f_i) \leq D$

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$$\mathfrak{B} = 2nD^s(D-1)^{n-s+1}\binom{n}{s-1} \qquad \in O\left(nD^n\binom{n}{s-1}\right)$$

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Arithmetic complexity for one block QE

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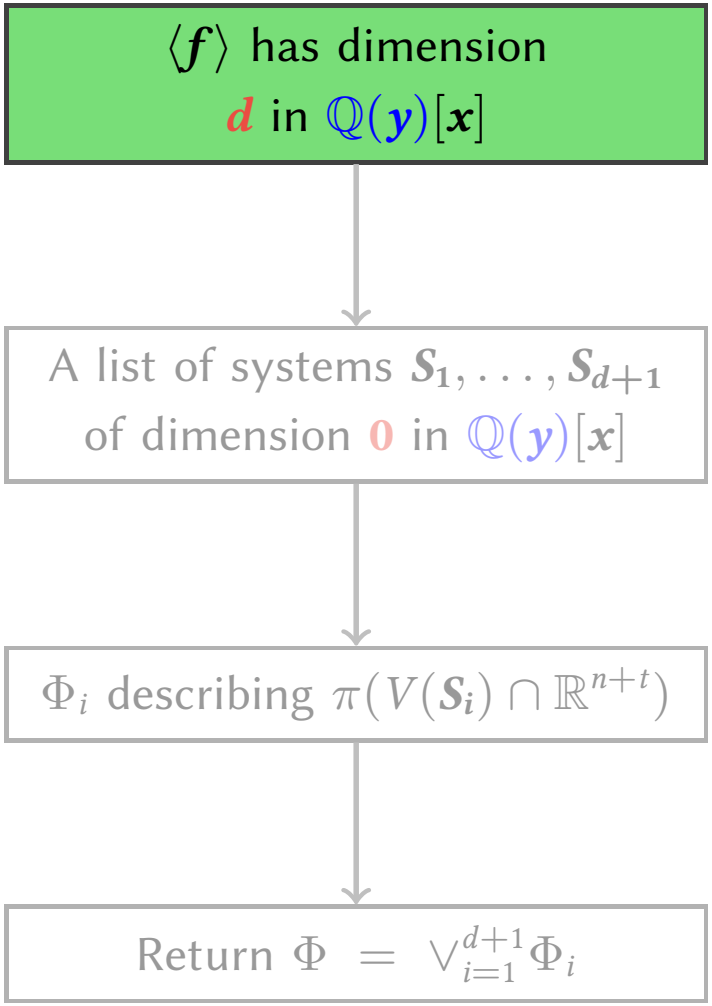
- Explicit constant in the exponent
- For $D = 2$ and s is fixed: $\mathfrak{B}$ is polynomial in n

First step: Reduction to dimension zero



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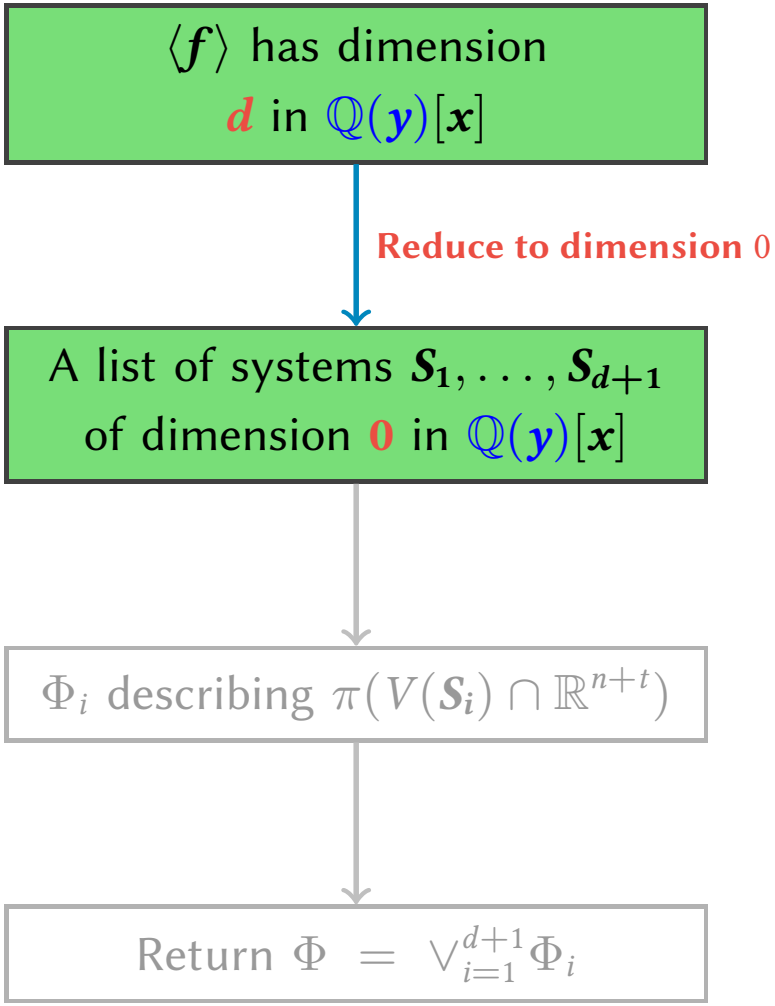

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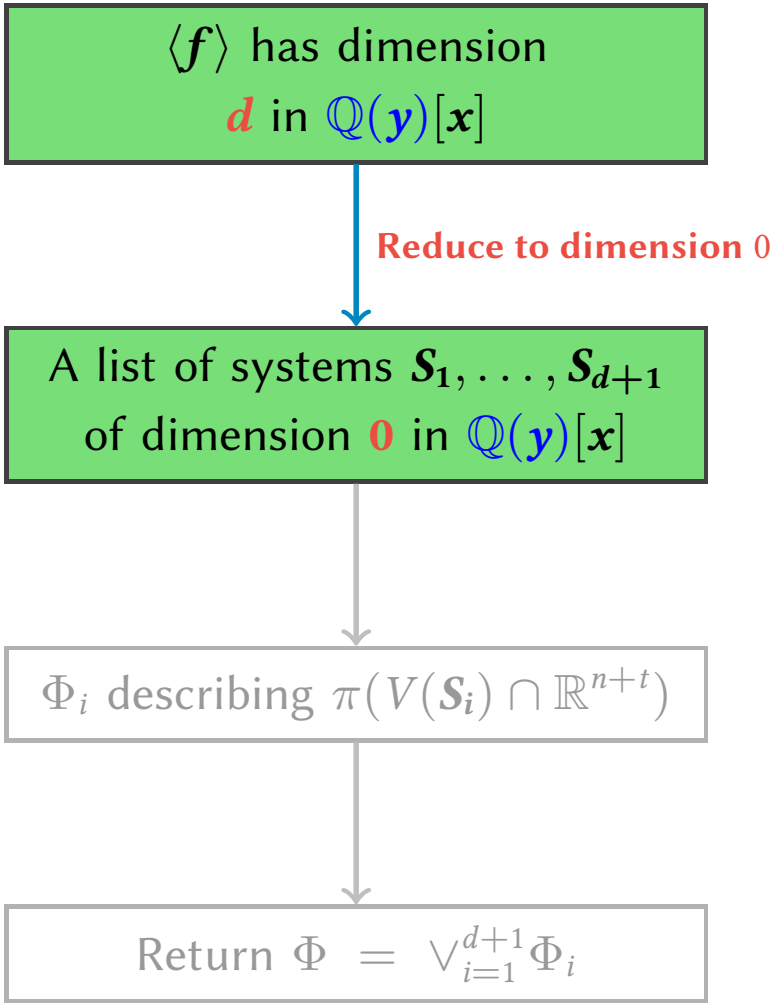

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Real root finding algorithm

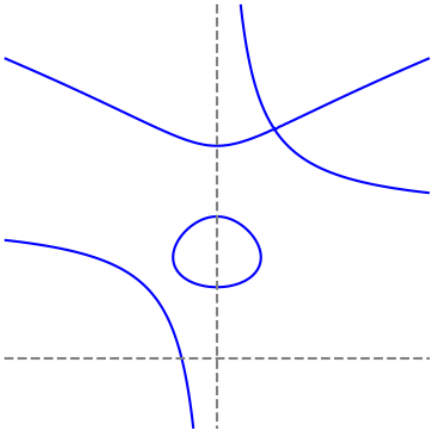
[Safey El Din - Schost '03]

Safey El Din - Schost algorithm

Compute points per connected components

Safey El Din - Schost algorithm

Compute points per connected components



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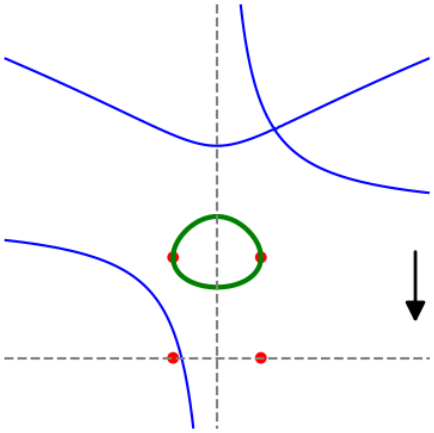

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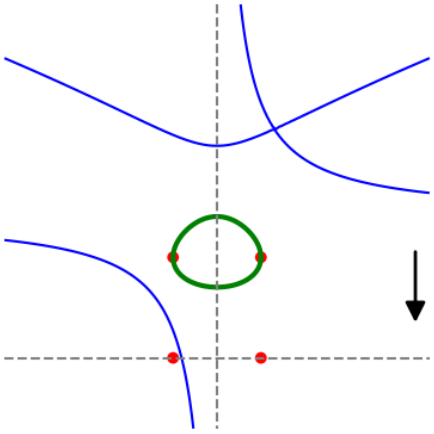

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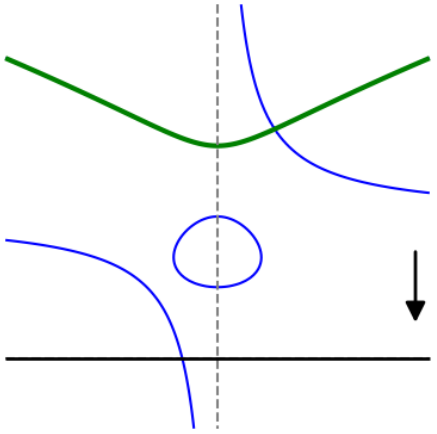

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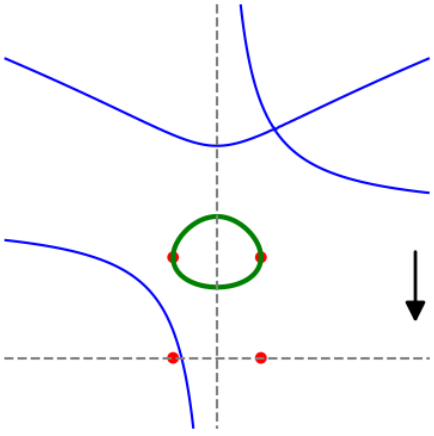

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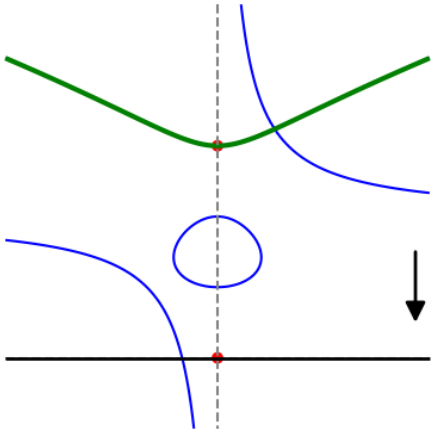

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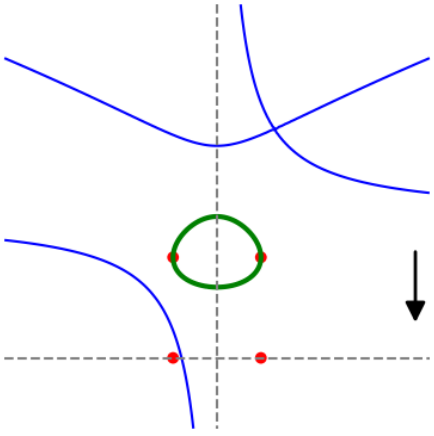

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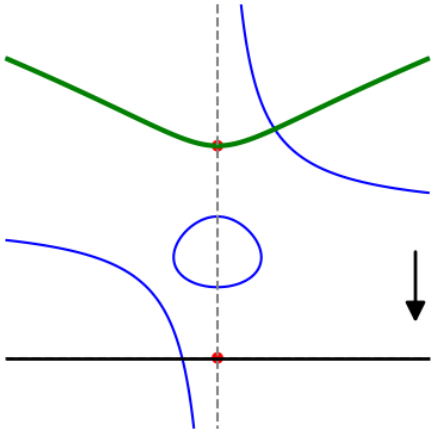

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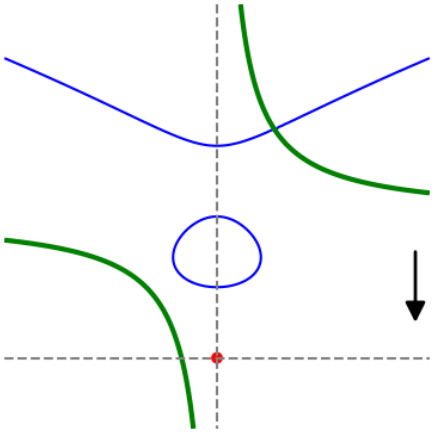

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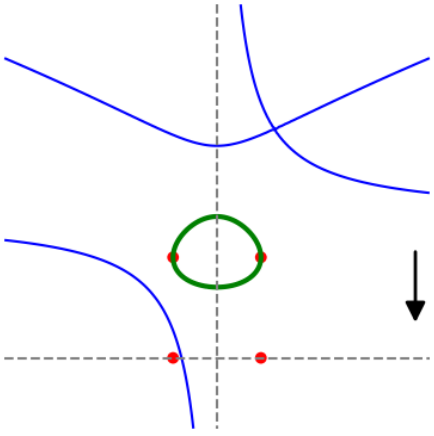

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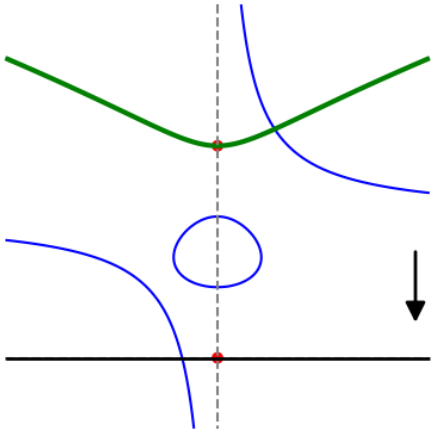

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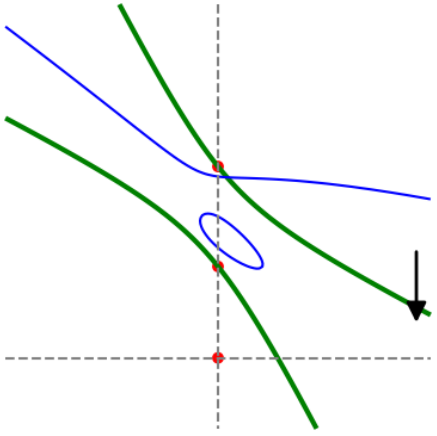

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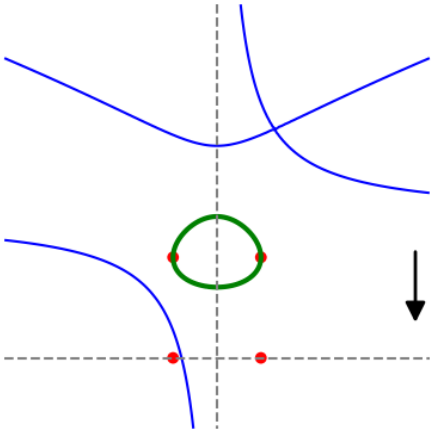

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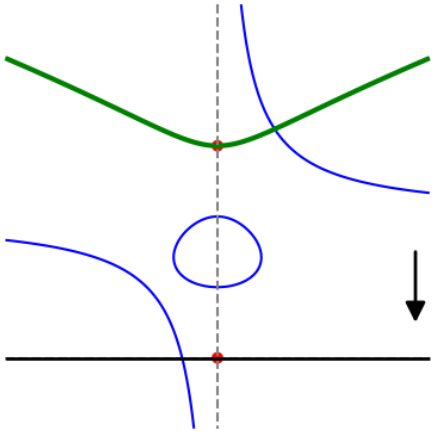

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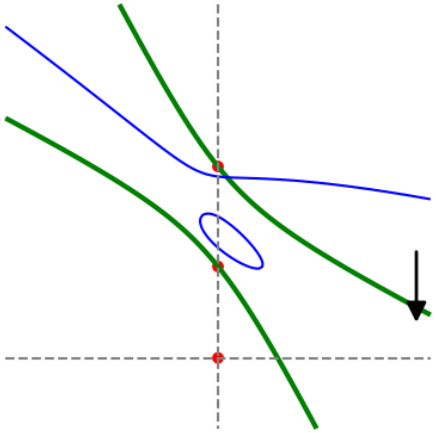

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Consider y as parameters

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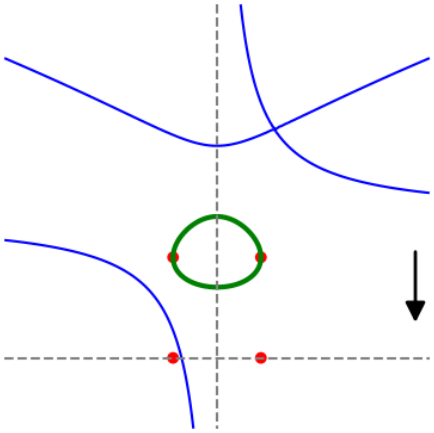

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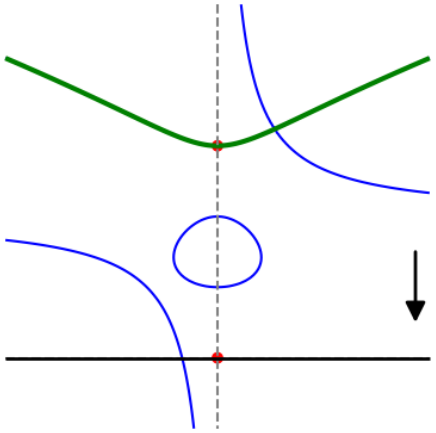

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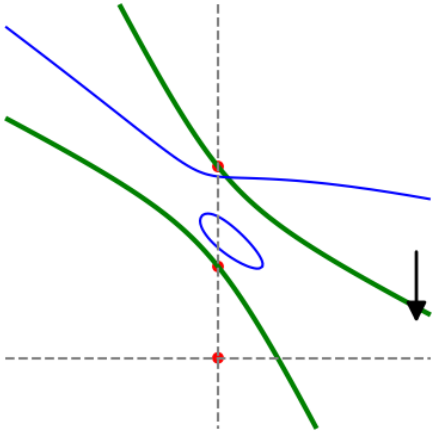

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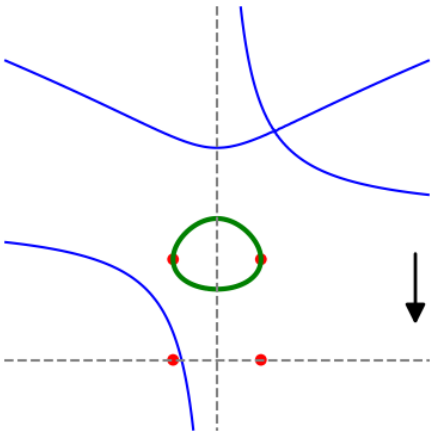

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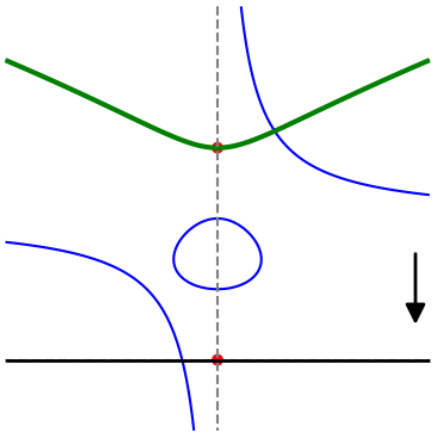

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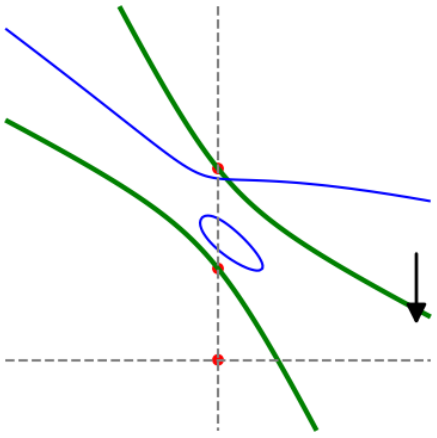

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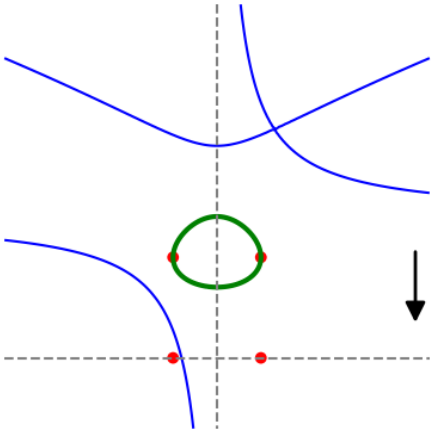

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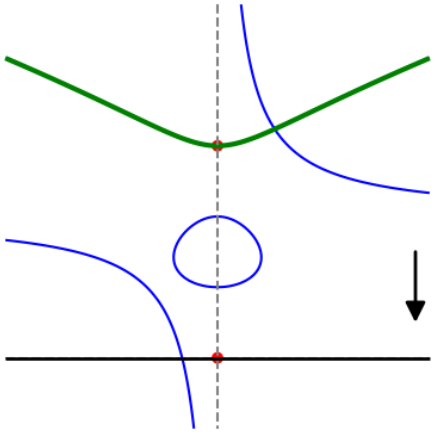

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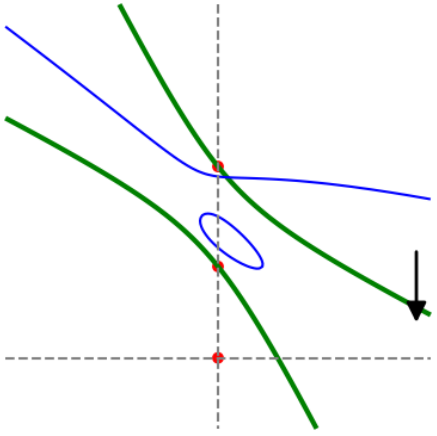

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S_i defines $\mathbf{crit}(\pi_i, f^A) \cap \pi_{i-1}^{-1}(0, \ldots, 0)$

Safey El Din - Schost algorithm

Jacobian criterion

$$S_i = \underbrace{\left\{ f^A,\ (n-d)\text{-minors of } \left[\frac{\partial f^A}{\partial x_{i+1}}, \ldots, \frac{\partial f^A}{\partial x_n} \right] \right\}}_{\mathrm{crit}(\pi_i, f^A)} \cup \underbrace{\{x_1, \ldots, x_{i-1}\}}_{\pi_{i-1}^{-1}(0,\ldots,0)}$$

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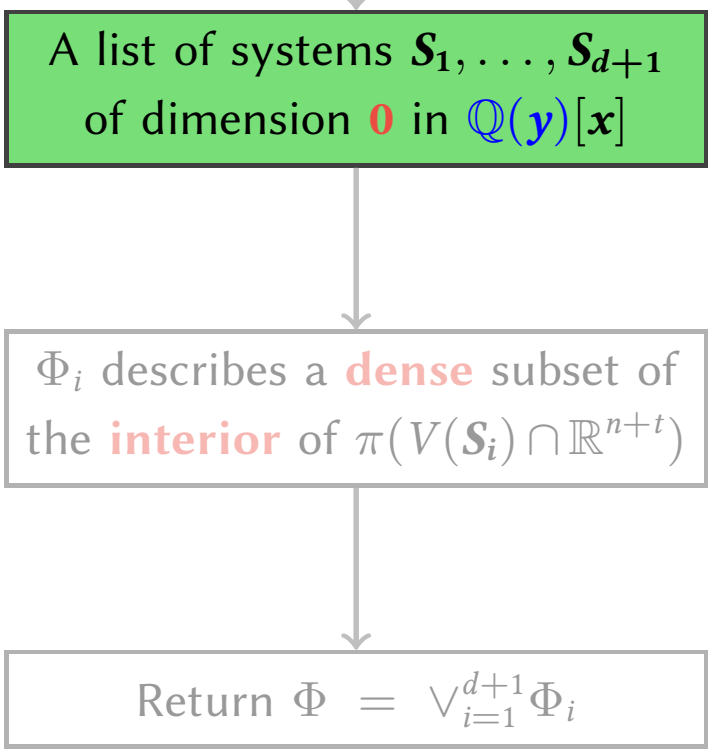
S_i is a determinantal system: Useful for complexity analysis!

Second step: Classify real roots of S_i

S_i generates a radical zero-dimensional ideal in $\mathbb{Q}(y)[x]$

$\langle f \rangle$ has dimension d in $\mathbb{Q}(y)[x]$



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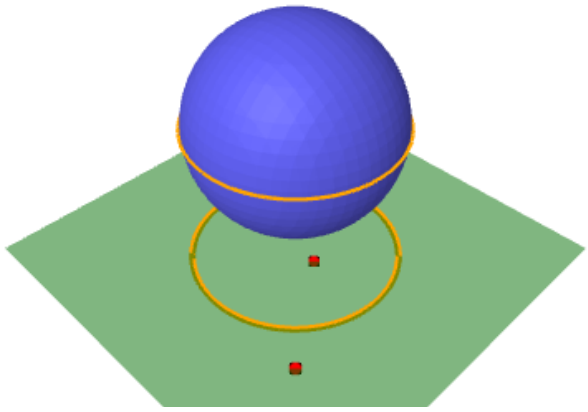
A list of systems $S_1, \ldots, S_{d+1}$ of dimension 0 in $\mathbb{Q}(y)[x]$

Classify real roots of S_i

Φ_i describes a dense subset of the interior of $\pi(V(S_i) \cap \mathbb{R}^{n+t})$

Return $\Phi = \vee_{i=1}^{d+1} \Phi_i$



Real root classification

$$x^2 + {y_1}^2 + {y_2}^2 = 1$$

Output:
$$\begin{cases} {y_1}^2 + {y_2}^2 < 1 & : \text{ 2 solutions} \\ {y_1}^2 + {y_2}^2 > 1 & : \text{ 0 solution} \\ {y_1}^2 + {y_2}^2 = 1 & : \text{ 1 solution} \end{cases}$$

Omit the index i: $S_i \to S$

Classic algorithms and drawbacks

- Sturm-based algorithms: Univariate polynomials with parameters
  [González-Vega, Recio, Lombardi, Roy '98; Liang, Jeffrey, Moreno Maza '08]

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An alternative to Sturm's theorem: **Hermite's quadratic forms**

Hermite's matrices

Ch. Hermite

Root counting using quadratic forms [Hermite 1856]
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Fix a basis $M = \{m_1, \ldots, m_\delta\}$. The **Hermite matrix** of S w.r.t. M is

$$H = (\underbrace{\operatorname{trace}(\mathcal{L}_{m_k \cdot m_\ell})}_{\in \mathbb{K}})_{1 \le k, \ell \le \delta}.$$

Parametric Hermite matrices

Take $\mathbb{K} = \mathbb{Q}(\boldsymbol{y})$ as coefficient field, $\boldsymbol{S}$ is zero-dimensional in $\mathbb{Q}(\boldsymbol{y})[\boldsymbol{x}]$

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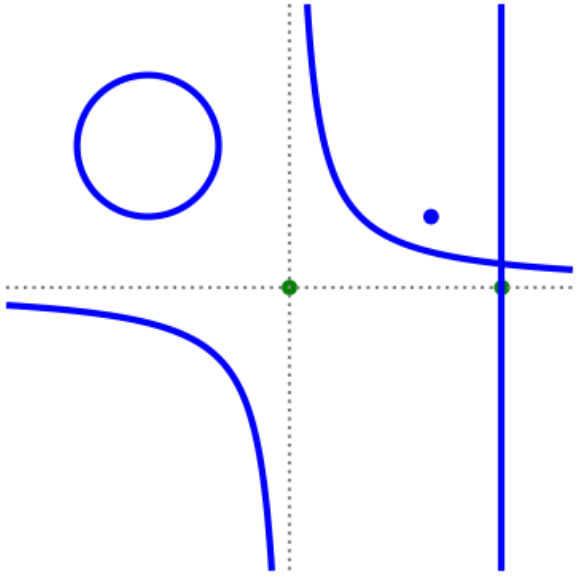
Number of real roots of $\boldsymbol{S}(\boldsymbol{x}, \eta)$	$=$	Signature of $\boldsymbol{H}(\eta)$

A better way to obtain the boundary!

$V(\boldsymbol{w}_\infty) \in \mathbb{C}^t$ contains:

- Non-proper points.



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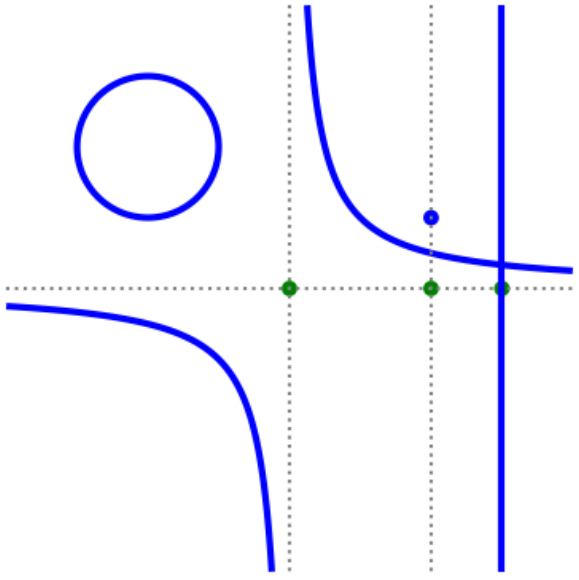

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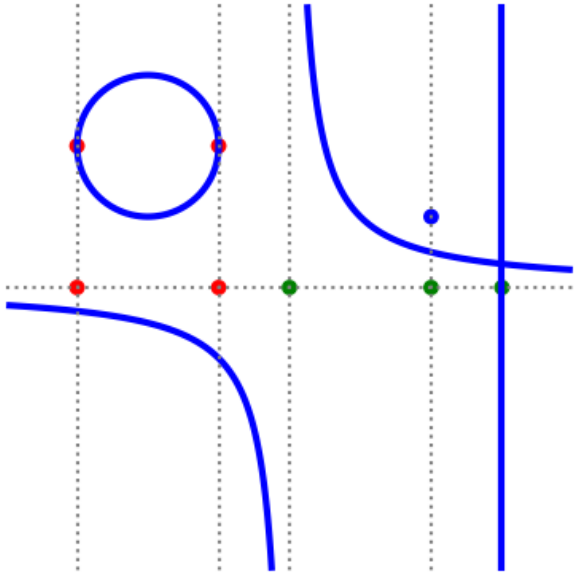

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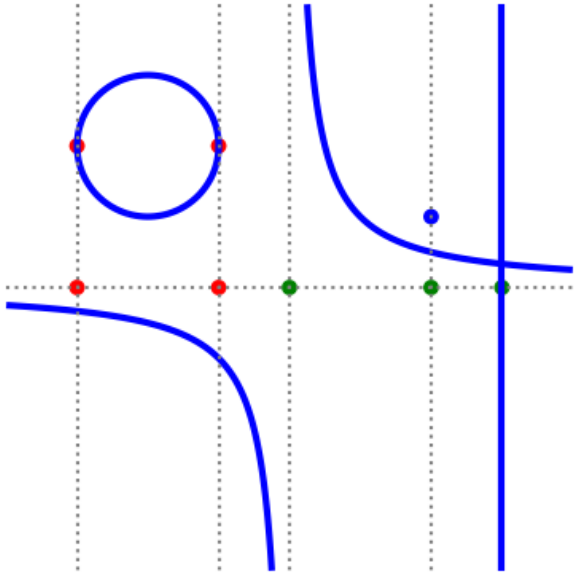

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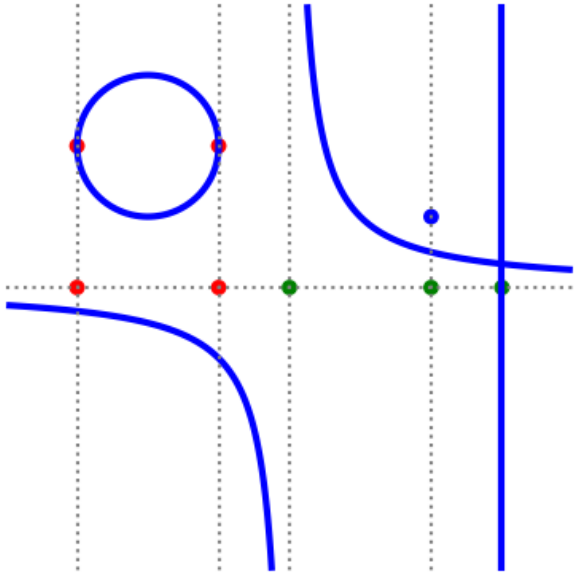

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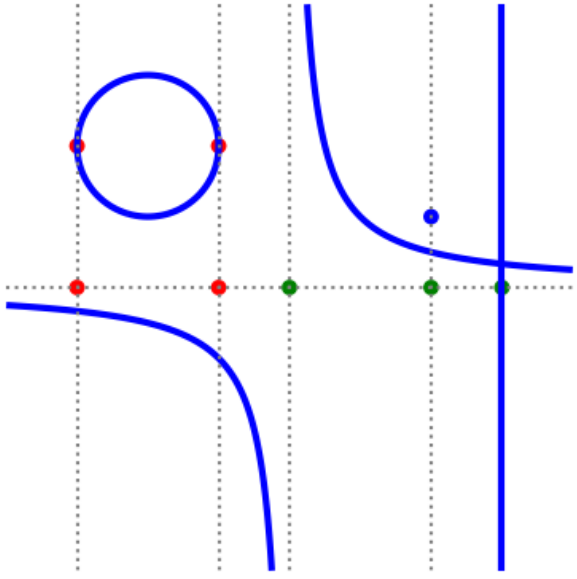

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Random dense systems $\longrightarrow$

t	D	HERMITE	MAPLE[DV]	MAPLE[BP]
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2	$[3,3]$	**1.1 s**	2 h	8 s
3	$[3,2]$	**.4 s**	2 h	3 s
3	$[2,2,2]$	**8 s**	$>$ 120 h	20 m
3	$[4,2]$	**12 s**	$>$ 120 h	15 m

Algorithm for classifying real roots

For $\eta \in \mathbb{R}^t$ s.t. $\boldsymbol{w}_{\infty}(\eta) \neq 0$:

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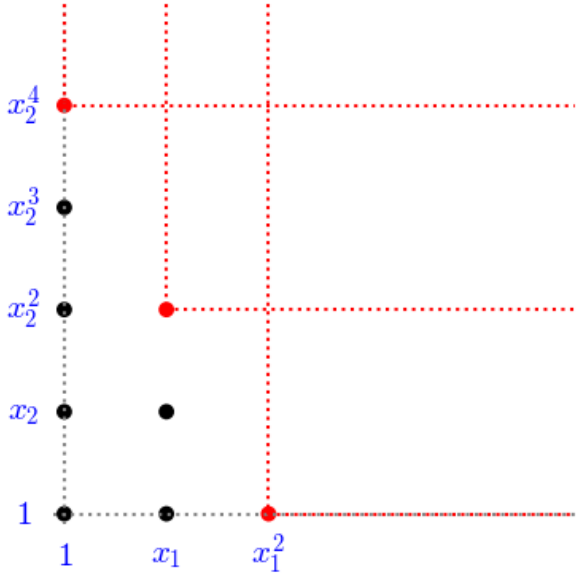
If we want also the real root classification:

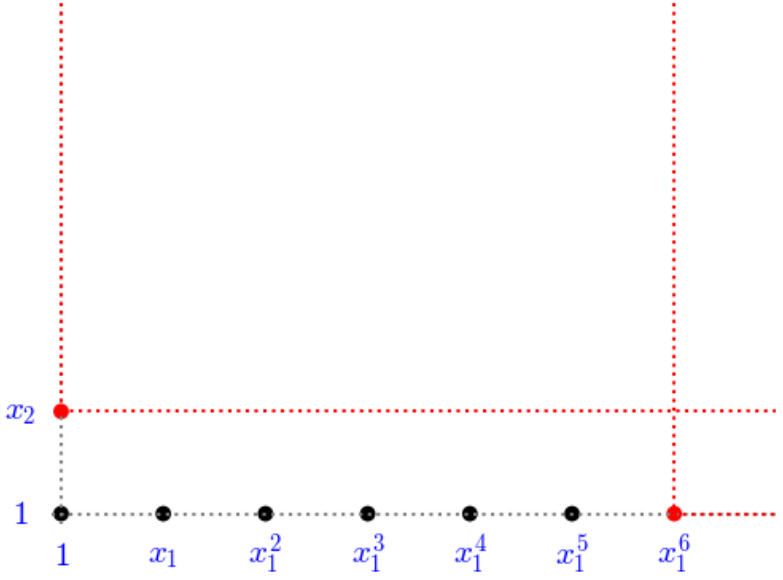
3'. Return all the sign conditions and the corresponding number of real solutions.

Choice of bases for quotient rings

Monomial order over $\boldsymbol{x}$ → Gröbner bases → Basis of quotient rings



grevlex staircase



lex staircase

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Degree in $\boldsymbol{y}$ of entries in Hermite matrices:

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We use **grevlex** Gröbner bases for constructing Hermite matrices

Degree bound & Complexity

Complexity? Calls to RRC are the **main cost** ⇝ Computing sample points?

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Classifying real roots of $\boldsymbol{S}$ (a determinantal system)

- $\boldsymbol{M} = \{m_1, \ldots, m_\delta\}$: **grevlex** basis of $\mathbb{Q}(\boldsymbol{y})[\boldsymbol{x}]/\langle \boldsymbol{S} \rangle$
- $\boldsymbol{H} = (\boldsymbol{h_{k,\ell}})_{1 \leq k,\ell \leq \delta}$ $\quad h_{k,\ell} = \operatorname{trace}(\mathcal{L}_{m_k \cdot m_\ell})$

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Complexity for one block QE

- **Sharp** degree bound for the minors of H:

$$\mathfrak{B} = D^s(D-1)^{n-s}\left(2(n-s)(D-1)\binom{n-1}{s-2} + (n(D-2)+s)\binom{n-1}{s-1}\right)$$

- Arithmetic complexity [L. , Safey El Din '20]

$$O^{\sim}\left(8^t \mathfrak{B}^{3t}\binom{t+\mathfrak{B}}{t}\right)$$

Sketch of the proof

1. Generic f
   $\rightsquigarrow$ S in **Noether position**

$\longrightarrow$

Entries of H are in $\mathbb{Q}[y]$
$\deg(h_{k,\ell}) \leq \deg(m_k) + \deg(m_\ell)$

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$$\deg(\det H) \leq 2\ \mathrm{HS}'(1)$$

Degree bound & Complexity

[Faugère, Safey El Din, Spaenlehauer '13]

$$\mathrm{HS}(z) = \frac{\det(A(z^{D-1}))}{z^{(D-1)\binom{s-1}{2}}} \frac{(1-z^D)^s(1-z^{D-1})^{n-s}}{(1-z)^n}$$

where $A(z)$ is the $(s-1)\times(s-1)$ matrix whose (i,j)-th entry is $\sum_k \binom{s-i}{k}\binom{n-1-j}{k} z^k$

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Simplification [Berthomieu, Bostan, Ferguson, Safey El Din '21]

$$\mathrm{HS}(z) = (1+\ldots+z^{D-1})^s(1+\ldots+z^{D-2})^{n-s}\sum_{k=0}^{s-1}\binom{n-s-1+k}{k} z^{(D-1)k}$$

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Experiments

Implemented in Maple with FGb, RAGLib, msolve

Intel Xeon Gold 6244 3.6GHz + 754GB RAM

∞: stopped after 5 days

t	D	HERMITE					MAPLE[RF]			MAPLE[RC]		
		MAT	DET	SP	**total**	DEG	DV	CAD	**total**	BP	RRC	**total**
2	$[2,2]$	.07 s	.01 s	.3 s	**.4 s**	8	.1 s	.3 s	.4 s	.1 s	1 s	1.1 s
2	$[3,2]$	.1 s	.2 s	4.8 s	**5 s**	18	1 m	5 s	1 m	.3 s	12 s	12 s
2	$[2,2,2]$	.3 s	.3 s	33 s	**34 s**	24	17m	32 s	17m	23 s	2 m	2 m
2	$[3,3]$	.3 s	.8 s	3 m	**3 m**	36	2 h	4 m	2 h	8 s	4 m	4 m
3	$[2,2]$	.1 s	.02 s	26 s	**27 s**	8	1 s	35 s	36 s	.2 s	12m	12m
3	$[3,2]$	.2 s	.2 s	3 h	**3 h**	18	2 h	84 h	86 h	3 s	37 h	37 h
3	$[2,2,2]$	.5 s	7 s	32 h	**32 h**	24	∞	∞	∞	20m	∞	∞
3	$[4,2]$	.6 s	12 s	90 h	**90 h**	32	∞	∞	∞	12m	∞	∞
4	$[2,2]$	.2 s	.1 s	8 m	**8 m**	8	4 s	∞	∞	1 s	∞	∞

Random dense systems with $n = s$ (RRC)

t	n	s	MAT+DET	SP	**TOTAL**	DEG	MAPLE[RF]	MAPLE[RC]
3	6	6	1 h.	85 h.	**86 h**	48	∞	∞

Kuramoto model for 4 oscillators

MAT: compute Hermite matrices

DET: compute the minors

SP: compute sample points

DEG: highest degree in the output

RF[DV]: RootFinding[DiscriminantVariety]

RF[CAD]: RootFinding[RRC]

RC[BP]: RegularChains[BorderPolynomial]

RC[RRC]: RegularChains[RRC]

Experiments

t	n	s	MAT+DET	SP	TOTAL	DEG	MAPLE	MATHEMATICA
2	3	2	1 s	6 s	**7 s**	24	∞	∞
2	4	2	9 s	2 m	**2 m**	40	∞	∞
2	5	2	2 m	18 m	**20 m**	56	∞	∞
2	6	2	20 m	2.5 h	**3 h**	72	∞	∞
2	7	2	1.5 h	6.5 h	**8 h**	88	∞	∞
3	3	2	6 s.	2 m.	**2 m**	24	∞	∞
3	4	2	5 m.	20 m.	**25 m**	40	∞	∞
3	5	2	2 h.	8 h.	**10 h**	56	∞	∞
4	3	2	40 s.	30 m.	**30 m**	24	∞	∞
4	4	2	5 h.	45 h.	**50 h**	40	∞	∞

Random dense systems with $D = 2$

D	t	n	s	MAT+DET	SP	TOTAL	DEG	MAPLE	MATHEMATICA
2	3	3	2	3 s.	37 s.	**40 s**	22	∞	∞
2	3	4	2	2 m.	10 m.	**12 m**	34	∞	∞
2	3	5	2	2 m.	10 m.	**12 m**	32	∞	∞
[3, 2]	3	3	2	15 m.	2 h.	**2 h**	48	∞	∞
2	4	3	2	20 s.	20 m.	**21 m**	22	∞	∞
2	4	4	2	15 s.	18 m.	**19 m**	20	∞	∞

Random sparse systems

Conclusions & Perspectives

Conclusions

- New **practical** algorithms for RRC and one-block QE
- **Better** degree bound and complexity results for **generic** inputs
- **Easier evaluation** of formulas using the matrices
- Solve **applications**: Kuramoto model with 4 oscillators

[Kuramoto '75; Harris and Hauenstein '20]

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[Kuramoto '75; Harris and Hauenstein '20]

Perspectives

- Hermite matrices for computing **totally real hyperplane section**
- Work on the **implementation** to tackle **applications**
- Control the degree of denominators in non-generic case
- Exploit the **structure** of Hermite matrices
- Handle **inequalities**