Surreal numbers with exponential function and omega-exponentiation.

M. Matusinski (U. Bordeaux)
joint work with A. Berarducci, S. Kuhlmann and V. Mantova

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Simple and complicated numbers.

Gonshor’s alternative definition (1986):

\[ a \in \text{No} \iff a : \alpha \rightarrow \{\ominus, \oplus\} \text{ pour un certain } \alpha \in \text{On} \]
\[ \iff a = \bigoplus \bigoplus \bigoplus \bigoplus \bigoplus \cdots \]
\[ \text{length } \alpha \]
Simple and complicated numbers.

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\[ a \in \text{No} \iff a : \alpha \rightarrow \{\ominus, \oplus\} \text{ pour un certain } \alpha \in \text{On} \]
\[ \iff a = \underline{\oplus \oplus \ominus \ominus \ominus \cdots} \]

length \(\alpha\)

Examples:
0 = empty sequence, 1 = \(\oplus\), \(-1 = \ominus\), \(1/2 = \oplus \ominus\),...
Simple and complicated numbers.

Gonshor’s alternative definition (1986):

\[ a \in \text{No} :\iff a : \alpha \to \{\emptyset, \oplus\} \text{ pour un certain } \alpha \in \text{On} \]
\[ \iff a = \oplus \oplus \oplus \oplus \oplus \cdots \]
\[ \text{length } \alpha \]

\[ \rightarrow \text{ Lexicographical total ordering: } \& = \text{concatenation} \]
\[ a \& \emptyset < a < a \& \oplus \]

\[ \rightarrow \text{ Partial ordering } \textit{simplicity}: \]
\[ a \leq_s b :\iff a \text{ is an initial subsequence of } b. \]
Lex-ordered full rooted binary tree of depth $\text{On}$. 

M. Matusinski

Surreal numbers, exp and $\omega$-map.
Structure and...

Theorem (Conway 76, Ehrlich 89, 2001)

\[ \mathbb{No} \text{ is a dense linear ordering without endpoints which canonically contains } \mathbb{R} \text{ and } \mathbb{On}. \]
Structure and... universality!

Theorem (Conway 76, Ehrlich 89, 2001)

\textbf{No} is a dense linear ordering without endpoints which canonically contains \( \mathbb{R} \) and \( \mathrm{On} \).

\( \sim \rightarrow \) \textbf{No} is the \textit{universal domain} for linear orderings.

+ initial embeddings.
Structure and...

Theorem (Conway 76, Ehrlich 89, 2001)

No is a dense linear ordering without endpoints which canonically contains \( \mathbb{R} \) and On.

\( \rightsquigarrow \text{No is the universal domain for linear orderings. + initial embeddings.} \)

Model theory with set theory NBG with Global Choice: the unique (up to isom) monster model (\( \kappa \)-saturated, \( \kappa \)-homogenous, \( \kappa \)-universal for any \( \kappa \))

No \( \succ \mathbb{Q}, \mathbb{R} \) (DLO without endpoints)
Conway’s approach: on games...

How to win a partisan combinatorial games?
Conway’s approach: on games... and numbers

How to win a partisan combinatorial games?

Any GAME has a NUMBER (not a nimber!):

- $n(G) < 0 \iff \text{Left Player} \text{ has a winning way};$
- $n(G) > 0 \iff \text{Right Player} \text{ has a winning way;}
- n(G) = 0 \iff \text{the first to play loses.}$

Any NUMBER is an EQUIVALENCE CLASS OF GAMES
Surreal numbers with algebraic structure.

Surreal numbers with **commutative algebraic operations** recursively defined.
Surreal numbers with algebraic structure.

Surreal numbers with **commutative algebraic operations** recursively defined.

Conway’s original recursive definition (1976):

Dedekind + Von Neumann = Conway’s surreals
Old and young numbers.

RULE 1

\[ a := \{ A_L \mid A_R \} \]

where: \((a^L \in A_L \land a^R \in A_R) \Rightarrow a^L < a^R\).

Examples:

\[ 0 := \{ \emptyset \mid \emptyset \}, \quad 1 := \{ 0 \mid \emptyset \}, \quad 1/2 := \{ 0 \mid 1 \}, \quad \text{etc}... \]
Old and young numbers.

RULE 1

\[ a := \{ A_L \mid A_R \} \]

where: \((a^L \in A_L \land a^R \in A_R) \Rightarrow a^L < a^R.\)

RULE 2  Given \(a = \{ A_L \mid A_R \}\) and \(b = \{ B_L \mid B_R \}\),

\[ a \leq b :\iff (a^L \in A_L \land b^R \in B_R) \Rightarrow (a^L < b \land a < b^R) \]
Algebraic structure.

▶ Addition:

\[ a + b := \{ a^L + b, a + b^L \mid a^R + b, a + b^R \} \]
Algebraic structure.

- Addition: 
  \[ a + b := \{ a^L + b, a + b^L | a^R + b, a + b^R \} \]

- Inverse element: 
  \[ -a := \{ -a^R | -a^L \} \]

- Neutral element: 
  \[ 0 = \{ \emptyset | \emptyset \} \]
Algebraic structure.

- **Addition:**
  \[ a + b := \{ a^L + b, a + b^L \mid a^R + b, a + b^R \} \]

- **Multiplication:**
  \[ a \cdot b := \ldots \]

  - **Inverse element:**
    \[ a^{-1} := \ldots \]

  - **Neutral element:**
    \[ 1 = \{ 0 \mid \emptyset \} \]
From real to surreal.

On **DAY** $\omega$, the first infinite ordinal:

$$\omega := \{1, 2, 3, \ldots \mid \emptyset\}$$
From real to surreal.

has now an *infinitesimal inverse*:

\[ \omega^{-1} = \frac{1}{\omega} := \left\{ 0 \mid \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots \right\} \]
Theorem (Conway 76, Ehrlich 89, 2001)

**No** is an ordered real closed Field which extends:

- the ordered real closed field \( \mathbb{R} \);
- the ordered commutative Semiring \( \mathbb{On} \) (via Hessenberg operations).
Structure and... universality!

Theorem (Conway 76, Ehrlich 89, 2001)

\( \mathbb{N}_o \) is an ordered real closed Field which extends:

- the ordered real closed field \( \mathbb{R} \);
- the ordered commutative Semiring \( \mathcal{O}_n \) (via Hessenberg operations).

\( \mathbb{N}_o \) is the universal domain for:

- ordered Abelian groups;
- real fields.

+ initial embeddings.

\( \mathbb{N}_o \supseteq \mathbb{R} \) (ordered real closed fields)
Conway’s $\omega$-map.

The natural valuation on $\text{No}$:

$$\text{val} : (\text{No}, \cdot, \leq) \to (\text{No}/\sim_+ \cup \{\infty\}, +, \leq)$$

$$a \mapsto [a]_+$$

via the Archimedean equivalence relation $\sim_+$.

$$a \sim_+ b \iff \exists n, \frac{1}{n}|a| \leq |b| \leq n|a|$$
Conway’s $\omega$-map.

▶ The **natural valuation** on $\mathbf{No}$:

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via the Archimedean equivalence relation $\sim_+$.

\[ a \sim_+ b \equiv \exists n, \frac{1}{n} |a| \leq |b| \leq n |a| \]

▶ Conway’s $\omega$-map:

\[ \Omega : (\mathbf{NO}, +, \leq) \leftrightarrow (\mathbf{NO}_{>0}, \cdot, \leq) \]
\[ a \mapsto \omega^a := \{0, \ n \omega^a_L \mid \omega^a_R / 2^n\} \]
Conway's $\omega$-map.

Examples:

1. $\omega^0 = \{0 \mid \emptyset\} = 1$
2. $\omega^1 = \{0, n \mid \emptyset\} = \omega$
3. $\omega^{-1} = \{0 \mid 1/2^n\} = \frac{1}{\omega}$
Conway’s ω-map.

Theorem (Conway)

*The ω-map is a canonical section of val - therefore an exponentiation - which extends ordinal exponentiation.*

So in particular:

\[
\mathbb{N}_0 \simeq \text{val} (\mathbb{N}_0 \setminus \{0\}).
\]
Conway’s $\omega$-map.

Theorem (Conway)

*The $\omega$-map is a canonical section of $\mathrm{val}$ - therefore an exponentiation - which extends ordinal exponentiation.*

So in particular:

\[ \mathbb{N}^{\omega} \simeq \mathrm{val} (\mathbb{N} \setminus \{0\}) . \]

$\omega^{\mathbb{N}}$ is the group of **monomials** and $\mathbb{R} \simeq$ residue field
Canonical Kaplansky’s embedding.

Canonical expansion of surreal numbers as **generalized power series**:

\[ a = \sum_{i < \lambda} r_i \omega^{a_i} \]

uniquely for some \( \lambda \in \text{On} \), \((a_i)_{i < \lambda}\) decreasing in \( \text{No} \) and \((r_i)_{i < \lambda}\) in \( \mathbb{R} \setminus \{0\} \).
Canonical Kaplansky’s’s embedding.

Canonical expansion of surreal numbers as generalized power series:

\[ a = \sum_{i < \omega} r_i \omega^{a_i} \]

\[ \mathbb{No} = \mathbb{R}(\omega^{\mathbb{No}})_{\text{On}} \]

\[ \Rightarrow \text{On-bounded series} \]
Restricted analytic functions

Alling (1987) + van den Dries - Macintyre - Marker (1994): No carries restricted analytic functions: for any $a = a_0 + \varepsilon$,

$$f(a) = f(a_0 + \varepsilon) = \sum_{n \in \mathbb{N}} \frac{f^{(n)}(a_0)}{n!} \varepsilon^n$$
Restricted analytic functions

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In particular, exp for bounded surreal numbers:

$$\exp(a) = \exp(a_0 + \varepsilon) = \sum_{n \in \mathbb{N}} \frac{e^{a_0}}{n!} \varepsilon^n$$
Gonshor’s exp.

**Theorem (Gonshor (1986))**

No carries an **exponential** (and therefore a **logarithm** \( \log = \exp^{-1} \)) which extends the analytic exp:

\[
\exp: (\text{NO}, +, \leq) \rightarrow (\text{NO}_{>0}, \cdot, \leq)
\]
Gonshor’s exp.

**Theorem (Gonshor (1986))**

No carries an exponential (and therefore a logarithm $\log = \exp^{-1}$) which extends the analytic exp:

$$\exp : (\text{NO}, +, \leq) \rightarrow (\text{NO}_{>0}, \cdot, \leq)$$

**Global definition:**  \[ \forall a \in \text{No}, \exp(a) := \cdots \]
Gonshor’s exp.

Theorem (Gonshor (1986))

No carries an exponential (and therefore a logarithm log = exp⁻¹) which extends the analytic exp:

\[ \exp : (NO, +, \leq) \rightarrow (NO_{>0}, \cdot, \leq) \]

Examples:

\[ \exp(0) = 1, \exp(1) = e \text{ and } \exp(r) = e^r \]

\[ \exp(a_0 + \varepsilon) = \sum_{n \in \mathbb{N}} \frac{e^{a_0}}{n!} \varepsilon^n = e^{a_0} \sum_{n \in \mathbb{N}} \frac{1}{n!} \varepsilon^n \]

\[ \exp(\omega) = \omega^\omega \]
Exponential and $\omega$-map.

For any surreal number $a$:

$$\log \left( \omega^{\omega^a} \right) = \omega^{h(a)}$$

where

$$h : \mathbb{No} \cong \mathbb{No}_{>0} \quad \text{(as ordered classes)}$$

$$a \mapsto h(a) := \{0, h(a^L) \mid h(a^R), \omega^a/2^n\}$$
Universality again: $\mathbb{R}_{\text{an,exp}}$

Theorem (Alling 87, Gonshor 86, van den Dries-Ehrlich 2001)

No is a real analytic and real exponential Field which extends $\mathbb{R}_{\text{an,exp}}$
Universality again: $\mathbb{R}_{\text{an},\text{exp}}$

**Theorem (Alling 87, Gonshor 86, van den Dries-Ehrlich 2001)**

*No* is a real analytic and real exponential Field which extends $\mathbb{R}_{\text{an},\text{exp}}$

$\leadsto$ *No* is the **universal domain** for real analytic and exponential fields.

+ initial embeddings (Ehrlich-Kaplan preprint)

*No* $\supseteq \mathbb{R}_{\text{an},\text{exp}}$ (real analytic and exponential fields)
Surreal numbers with derivation.

Next step?

⇝ No as a universal domain for non oscillating differentiable (germs of) real functions:

Hardy fields
Surreal derivation and transseries.

**Surreal derivation:** a derivation $d$ such that

- $\ker d = \mathbb{R} \simeq$ residue field of the natural valuation;
- $a > \mathbb{R} \Rightarrow a' > 0$;
- $\rightarrow H$-field (Aschenbrenner - van den Dries)
- strong linearity;
- $d(\exp(a)) = \exp(a) \cdot d(a)$;

This implies: strong l’Hospital’s rule, rule for the logarithmic derivative and strong Leibniz rule (cf Kuhlmann-M.)

**Transseries field:** in the sense of M. Schmeling
Surreal derivation and transseries.

Surreal derivation:

**Transseries field** (Schmeling): field \((T \subseteq \mathbb{R}((G)), \log)\) such that:

1. \((T1)\) the domain of \(\log\) consists of the positive series;
2. \((T2)\) \(\log(G) \subseteq \mathbb{R}((G_{>1}))\);
3. \((T3)\) \(\log(1 + \varepsilon) = \sum_{n \geq 1}(-1)^{n+1} \frac{\varepsilon^n}{n}\) for any \(\varepsilon \in \mathbb{R}((G_{>1}))\);
4. \((T4)\) about *log-atomic elements*, technical...
Results of Berarducci - Mantova.

Theorem (Berarducci - Mantova 2018)  
\textbf{No} carries a surreal derivation \( d_{BM} \) which is "the simplest".

HOW? By proving that \textbf{No} is a field of transseries

- axioms of Schmeling
- identifying the \textit{log-atomic elements} = lambda-numbers extending Kuhlmann–M. kappa-numbers
- etc…
Universality continued: Hardy fields and transseries.

Theorem (Aschenbrenner-van den Dries-van der Hoeven 2019, Ehrlich-van den Dries preprint)

*We have*

\[ \mathbb{N}_o \succcurlyeq \mathbb{T}, \]

*i.e. \( \mathbb{N}_o \) is a Liouville closed \( H \)-fields with DIVP and with small derivation.*

\( \succcurlyeq \) \( \mathbb{N}_o \) is the universal domain for \( H \)-fields.

In particular:

- **Hardy fields:** \( \mathcal{H} \hookrightarrow \mathbb{N}_o \).

- Transseries, LE-series, (some) EL-series.
Theorem (Berarducci - Mantova 2019)

There is a (partial) composition on $\mathbb{No}$ by the subfield of $\omega$-series, in particular by classical transseries / LE-series / EL-series:

$$\circ : \mathbb{R}\{\{\omega\}\} \times \mathbb{No}^{>\mathbb{R}} \rightarrow \mathbb{No}$$

(Idea: $\Omega(1) = \omega \leftrightarrow$ germ of identity at $+\infty$)

BUT $d_{BM}$ is not compatible with (a global extension of) it!
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Omega-fields.

**Problem:** to find a compatible derivation on $\mathbb{No}$.

$\Rightarrow$ adapt Kuhlmann-M.2011 on EL-series to $\kappa$-bounded EL-series fields (Kuhlmann-Shelah 2005) + composition.

**Omega-field** (Berarducci-Kuhlmann-Mantova-M.): a real-closed field which is isomorphic to the value group of its natural valuation:

$$\mathbb{K} \cong \text{val}(\mathbb{K} \setminus \{0\})$$
Omega-fields.

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**Omega-field** (Berarducci-Kuhlmann-Mantova-M.): a real-closed field which is isomorphic to the value group of its natural valuation:

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Our results...

In particular, for $K = \mathbb{R}((G))_\kappa$ $\kappa$-bounded series field ($\kappa$ regular uncountable cardinal).

**Theorem (Berarducci - Kuhlmann - Mantova - M. 2020)**

*There are omega-fields $\mathbb{R}((G))_\kappa$.*

Any $(\mathbb{R}((G))_\kappa, \Omega)$ admits log (and therefore an exp) determined by $\Omega$ and by any $h : \mathbb{R}((G))_\kappa \simeq (\mathbb{R}((G))_\kappa)_{>0}$ (ordered sets). Depending on $h$, either $K \models T_{an,exp}$, or not even o-minimal.

Conversely, $(\mathbb{R}((G))_\kappa, \log)$ admits $\Omega$ if and only if $G \cong G_{>1}$ (as ordered sets). In this case, $\exists h : \mathbb{R}((G))_\kappa \simeq (\mathbb{R}((G))_\kappa)_{>0}$ linking $\Omega$ and $\log$.

In particular, $\exists \mathbb{R}((G))_\kappa$ with log but no $\Omega$.
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In particular, $\exists \mathbb{R}((G))_\kappa$ with log but no $\Omega$. 
Ideas of the proofs.

- existence of omega-fields $\mathbb{R}((G))_\kappa$ with strong $\Omega$;
- log compatible with omega based on:

$$\forall a, \log (\omega^a) = \omega^{h(a)}$$

We put:

$$\log (\omega \sum_i r_i \omega^{a_i}) := \sum_i r_i \omega^{h(a_i)}$$

and

$$\log (r \omega^a (1 + \varepsilon)) = \ln(r) + \log (\omega^a) + \sum_n \frac{1}{n} \varepsilon^n.$$
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$$\log \left( r \omega^a (1 + \varepsilon) \right) = \ln(r) + \log \left( \omega^a \right) + \sum_n \frac{1}{n} \varepsilon^n.$$
Ideas of the proofs.

- Choice of $h : K \cong K > 0$ determines log via the Growth Axioms Scheme (Ressayre).
  
  E.g. $h(a) = (-a + 1)^{-1}$ if $a \leq 0$ and $h(a) = a + 1$ if $a \geq 0$.

  Construction of examples with (GA): $h(a) < r \omega^a$
  (compare with Gonshor’s $h$).

- $\psi : G \cong G > 1$ s.t. $\omega^g = \exp(\psi(g))$.

- To get $G \not\cong G > 1$, start with the Hahn group over $\Gamma_0 = \omega_1 \times_{\text{lex}} \mathbb{Z}$...
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- **ψ :** $G \cong G_{>1}$ s.t. $\omega^g = \exp(\psi(g))$. .

- To get $G \not\cong G_{>1}$, start with the Hahn group over $\Gamma_0 = \omega_1 \times \text{lex } \mathbb{Z}$. .
To be continued...

Theorem (Berarducci - Freni preprint)

*The field of transseries $\mathbb{T}$ is an omega-field.*

To do list:

- compatible derivations for $\kappa$-bounded-omega-fields
- composition for $\kappa$-bounded-omega-fields
- classification of omega-groups
- model theory of omega-fields

Question: *is there a transexponential o-minimal structure?*
Announcement
This will be one of the topics of the

Thematic Program on
Tame Geometry, Transseries and Applications to Analysis and Geometry
(Fields Inst., January–June, 2022)
Thank you for your attention!

...and would very much like to see you at the Fields in Toronto!