Surreal numbers with exponential function and omega-exponentiation.

M. Matusinski (U. Bordeaux)

joint work with A. Berarducci, S. Kuhlmann and V. Mantova

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Simple and complicated numbers.

Gonshor's alternative definition (1986):

$$\begin{array}{rcl} a \in \mathbf{No} & :\Leftrightarrow & a : \alpha \to \{\ominus, \oplus\} \text{ pour un certain } \alpha \in \mathbf{On} \\ & \Leftrightarrow & a = \underbrace{\oplus \oplus \ominus \oplus \ominus \ominus \cdots}_{\textbf{length } \alpha} \end{array}$$

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Examples:
0= empty sequence, $1 = \oplus$, $-1 = \ominus$, $1/2 = \oplus\ominus$,...

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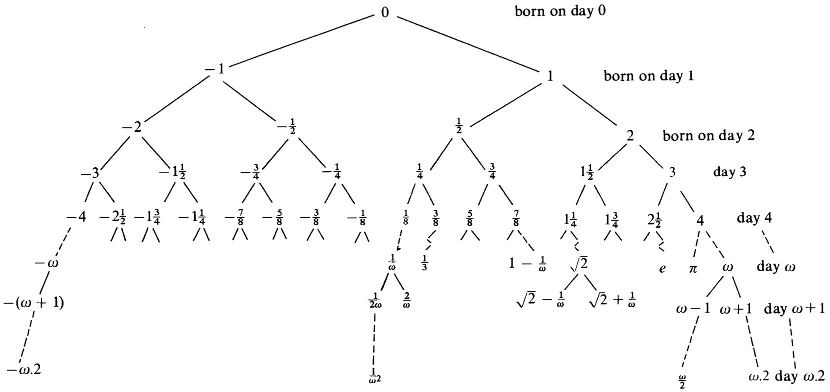
$\rightarrow$ **Lexicographical total ordering**: & = concatenation

$$a\&\ominus \;<\; a \;<\; a\&\oplus$$

$\rightarrow$ **Partial ordering *simplicity***:

$$a \leq_s b :\Leftrightarrow a \text{ is an initial subsequence of } b.$$

Lex-ordered full rooted binary tree of depth **On**.



Structure and...

Theorem (Conway 76, Ehrlich 89, 2001)

No *is a dense linear ordering without endpoints which canonically contains* $\mathbb{R}$ *and* **On**.

Structure and... universality!

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+ initial embeddings.

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Model theory with set theory NBG with Global Choice : the unique (up to isom) monster model (κ*-saturated,* κ*-homogenous,* κ*-universal for any* κ*)*

$$\mathbf{No} \succcurlyeq \mathbb{Q},\ \mathbb{R} \quad (DLO\ without\ endpoints)$$

Conway's approach: on games...

How to win a

partisan combinatorial games?

Conway's approach: on games... and numbers

How to win a

partisan combinatorial games?

Any GAME has a NUMBER (not a nimber!):

- $n(G) < 0 \Leftrightarrow$ Left Player has a winning way;
- $n(G) > 0 \Leftrightarrow$ Right Player has a winning way;
- $n(G) = 0 \Leftrightarrow$ the first to play loses.

Any NUMBER is an EQUIVALENCE CLASS OF GAMES

Surreal numbers with algebraic structure.

Surreal numbers with **commutative algebraic operations** $\rightsquigarrow$ recursively defined.

Surreal numbers with algebraic structure.

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Conway's original recursive definition (1976):

Dedekind + Von Neumann = Conway's surreals

Old and young numbers.

RULE 1

$$a := \{A_L \mid A_R\}$$

where: $(a^L \in A_L \wedge a^R \in A_R) \Rightarrow a^L < a^R$.

Examples:

$$0 := \{\emptyset \mid \emptyset\}, \quad 1 := \{0 \mid \emptyset\}, \quad 1/2 := \{0 \mid 1\}, \quad \text{etc...}$$

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RULE 1

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where: $(a^L \in A_L \wedge a^R \in A_R) \Rightarrow a^L < a^R$.

RULE 2 Given $a = \{A_L \mid A_R\}$ and $b = \{B_L \mid B_R\}$,

$$a \leq b \;:\Longleftrightarrow$$

$$(a^L \in A_L \wedge b^R \in B_R) \Rightarrow (a^L < b \wedge a < b^R)$$

Algebraic structure.

- **Addition:**

$$a + b := \{a^L + b,\ a + b^L \mid a^R + b,\ a + b^R\}$$

Algebraic structure.

- **Addition:**

$$a + b := \{a^L + b,\, a + b^L \mid a^R + b,\, a + b^R\}$$

inverse element:

$$-a := \{-a^R \mid -a^L\}$$

neutral element:

$$0 = \{\emptyset \mid \emptyset\}$$

Algebraic structure.

- **Addition:**

$$a + b := \{a^L + b,\, a + b^L \mid a^R + b,\, a + b^R\}$$

- **Multiplication:**

$$a \cdot b := \cdots$$

inverse element:

$$a^{-1} := \cdots$$

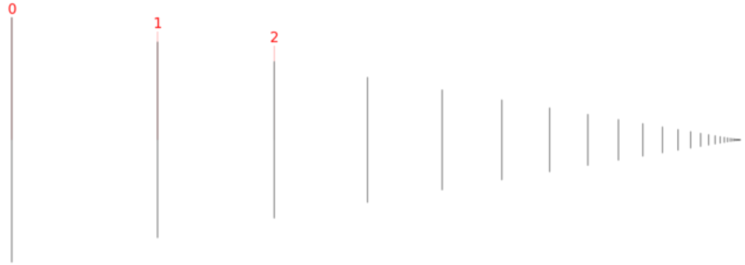
neutral element:

$$1 = \{0 \mid \emptyset\}$$

From real to surreal.

On **DAY** ω, the first infinite ordinal:

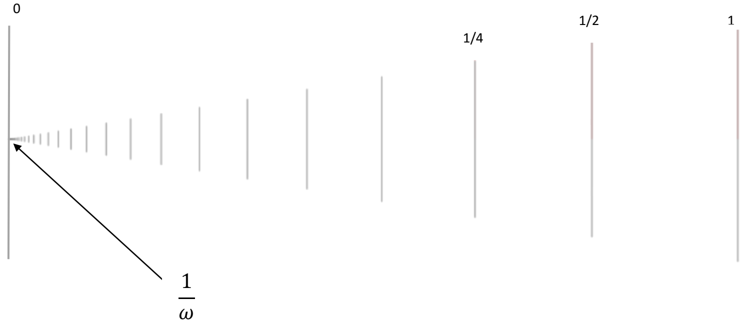
$$\omega := \{1, 2, 3, \dots \mid \emptyset\}$$



From real to surreal.

has now an *infinitesimal inverse*:

$$\omega^{-1} = \frac{1}{\omega} := \left\{ 0 \mid \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots \right\}$$



Structure and...

Theorem (Conway 76, Ehrlich 89, 2001)

No *is an ordered real closed Field which extends:*

- *the ordered real closed field* $\mathbb{R}$*;*
- *the ordered commutative Semiring* **On** *(via Hessenberg operations).*

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$\rightsquigarrow$ **No** *is the* ***universal domain*** *for:*

- *ordered Abelian groups;*
- *real fields.*

+ initial embeddings.

$$\textbf{No} \succcurlyeq \mathbb{R} \text{ (ordered real closed fields)}$$

Conway's ω-map.

- The **natural valuation** on **No**:

$$\begin{array}{rccl} \text{val}: & (\text{NO}, \cdot, \leq) & \to & (\text{NO}_{/\sim_+} \cup \{\infty\}, +, \leq) \\ & a & \mapsto & [a]_+ \end{array}$$

via the Archimedean equivalence relation $\sim_+$.

$$a \sim_+ b \Leftrightarrow \exists n,\ \frac{1}{n}|a| \leq |b| \leq n|a|$$

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- **Conway's ω-map**:

$$\begin{array}{rccl} \Omega: & (\mathrm{NO}, +, \leq) & \hookrightarrow & (\mathrm{NO}_{>0}, \cdot, \leq) \\ & a & \mapsto & \omega^a := \{0,\ n\omega^{a^L} \mid \omega^{a^R}/2^n\} \end{array}$$

Conway's ω-map.

Examples:

- $\omega^0 = \{0 \mid \emptyset\} = 1$
- $\omega^1 = \{0,\, n \mid \emptyset\} = \omega$
- $\omega^{-1} = \{0 \mid 1/2^n\} = \dfrac{1}{\omega}$

Conway's ω-map.

Theorem (Conway)

The ω-map is a canonical section of val - therefore an exponentiation - which extends ordinal exponentiation.

So in particular:

$$\boxed{\mathbf{No} \simeq \mathrm{val}\left(\mathbf{No} \setminus \{0\}\right).}$$

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$\omega^{\mathbf{No}}$ is the group of **monomials** and $\mathbb{R} \simeq$ residue field

Canonical Kaplansky's embedding.

Canonical expansion of surreal numbers as **generalized power series**:

$\forall a \in \mathbf{No}$,

$$a = \sum_{i<\lambda} r_i \, \omega^{a_i}$$

uniquely for some $\lambda \in \mathbf{On}$, $(a_i)_{i<\lambda}$ *decreasing* in $\mathbf{No}$ and $(r_i)_{i<\lambda}$ in $\mathbb{R} \setminus \{0\}$.

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$$a = \sum_{i<\lambda} r_i \, \omega^{a_i}$$

$$\boxed{\mathbf{No} = \mathbb{R}((\omega^{\mathbf{No}}))_{\mathbf{On}}}$$

$\Rightarrow$ **On**-bounded series

Restricted analytic functions

Alling (1987) + van den Dries - Macintyre - Marker (1994):
No carries **restricted analytic functions**: for any $a = a_0 + \varepsilon$,

$$f(a) = f(a_0 + \varepsilon) = \sum_{n \in \mathbb{N}} \frac{f^{(n)}(a_0)}{n!} \varepsilon^n$$

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In particular, exp for *bounded* surreal numbers:

$$\exp(a) = \exp(a_0 + \varepsilon) = \sum_{n \in \mathbb{N}} \frac{e^{a_0}}{n!} \varepsilon^n$$

Gonshor's exp.

Theorem (Gonshor (1986))

No *carries an* ***exponential*** *(and therefore a* ***logarithm*** $\log = \exp^{-1}$*) which extends the analytic exp:*

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Global definition: $\forall a \in \mathbf{No},\ \exp(a) := \cdots$

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Examples:

$\exp(0) = 1$, $\exp(1) = e$ and $\exp(r) = e^r$

$$\exp(a_0 + \varepsilon) = \sum_{n \in \mathbb{N}} \frac{e^{a_0}}{n!}\varepsilon^n = e^{a_0} \sum_{n \in \mathbb{N}} \frac{1}{n!}\varepsilon^n$$

$\exp(\omega) = \omega^\omega$

Exponential and ω-map.

- For any surreal number a:

$$\boxed{\log\left(\omega^{\omega^{a}}\right) = \omega^{h(a)}}$$

where

$$\begin{array}{rccl} h: & \mathbf{No} & \simeq & \mathbf{No}_{>0} \quad \text{(as ordered classes)} \\ & a & \mapsto & h(a) := \left\{0,\ h(a^L) \mid h(a^R),\ \omega^a/2^n\right\} \end{array}$$

Universality again: $\mathbb{R}_{\mathrm{an},\exp}$

Theorem (Alling 87, Gonshor 86, van den Dries-Ehrlich 2001)

No *is a real analytic and real exponential Field which extends* $\mathbb{R}_{\mathrm{an},\exp}$

Universality again: $\mathbb{R}_{\mathrm{an},\exp}$

Theorem (Alling 87, Gonshor 86, van den Dries-Ehrlich 2001)

No *is a real analytic and real exponential Field which extends* $\mathbb{R}_{\mathrm{an},\exp}$

$\rightsquigarrow$ **No** *is the **universal domain** for real analytic and exponential fields.*

+ initial embeddings (Ehrlich-Kaplan preprint)

$$\mathbf{No} \succcurlyeq \mathbb{R}_{\mathrm{an},\exp} \text{ (real analytic and exponential fields)}$$

Surreal numbers with derivation.

Next step?

$\rightsquigarrow$ **No** as a universal domain for non oscillating differentiable (germs of) real functions:

Hardy fields

Surreal derivation and transseries.

Surreal derivation: a derivation d such that

- $\ker d = \mathbb{R} \simeq$ residue field of the natural valuation;
- $a > \mathbb{R} \Rightarrow a' > 0$;
  $\rightarrow$ ***H*-field** (Aschenbrenner - van den Dries)
- strong linearity;
- $d(\exp(a)) = \exp(a) \cdot d(a)$;

This implies: strong l'Hospital's rule, rule for the logarithmic derivative and strong Leibniz rule (cf Kuhlmann-M.)

Transseries field: in the sense of M. Schmeling

Surreal derivation and transseries.

Surreal derivation:

Transseries field (Schmeling): field $(\mathbb{T} \subseteq \mathbb{R}((G)), \log)$ such that:

(T1) the domain of $\log$ consists of the positive series;

(T2) $\log(G) \subseteq \mathbb{R}((G_{>1}))$;

(T3) $\log(1+\varepsilon) = \sum_{n\geq 1}(-1)^{n+1}\frac{\varepsilon^n}{n}$ for any $\varepsilon \in \mathbb{R}((G_{>1}))$;

(T4) about *log-atomic elements*, technical...

Results of Berarducci - Mantova.

Theorem (Berarducci - Mantova 2018)
No *carries a surreal derivation d_{BM} which is "the simplest".*

HOW? By proving that **No** is a **field of transseries**

- axioms of Schmeling
- identifying the *log-atomic elements* = lambda-numbers extending Kuhlmann–M. kappa-numbers
- etc...

Universality continued: Hardy fields and transseries.

Theorem (Aschenbrenner-van den Dries-van der Hoeven 2019, Ehrlich-van den Dries preprint)

We have

$$\mathbf{No} \succcurlyeq \mathbb{T},$$

i.e. **No** *is a* ***Liouville closed H-fields with DIVP and with small derivation****.*

$\rightsquigarrow$ **No** *is the universal domain for H-fields.*

In particular:

Hardy fields: $\mathcal{H} \hookrightarrow$ **No.**

Transseries, LE-series, (some) EL-series.

BUT...

Theorem (Berarducci - Mantova 2019)

There is a (partial) composition on **No** *by the subfield of ω-series, in particular by classical transseries / LE-series / EL-series:*

$$\circ : \mathbb{R}\{\{\omega\}\} \times \mathbf{No}^{>\mathbb{R}} \to \mathbf{No}$$

(Idea: $\Omega(1) = \omega \leftrightarrow$ germ of identity at $+\infty$)

BUT d_{BM} is not compatible with (a global extension of) it!

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Omega-fields.

Problem: to find a compatible derivation on **No**.

$\rightsquigarrow$ adapt Kuhlmann-M.2011 on EL-series to κ-bounded EL-series fields (Kuhlmann-Shelah 2005) + composition.

Omega-field (Berarducci-Kuhlmann-Mantova-M.): a real-closed field which is isomorphic to the value group of its natural valuation:

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Our results...

In particular, for $\mathbb{K} = \mathbb{R}((G))_\kappa$ κ-bounded series field (κ regular uncountable cardinal).

Theorem (Berarducci - Kuhlmann - Mantova - M. 2020)

There are omega-fields $\mathbb{R}((G))_\kappa$.
Any $(\mathbb{R}((G))_\kappa, \Omega)$ admits log (and therefore an exp) determined by Ω and by any $h : \mathbb{R}((G))_\kappa \simeq (\mathbb{R}((G))_\kappa)_{>0}$ (ordered sets). Depending on h, either $\mathbb{K} \models T_{\mathrm{an,exp}}$, or not even o-minimal.

Conversely, $(\mathbb{R}((G))_\kappa, \log)$ admits Ω if and only if $G \cong G_{>1}$ (as ordered sets). In this case, $\exists\ h : \mathbb{R}((G))_\kappa \simeq (\mathbb{R}((G))_\kappa)_{>0}$ linking Ω and log.

In particular, $\exists\ \mathbb{R}((G))_\kappa$ with log but no Ω.

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Ideas of the proofs.

- existence of omega-fields $\mathbb{R}((G))_\kappa$ with strong Ω;
- log compatible with omega based on:

$$\forall a, \log\left(\omega^{\omega^a}\right) = \omega^{h(a)}$$

We put:

$$\log\left(\omega^{\sum_i r_i \omega^{a_i}}\right) := \sum_i r_i \omega^{h(a_i)}$$

and

$$\log\left(r\omega^a(1+\varepsilon)\right) = \ln(r) + \log\left(\omega^a\right) + \sum_n \frac{1}{n}\varepsilon^n.$$

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- Choice of $h : \mathbb{K} \simeq \mathbb{K}_{>0}$ determines log via the Growth Axioms Scheme (Ressayre).
  E.g. $h(a) = (-a+1)^{-1}$ if $a \leq 0$ and $h(a) = a+1$ if $a \geq 0$.
  Construction of examples with (GA): $h(a) < r\omega^a$ (compare with Gonshor's h).
- $\psi : G \cong G_{>1}$ s.t. $\omega^g = \exp(\psi(g))$. .
- To get $G \ncong G_{>1}$, start with the Hahn group over $\Gamma_0 = \omega_1 \times_{\text{lex}} \mathbb{Z}$...

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- To get $G \not\cong G_{>1}$, start with the Hahn group over $\Gamma_0 = \omega_1 \times_{\text{lex}} \mathbb{Z}$...

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To be continued...

Theorem (Berarducci - Freni preprint)
The field of transseries $\mathbb{T}$ is an omega-field.

To do list:

- compatible derivations for κ-bounded-omega-fields
- composition for κ-bounded-omega-fields
- classification of omega-groups
- model theory of omega-fields

Question: *is there a transexponential o-minimal structure?*

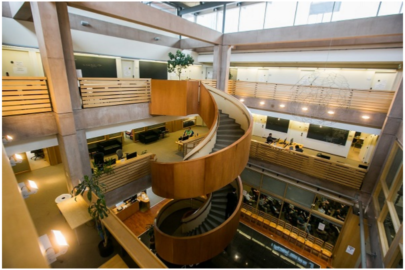
Announcement

This will be one of the topics of the

Thematic Program on

Tame Geometry, Transseries and Applications to Analysis and Geometry

(Fields Inst., January–June, 2022)



Thank you for your attention!

...and would very much like to see you at the Fields in Toronto!