Primary ideals and differential operators

Yairon Cid-Ruiz

Ghent University

Includes joint works with: Bernd Sturmfels and Roser Homs.

MAX seminar

February 22, 2021

Lasker–Noether theorem

Theorem

Let R be a Noetherian commutative ring and $I \subset R$ be an ideal. Then, there exists a **primary decomposition**

$$I = Q_1 \cap Q_2 \cap \cdots \cap Q_m$$

(where each Q_i is a primary ideal).

Lasker–Noether theorem

Theorem

Let R be a Noetherian commutative ring and $I \subset R$ be an ideal. Then, there exists a **primary decomposition**

$$I = Q_1 \cap Q_2 \cap \cdots \cap Q_m$$

(where each Q_i is a primary ideal).

An ideal Q is primary if $\text{Ass}(Q) = \{P\}$ (where $\sqrt{Q} = P$).

Lasker–Noether theorem

Theorem

Let R be a Noetherian commutative ring and $I \subset R$ be an ideal. Then, there exists a **primary decomposition**

$$I = Q_1 \cap Q_2 \cap \cdots \cap Q_m$$

(where each Q_i is a primary ideal).

An ideal Q is primary if $\mathrm{Ass}(Q) = \{P\}$ (where $\sqrt{Q} = P$).

Slogan

Primary ideals are the **basic building blocks** in ideal theory.

Theorem (Palamodov, 1964)

Let $R = \mathbb{C}[x_1, \ldots, x_n]$, $\mathfrak{p} \subset R$ be a prime ideal and $Q \subset R$ be a $\mathfrak{p}$-primary ideal. Then, there exist **(Noetherian)** differential operators

$$A_1, \ldots, A_t \in D_n(\mathbb{C}) = R\langle \partial_{x_1}, \ldots, \partial_{x_n} \rangle,$$

such that

$$Q = \left\{ f \in R \mid A_j \cdot f \in \mathfrak{p} \text{ for all } 1 \leq j \leq t \right\}.$$

Theorem (Palamodov, 1964)

Let $R = \mathbb{C}[x_1, \ldots, x_n]$, $\mathfrak{p} \subset R$ be a prime ideal and $Q \subset R$ be a $\mathfrak{p}$-primary ideal. Then, there exist **(Noetherian)** differential operators

$$A_1, \ldots, A_t \in D_n(\mathbb{C}) = R\langle \partial_{x_1}, \ldots, \partial_{x_n} \rangle,$$

such that

$$Q = \left\{ f \in R \mid A_j \cdot f \in \mathfrak{p} \text{ for all } 1 \leq j \leq t \right\}.$$

$A_1, \ldots, A_t$ are referred to as **Noetherian operators** that encode the $\mathfrak{p}$-primary ideal Q.

Theorem (Palamodov, 1964)

Let $R = \mathbb{C}[x_1, \ldots, x_n]$, $\mathfrak{p} \subset R$ be a prime ideal and $Q \subset R$ be a $\mathfrak{p}$-primary ideal. Then, there exist **(Noetherian)** differential operators

$$A_1, \ldots, A_t \in D_n(\mathbb{C}) = R\langle \partial_{x_1}, \ldots, \partial_{x_n} \rangle,$$

such that

$$Q = \left\{ f \in R \mid A_j \cdot f \in \mathfrak{p} \text{ for all } 1 \leq j \leq t \right\}.$$

$A_1, \ldots, A_t$ are referred to as **Noetherian operators** that encode the $\mathfrak{p}$-primary ideal Q.

Interesting for (at least) two reasons:

- Describe primary ideals via the use of differential operators.
- Fundamental Principle of Ehrenpreis and Palamodov: solutions of linear systems of PDE with constant coefficients.

A little bit of history and context

- The existence of "Noetherian operators" for **zero dimensional primary ideals** is due to Gröbner in the 1930's.

A little bit of history and context

- The existence of "Noetherian operators" for **zero dimensional primary ideals** is due to Gröbner in the 1930's.
- Gröbner (1952), "La théorie des idéaux et la géométrie algébrique".

A little bit of history and context

- The existence of "Noetherian operators" for **zero dimensional primary ideals** is due to Gröbner in the 1930's.

- Gröbner (1952), "La théorie des idéaux et la géométrie algébrique".

⋆ *In 1960, Ehrenpreis announced his Fundamental Principle, which states that the solutions of a linear system of PDE with constant coefficients can be represented in terms of certain integrals. At the core of the Fundamental Principal, one has **the existence of Noetherian operators for primary ideals in** $\mathbb{C}[x_1, \ldots, x_n]$.*

But, there was a mistake in his proof. Palamodov pointed out the mistake. Then, complete and correct proofs were presented (independently) by Ehrenpreis and Palamodov.

Later, other slightly different proofs were given by Björk and Hörmander.

Later algebraic developments

- Brumfiel (1978), "Differential operators and primary ideals".

Later algebraic developments

- Brumfiel (1978), "Differential operators and primary ideals".
- Oberst (1999), "The construction of Noetherian operators".

(after 2020)

Later algebraic developments

- Brumfiel (1978), "Differential operators and primary ideals".
- Oberst (1999), "The construction of Noetherian operators".

(after 2020)

- Cid-Ruiz (2020), "Noetherian operators, primary submodules and symbolic powers", `Collectanea Mathematica`.

Later algebraic developments

- Brumfiel (1978), “Differential operators and primary ideals”.
- Oberst (1999), “The construction of Noetherian operators”.

(after 2020)

- Cid-Ruiz (2020), “Noetherian operators, primary submodules and symbolic powers”, `Collectanea Mathematica`.
- Cid-Ruiz, Homs, Sturmfels (2020), “Primary ideals and their differential equations”, `Foundations of Computational Mathematics`.

Later algebraic developments

- Brumfiel (1978), "Differential operators and primary ideals".
- Oberst (1999), "The construction of Noetherian operators".

(after 2020)

- Cid-Ruiz (2020), "Noetherian operators, primary submodules and symbolic powers", `Collectanea Mathematica`.
- Cid-Ruiz, Homs, Sturmfels (2020), "Primary ideals and their differential equations", `Foundations of Computational Mathematics`.
- Chen, Härkönen, Krone, Leykin (2020), "Noetherian operators and primary decomposition".

Later algebraic developments

- Brumfiel (1978), "Differential operators and primary ideals".
- Oberst (1999), "The construction of Noetherian operators".

(after 2020)

- Cid-Ruiz (2020), "Noetherian operators, primary submodules and symbolic powers", `Collectanea Mathematica`.
- Cid-Ruiz, Homs, Sturmfels (2020), "Primary ideals and their differential equations", `Foundations of Computational Mathematics`.
- Chen, Härkönen, Krone, Leykin (2020), "Noetherian operators and primary decomposition".
- Chen, Cid-Ruiz, Härkönen, Krone, Leykin (2021),"Noetherian Operators in Macaulay2".

Later algebraic developments

- Brumfiel (1978), "Differential operators and primary ideals".
- Oberst (1999), "The construction of Noetherian operators".

(after 2020)

- Cid-Ruiz (2020), "Noetherian operators, primary submodules and symbolic powers", `Collectanea Mathematica`.
- Cid-Ruiz, Homs, Sturmfels (2020), "Primary ideals and their differential equations", `Foundations of Computational Mathematics`.
- Chen, Härkönen, Krone, Leykin (2020), "Noetherian operators and primary decomposition".
- Chen, Cid-Ruiz, Härkönen, Krone, Leykin (2021),"Noetherian Operators in Macaulay2".
- Cid-Ruiz, Sturmfels (2021),"Primary decomposition with differential operators".

Palamodov's example

Let:

- $R = \mathbb{C}[x_1, x_2, x_3]$.
- $Q = (x_1x_3 - x_2, x_2^2, x_3^2) \subset R$.
- $\mathfrak{p} = (x_2, x_3) = \sqrt{I} \subset R$.

The Noetherian operators for I are $A_1 = 1$ and $A_2 = \partial_{x_3} + x_1\partial_{x_2}$, that is

$$Q = \left\{ f \in R \mid f \in \mathfrak{p} \;\text{ and }\; (\partial_{x_3} + x_1\partial_{x_2}) \cdot f \in \mathfrak{p} \right\}.$$

Palamodov's example

Let:

- $R = \mathbb{C}[x_1, x_2, x_3]$.
- $Q = (x_1x_3 - x_2, x_2^2, x_3^2) \subset R$.
- $\mathfrak{p} = (x_2, x_3) = \sqrt{I} \subset R$.

The Noetherian operators for I are $A_1 = 1$ and $A_2 = \partial_{x_3} + x_1\partial_{x_2}$, that is

$$Q = \left\{ f \in R \mid f \in \mathfrak{p} \ \text{ and } \ (\partial_{x_3} + x_1\partial_{x_2}) \cdot f \in \mathfrak{p} \right\}.$$

This example was given by Palamodov as a counter-example to Ehrenpreis' **wrong** claim:

"*Noetherian operators can always be chosen with constant coefficients*".

Probably the simplest example, because the claim holds for the monomial and zero-dimensional cases (see Sturmfels (2002)).

Palamodov's example (continuation)

Find all the smooth solutions $f : \mathbb{C}^3 \to \mathbb{C}$ of the system

$$\frac{\partial^2 f}{\partial_{x_1}\partial_{x_3}} - \frac{\partial f}{\partial_{x_2}} = \frac{\partial^2 f}{\partial^2_{x_2}} = \frac{\partial^2 f}{\partial^2_{x_3}} = 0.$$

We compute all the solutions with the Fundamental Principle:

1. The system corresponds to $Q = \left(x_1x_3 - x_2, x_2^2, x_3^2\right)$.

Palamodov's example (continuation)

Find all the smooth solutions $f : \mathbb{C}^3 \to \mathbb{C}$ of the system

$$\frac{\partial^2 f}{\partial_{x_1}\partial_{x_3}} - \frac{\partial f}{\partial_{x_2}} = \frac{\partial^2 f}{\partial_{x_2}^2} = \frac{\partial^2 f}{\partial_{x_3}^2} = 0.$$

We compute all the solutions with the Fundamental Principle:

1. The system corresponds to $Q = \left(x_1x_3 - x_2, x_2^2, x_3^2\right)$.
2. $Q = \{f \in R \mid f \in \mathfrak{p} \text{ and } \left(\partial_{x_3} + x_1\partial_{x_2}\right) \cdot f \in \mathfrak{p}\}$ where $\mathfrak{p} = (x_2, x_3)$.

Palamodov's example (continuation)

Find all the smooth solutions $f : \mathbb{C}^3 \to \mathbb{C}$ of the system

$$\frac{\partial^2 f}{\partial_{x_1}\partial_{x_3}} - \frac{\partial f}{\partial_{x_2}} = \frac{\partial^2 f}{\partial^2_{x_2}} = \frac{\partial^2 f}{\partial^2_{x_3}} = 0.$$

We compute all the solutions with the Fundamental Principle:

1. The system corresponds to $Q = \left(x_1x_3 - x_2, x_2^2, x_3^2\right)$.
2. $Q = \{f \in R \mid f \in \mathfrak{p} \;\text{ and }\; (\partial_{x_3} + x_1\partial_{x_2}) \cdot f \in \mathfrak{p}\}$ where $\mathfrak{p} = (x_2, x_3)$.
3. By the **Fundamental Principle** there exist measures $\mu_i(\xi) = \mu_i(\xi_1, \xi_2, \xi_3)$ with $i = 1, 2$ supported on $V(\xi_2, \xi_3)$ such that

$$\begin{aligned} f(x_1, x_2, x_3) &= \int_{V(\xi_2,\xi_3)} 1 \cdot \exp(\mathbf{x} \cdot \xi) d\mu_1(\xi) + \int_{V(\xi_2,\xi_3)} (x_3 + \xi_1 x_2) \cdot \exp(\mathbf{x} \cdot \xi) d\mu_2(\xi) \\ &= \int \exp(x_1\xi_1) d\mu_1(\xi_1) + \int (x_3 + \xi_1 x_2) \exp(x_1\xi_1) d\mu_2(\xi_1) \\ &= \varphi(x_1) + \int (x_3 + \xi_1 x_2) \exp(x_1\xi_1) d\mu_2(\xi_1). \end{aligned}$$

Objectives

1. Characterize primary ideals with the "use of differential operators" in more general classes of ring.
2. Parametrize primary ideals.
3. Characterize ideal membership with differential conditions.
4. Obtain algorithms for the above tasks in the case of polynomial rings over a field of characteristic zero.

Definition

Let R be a commutative ring and A be a subring. Let M, N be R-modules. The n-th order A-linear differential operators $\mathrm{Diff}^n_{R/A}(M,N) \subseteq \mathrm{Hom}_A(M,N)$ are defined inductively by:

1. $\mathrm{Diff}^0_{R/A}(M,N) := \mathrm{Hom}_R(M,N)$.
2. $\mathrm{Diff}^n_{R/A}(M,N) := \left\{\delta \in \mathrm{Hom}_A(M,N) \mid [\delta, r] \in \mathrm{Diff}^{n-1}_{R/A}(M,N) \text{ for all } r \in R\right\}$, where $[\delta, r](m) = \delta(rm) - r\delta(m)$ for all $m \in M$.

The A-linear differential operators are given by

$$\mathrm{Diff}_{R/A}(M,N) := \bigcup_{n=0}^{\infty} \mathrm{Diff}^n_{R/A}(M,N).$$

Definition

Let R be a commutative ring and A be a subring. Let M, N be R-modules. The n-th order A-linear differential operators $\text{Diff}^n_{R/A}(M,N) \subseteq \text{Hom}_A(M,N)$ are defined inductively by:

1. $\text{Diff}^0_{R/A}(M,N) := \text{Hom}_R(M,N)$.
2. $\text{Diff}^n_{R/A}(M,N) := \left\{\delta \in \text{Hom}_A(M,N) \mid [\delta, r] \in \text{Diff}^{n-1}_{R/A}(M,N) \text{ for all } r \in R\right\}$, where $[\delta, r](m) = \delta(rm) - r\delta(m)$ for all $m \in M$.

The A-linear differential operators are given by

$$\text{Diff}_{R/A}(M,N) := \bigcup_{n=0}^{\infty} \text{Diff}^n_{R/A}(M,N).$$

Example

For $R = \mathbb{C}[x_1, \ldots, x_n]$ we have that

$$D_n(\mathbb{C}) = R\langle \partial_1, \ldots, \partial_n \rangle = \text{Diff}_{R/\mathbb{C}}(R,R).$$

What "kind of differential operators" can we use to describe primary ideals?

Life can be hard outside polynomial rings...

Consider $R = \frac{\mathbb{C}[x,y,z]}{(x^3+y^3+z^3)}$, then Bernstein, Gelfand, Gelfand (1972) showed that:

1. $\mathrm{Diff}_{R/\mathbb{C}}(R,R)$ is not a Noetherian ring.
2. Let $\mathfrak{m} = (x,y,z) \subset R$. For all $\delta \in \mathrm{Diff}_{R/\mathbb{C}}(R,R)$, we have that $\delta(\mathfrak{m}) \subset \mathfrak{m}$.

What "kind of differential operators" can we use to describe primary ideals?

Life can be hard outside polynomial rings...

Consider $R = \frac{\mathbb{C}[x,y,z]}{(x^3+y^3+z^3)}$, then Bernstein, Gelfand, Gelfand (1972) showed that:

1. $\text{Diff}_{R/\mathbb{C}}(R, R)$ is not a Noetherian ring.
2. Let $\mathfrak{m} = (x, y, z) \subset R$. For all $\delta \in \text{Diff}_{R/\mathbb{C}}(R, R)$, we have that $\delta(\mathfrak{m}) \subset \mathfrak{m}$.

Observation

The differential operator "∂_x" does not exist in $\text{Diff}_{R/\mathbb{C}}(R, R)$ because we would obtain $0 = \partial_x(0) = \partial_x(x^3 + y^3 + z^3) = 3x^2$

Life can be hard outside polynomial rings...

Consider $R = \frac{\mathbb{C}[x,y,z]}{(x^3+y^3+z^3)}$, then Bernstein, Gelfand, Gelfand (1972) showed that:

1. $\mathrm{Diff}_{R/\mathbb{C}}(R, R)$ is not a Noetherian ring.
2. Let $\mathfrak{m} = (x, y, z) \subset R$. For all $\delta \in \mathrm{Diff}_{R/\mathbb{C}}(R, R)$, we have that $\delta(\mathfrak{m}) \subset \mathfrak{m}$.

Observation

The differential operator “∂_x” does not exist in $\mathrm{Diff}_{R/\mathbb{C}}(R, R)$ because we would obtain $0 = \partial_x(0) = \partial_x(x^3 + y^3 + z^3) = 3x^2$ (a contradiction).

What "kind of differential operators" can we use to describe primary ideals?

Life can be hard outside polynomial rings...

Consider $R = \frac{\mathbb{C}[x,y,z]}{(x^3+y^3+z^3)}$, then Bernstein, Gelfand, Gelfand (1972) showed that:

1. $\text{Diff}_{R/\mathbb{C}}(R, R)$ is not a Noetherian ring.
2. Let $\mathfrak{m} = (x, y, z) \subset R$. For all $\delta \in \text{Diff}_{R/\mathbb{C}}(R, R)$, we have that $\delta(\mathfrak{m}) \subset \mathfrak{m}$.

Observation

The differential operator "∂_x" does not exist in $\text{Diff}_{R/\mathbb{C}}(R, R)$ because we would obtain $0 = \partial_x(0) = \partial_x(x^3 + y^3 + z^3) = 3x^2$ (a contradiction).

In $R = \frac{\mathbb{C}[x,y,z]}{(x^3+y^3+z^3)}$, $\mathfrak{m}^2$ cannot be described with differential operators

If $\delta_1, \ldots, \delta_m \in \text{Diff}_{R/\mathbb{C}}(R, R)$ were Noetherian operators for $\mathfrak{m}^2$, we would have

$$\mathfrak{m}^2 = \left\{ f \in R \mid \delta_i \cdot f \in \mathfrak{m} \text{ for all } 1 \leq i \leq m \right\}$$

What "kind of differential operators" can we use to describe primary ideals?

Life can be hard outside polynomial rings...

Consider $R = \frac{\mathbb{C}[x,y,z]}{(x^3+y^3+z^3)}$, then Bernstein, Gelfand, Gelfand (1972) showed that:

1. $\text{Diff}_{R/\mathbb{C}}(R, R)$ is not a Noetherian ring.
2. Let $\mathfrak{m} = (x, y, z) \subset R$. For all $\delta \in \text{Diff}_{R/\mathbb{C}}(R, R)$, we have that $\delta(\mathfrak{m}) \subset \mathfrak{m}$.

Observation

The differential operator "∂_x" does not exist in $\text{Diff}_{R/\mathbb{C}}(R, R)$ because we would obtain $0 = \partial_x(0) = \partial_x(x^3 + y^3 + z^3) = 3x^2$ (a contradiction).

In $R = \frac{\mathbb{C}[x,y,z]}{(x^3+y^3+z^3)}$, $\mathfrak{m}^2$ cannot be described with differential operators

If $\delta_1, \ldots, \delta_m \in \text{Diff}_{R/\mathbb{C}}(R, R)$ were Noetherian operators for $\mathfrak{m}^2$, we would have

$$\mathfrak{m}^2 = \left\{ f \in R \mid \delta_i \cdot f \in \mathfrak{m} \text{ for all } 1 \le i \le m \right\} \supseteq \mathfrak{m}$$

What "kind of differential operators" can we use to describe primary ideals?

Life can be hard outside polynomial rings...

Consider $R = \frac{\mathbb{C}[x,y,z]}{(x^3+y^3+z^3)}$, then Bernstein, Gelfand, Gelfand (1972) showed that:

1. $\mathrm{Diff}_{R/\mathbb{C}}(R,R)$ is not a Noetherian ring.
2. Let $\mathfrak{m} = (x,y,z) \subset R$. For all $\delta \in \mathrm{Diff}_{R/\mathbb{C}}(R,R)$, we have that $\delta(\mathfrak{m}) \subset \mathfrak{m}$.

Observation

The differential operator "∂_x" does not exist in $\mathrm{Diff}_{R/\mathbb{C}}(R,R)$ because we would obtain $0 = \partial_x(0) = \partial_x(x^3+y^3+z^3) = 3x^2$ (a contradiction).

In $R = \frac{\mathbb{C}[x,y,z]}{(x^3+y^3+z^3)}$, $\mathfrak{m}^2$ cannot be described with differential operators

If $\delta_1, \ldots, \delta_m \in \mathrm{Diff}_{R/\mathbb{C}}(R,R)$ were Noetherian operators for $\mathfrak{m}^2$, we would have

$$\mathfrak{m}^2 = \left\{ f \in R \mid \delta_i \cdot f \in \mathfrak{m} \text{ for all } 1 \le i \le m \right\} \supseteq \mathfrak{m}$$

(a contradiction).

Noetherian operators for general algebras

Theorem (—)

Let $\Bbbk$ be a field, R be a $\Bbbk$-algebra of finite type, $\mathfrak{p} \in \mathrm{Spec}(R)$ be a prime ideal, and $I \subset R$ be a $\mathfrak{p}$-primary ideal. Then:

1. There exist $\delta_1, \ldots, \delta_m \in \mathrm{Diff}_{R/\Bbbk}(R, R/\mathfrak{p})$ such that

$$I = \{f \in R \mid \delta_i(f) = 0 \text{ for all } 1 \le i \le m\}.$$

Noetherian operators for general algebras

Theorem (—)

Let $\mathbb{k}$ be a field, R be a $\mathbb{k}$-algebra of finite type, $\mathfrak{p} \in \operatorname{Spec}(R)$ be a prime ideal, and $I \subset R$ be a $\mathfrak{p}$-primary ideal. Then:

1. There exist $\delta_1, \ldots, \delta_m \in \operatorname{Diff}_{R/\mathbb{k}}(R, R/\mathfrak{p})$ such that
$$I = \{f \in R \mid \delta_i(f) = 0 \text{ for all } 1 \le i \le m\}.$$
2. If R is **smooth** over $\mathbb{k}$, then there exist $\delta_1, \ldots, \delta_m \in \operatorname{Diff}^n_{R/\mathbb{k}}(R, R)$ such that
$$I = \{f \in R \mid \delta_i(f) \in \mathfrak{p} \text{ for all } 1 \le i \le m\}.$$

Let's go back to polynomial rings...

Data

- $\mathbb{k}$ a field of **characteristic zero**.
- $R = \mathbb{k}[x_1, \ldots, x_n]$.
- $P \in \operatorname{Spec}(R)$ is a prime ideal with codimension $\operatorname{ht}(P) = c$.
- $\mathbb{F} = \operatorname{Quot}(R/P) = R_P/PR_P$ residue field of P.

Goal

Parametrize **all** the P-primary ideals $Q \subset R$ with multiplicity

$$m = \operatorname{length}\left(R_P/QR_P\right)$$

over P. In `Macaulay2`, we can use $m = \texttt{degree}(Q)/\texttt{degree}(P)$.

Data

$\mathbb{k}$ a field of **characteristic zero**, $R = \mathbb{k}[x_1, \ldots, x_n]$, $P \in \mathrm{Spec}(R)$ is a prime ideal with codimension $\mathrm{ht}(P) = c$, $\mathbb{F} = \mathrm{Quot}(R/P) = R_P/PR_P$ residue field of P.

Data

$\Bbbk$ a field of **characteristic zero**, $R = \Bbbk[x_1, \ldots, x_n]$, $P \in \operatorname{Spec}(R)$ is a prime ideal with codimension $\operatorname{ht}(P) = c$, $\mathbb{F} = \operatorname{Quot}(R/P) = R_P/PR_P$ residue field of P.

Noether normalization

Since $\dim(R/P) = n - c$ and $\Bbbk$ is infinite, we **assume** that

$$\Bbbk[x_{c+1}, \ldots, x_n] \hookrightarrow R/P$$

is an integral extension.

Data

$\Bbbk$ a field of **characteristic zero**, $R = \Bbbk[x_1, \ldots, x_n]$, $P \in \mathrm{Spec}(R)$ is a prime ideal with codimension $\mathrm{ht}(P) = c$, $\mathbb{F} = \mathrm{Quot}(R/P) = R_P/PR_P$ residue field of P.

Noether normalization

Since $\dim(R/P) = n - c$ and $\Bbbk$ is infinite, we **assume** that

$$\Bbbk[x_{c+1}, \ldots, x_n] \hookrightarrow R/P$$

is an integral extension.

Punctual Hilbert scheme

$\mathrm{Hilb}^m(\mathbb{F}[[y_1, \ldots, y_c]]) := \left\{ I \subset \mathbb{F}[[y_1, \ldots, y_c]] \mid \dim_{\mathbb{F}} \left(\frac{\mathbb{F}[[y_1, \ldots, y_c]]}{I} \right) = m \right\}.$

Data

$\Bbbk$ a field of **characteristic zero**, $R = \Bbbk[x_1, \ldots, x_n]$, $P \in \mathrm{Spec}(R)$ is a prime ideal with codimension $\mathrm{ht}(P) = c$, $\mathbb{F} = \mathrm{Quot}(R/P) = R_P/PR_P$ residue field of P.

Noether normalization

Since $\dim(R/P) = n - c$ and $\Bbbk$ is infinite, we **assume** that

$$\Bbbk[x_{c+1}, \ldots, x_n] \hookrightarrow R/P$$

is an integral extension.

Punctual Hilbert scheme

$$\mathrm{Hilb}^m(\mathbb{F}[[y_1, \ldots, y_c]]) := \left\{ I \subset \mathbb{F}[[y_1, \ldots, y_c]] \mid \dim_{\mathbb{F}} \left(\frac{\mathbb{F}[[y_1, \ldots, y_c]]}{I} \right) = m \right\}.$$

An $\mathbb{F}$-vector space $V \subset \mathbb{F}[z_1, \ldots, z_c]$ is **closed under differentiation** if $A \bullet V \subset V$ for any $A \in \mathbb{F}[\partial_{z_1}, \ldots, \partial_{z_c}]$.

Weyl-Noether module

Relative Weyl algebra

$D_{n,c}(\Bbbk) = R\langle \partial_{x_1}, \ldots, \partial_{x_c} \rangle = \mathrm{Diff}_{R/\Bbbk[x_{c+1},\ldots,x_n]}(R, R).$

Weyl-Noether module

Relative Weyl algebra

$D_{n,c}(\mathbb{k}) = R\langle \partial_{x_1}, \ldots, \partial_{x_c} \rangle = \mathrm{Diff}_{R/\mathbb{k}[x_{c+1},\ldots,x_n]}(R, R).$

Weyl-Noether module

$\mathbb{F} \otimes_R D_{n,c}(\mathbb{k})$

Weyl-Noether module

Relative Weyl algebra

$D_{n,c}(\mathbb{k}) = R\langle \partial_{x_1}, \ldots, \partial_{x_c} \rangle = \mathrm{Diff}_{R/\mathbb{k}[x_{c+1},\ldots,x_n]}(R, R)$.

Weyl-Noether module

$\mathbb{F} \otimes_R D_{n,c}(\mathbb{k})$ ("computationally nice", $\mathbb{F}$-monomials : $\{w \otimes_R \partial_{x_1}^{\beta_1} \cdots \partial_{x_c}^{\beta_c} \mid w \in \mathbb{F}\}$).

Weyl-Noether module

Relative Weyl algebra

$D_{n,c}(\Bbbk) = R\langle \partial_{x_1}, \ldots, \partial_{x_c} \rangle = \mathrm{Diff}_{R/\Bbbk[x_{c+1},\ldots,x_n]}(R,R).$

Weyl-Noether module

$\mathbb{F} \otimes_R D_{n,c}(\Bbbk)$ ("computationally nice", $\mathbb{F}$-monomials : $\{w \otimes_R \partial_{x_1}^{\beta_1} \cdots \partial_{x_c}^{\beta_c} \mid w \in \mathbb{F}\}$).

Let $S = \Bbbk(x_{c+1}, \ldots, x_n)[x_1, \ldots, x_c]$. Then:

$$\begin{aligned}
\mathbb{F} \otimes_R D_{n,c}(\Bbbk) &\cong \mathbb{F} \otimes_R \mathrm{Diff}_{R/\Bbbk[x_{c+1},\ldots,x_n]}(R,R) \\
&\cong S/PS \otimes_S \mathrm{Diff}_{S/\Bbbk(x_{c+1},\ldots,x_n)}(S,S) \\
&\cong \mathrm{Diff}_{S/\Bbbk(x_{c+1},\ldots,x_n)}(S, S/PS).
\end{aligned}$$

Notice that: PS is a maximal ideal in S.

A representation theorem

char($\Bbbk$) $= 0$, $R = \Bbbk[x_1, \ldots, x_n]$, $P \in \text{Spec}(R)$ with $\text{ht}(P) = c$ and $\mathbb{F} = \text{Quot}(R/P)$.

Theorem (—, Homs, Sturmfels)

The following four sets of objects are in a natural bijective correspondence:

1. P-primary ideals Q in R of multiplicity m over P,
2. points in the punctual Hilbert scheme $\mathrm{Hilb}^m(\mathbb{F}[[y_1, \ldots, y_c]])$,
3. m-dimensional $\mathbb{F}$-subspaces of $\mathbb{F}[z_1, \ldots, z_c]$ that are closed under differentiation,
4. m-dimensional $\mathbb{F}$-subspaces of the Weyl-Noether module $\mathbb{F} \otimes_R D_{n,c}$ that are R-bi-modules.

Moreover, any basis of the $\mathbb{F}$-subspace in part (d) can be lifted to Noetherian operators $A_1, \ldots, A_m$ in the relative Weyl algebra $D_{n,c}$ that represent the ideal Q in part (a).

Some instances of $\mathrm{Hilb}^m(\mathbb{F}[[y_1, \ldots, y_c]])$

The case $c = 1$; $\mathrm{Hilb}^m(\mathbb{F}[[y_1]])$

$\mathrm{Hilb}^m(\mathbb{F}[[y_1]]) = \left\{ \mathbb{F}[[y_1]]/(y_1^m) \right\}$ (it's trivial; just one point)

This agrees with the following fact: **if** $\mathrm{ht}(P) = 1$**, then** P **is principal and so the only** P**-primary ideal of multiplicity** m **over** P **is** P^m.

Some instances of $\mathrm{Hilb}^m(\mathbb{F}[[y_1, \ldots, y_c]])$

The case $c = 1$; $\mathrm{Hilb}^m(\mathbb{F}[[y_1]])$

$\mathrm{Hilb}^m(\mathbb{F}[[y_1]]) = \left\{ \mathbb{F}[[y_1]]/(y_1^m) \right\}$ (it's trivial; just one point)

This agrees with the following fact: **if** $\mathrm{ht}(P) = 1$**, then** P **is principal and so the only** P**-primary ideal of multiplicity** m **over** P **is** P^m.

The case $c = 2$; $\mathrm{Hilb}^m(\mathbb{F}[[y_1, y_2]])$

- Briançon (1977): "Description de $\mathrm{Hilb}^n(\mathbb{C}[[x, y]])$".
- Iarrobino (1977): "Punctual Hilbert schemes".

Some instances of $\mathrm{Hilb}^m(\mathbb{F}[[y_1,\ldots,y_c]])$

The case $c = 1$; $\mathrm{Hilb}^m(\mathbb{F}[[y_1]])$

$\mathrm{Hilb}^m(\mathbb{F}[[y_1]]) = \left\{\mathbb{F}[[y_1]]/(y_1^m)\right\}$ (it's trivial; just one point)

This agrees with the following fact: **if** $\mathrm{ht}(P) = 1$**, then** P **is principal and so the only** P**-primary ideal of multiplicity** m **over** P **is** P^m.

The case $c = 2$; $\mathrm{Hilb}^m(\mathbb{F}[[y_1, y_2]])$

- Briançon (1977): "Description de $\mathrm{Hilb}^n(\mathbb{C}[[x,y]])$".
- Iarrobino (1977): "Punctual Hilbert schemes".

Symbolic powers $P^{(r)}$ and its multiplicity over P

$P^{(r)}$ corresponds with the point $(y_1,\ldots,y_c)^r \in \mathrm{Hilb}^m(\mathbb{F}[[y_1,\ldots,y_c]])$, where $m = \binom{c+r-1}{c}$.

Example: twisted cubic

We take $n = 4$, $c = 2$ and the prime:

$$P \quad = \quad \left(x_1^2 - x_2x_3,\ x_1x_2 - x_3x_4, x_2^2 - x_1x_4 \right) \quad \subset \quad \mathbb{C}[x_1, x_2, x_3, x_4].$$

(affine cone) *twisted cubic curve* $V(P) = \left\{(s^2t, st^2, s^3, t^3) \mid s, t \in \mathbb{C}\right\}$.

We take the following P-primary ideal of multiplicity $m = 3$ over P:

$$\begin{aligned} Q = \quad & \big(3x_1^2x_2^2 - x_2^3x_3 - x_1^3x_4 - 3x_1x_2x_3x_4 + 2x_3^2x_4^2\,,\ 3x_1^3x_2x_4 - 3x_1x_2^2x_3x_4 - 3x_1^2x_3x_4^2 + 3x_2x_3^2x_4^2 \\ & +2x_2^3 - 2x_3x_4^2\,,\ 3x_2^4x_3 - 6x_1x_2^2x_3x_4 + 3x_1^2x_3x_4^2 + x_2^3 - x_3x_4^2\,,\ 4x_1x_2^3x_3 + x_1^4x_4 - 6x_1^2x_2x_3x_4 \\ & -3x_2^2x_3^2x_4 + 4x_1x_3^2x_4^2\,,\ x_2^5 - x_1x_2^3x_4 - x_2^2x_3x_4^2 + x_1x_3x_4^3\,,\ x_1x_2^4 - x_2^3x_3x_4 - x_1x_2x_3x_4^2 + x_3^2x_4^3\,, \\ & x_1^4x_2 - x_2^3x_3^2 - 2x_1^3x_3x_4 + 2x_1x_2x_3^2x_4\,,\ x_1^5 - 4x_1^3x_2x_3 + 3x_1x_2^2x_3^2 + 2x_1^2x_3^2x_4 - 2x_2x_3^3x_4\,, \\ & 3x_1^4x_4^2 - 6x_1^2x_2x_3x_4^2 + 3x_2^2x_3^2x_4^2 + 4x_2^4 - 4x_2x_3x_4^2\,,\ x_2^3x_3^2x_4 + x_1^3x_3x_4^2 - 3x_1x_2x_3^2x_4^2 + x_3^3x_4^3 \\ & +x_1x_2^3 - x_1x_3x_4^2\,, 3x_1^4x_3x_4 - 6x_1^2x_2x_3^2x_4 + 3x_2^2x_3^3x_4 + 2x_1^3x_2 + 6x_1x_2^2x_3 - 6x_1^2x_3x_4 - 2x_2x_3^2x_4\,, \\ & 4x_2^3x_3^3 + 4x_1^3x_3^2x_4 - 12x_1x_2x_3^3x_4 + 4x_3^4x_4^2 - x_1^4 + 6x_1^2x_2x_3 + 3x_2^2x_3^2 - 8x_1x_3^2x_4 \big). \end{aligned}$$

Example: twisted cubic

The corresponding point in $\mathrm{Hilb}^3(\mathbb{F}[[y_1, y_2]])$

$$I = \left(y_1^3, y_2 + \overline{x_2} y_1^2\right) \quad \in \quad \mathrm{Hilb}^3(\mathbb{F}[[y_1, y_2]])$$

Example: twisted cubic

The corresponding point in $\text{Hilb}^3(\mathbb{F}[[y_1, y_2]])$

$$I = \left(y_1^3, y_2 + \overline{x_2} y_1^2\right) \quad \in \quad \text{Hilb}^3(\mathbb{F}[[y_1, y_2]])$$

The inverse system $I^{\perp}$ $\quad$ ("$y_1 = \partial_{z_1}, y_2 = \partial''_{z_2}$)

$$V = I^{\perp} = \mathbb{F} \cdot \left\{1,\ z_1,\ z_1^2 - 2\overline{x_2} z_2\right\} \in \mathbb{F}[z_1, z_2].$$

Example: twisted cubic

The corresponding point in $\mathrm{Hilb}^3(\mathbb{F}[[y_1, y_2]])$

$$I = (y_1^3, y_2 + \overline{x_2} y_1^2) \quad \in \quad \mathrm{Hilb}^3(\mathbb{F}[[y_1, y_2]])$$

The inverse system $I^\perp$ ("$y_1 = \partial_{z_1}, y_2 = \partial_{z_2}$")

$$V = I^\perp = \mathbb{F} \cdot \{1,\ z_1,\ z_1^2 - 2\overline{x_2} z_2\} \in \mathbb{F}[z_1, z_2].$$

A set of Noetherian operators

$$Q \ = \ \{\, f \in \mathbb{C}[x_1, x_2, x_3, x_4] \ : \ A_i \bullet f \in P \ \text{ for } \ i = 1, 2, 3\,\},$$

$$\text{where} \quad A_1 = 1\,,\ A_2 = \partial_{x_1} \ \text{ and } \ A_3 = \partial_{x_1}^2 - 2\,x_2\,\partial_{x_2}.$$

Differential conditions for ideal membership

Data

$\Bbbk$ field characteristic zero, $R = \Bbbk[x_1, \ldots, x_n]$, and $I \subset R$ an ideal with $\mathrm{Ass}(I) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_k\}$

Definition (—,Sturmfels): Differential primary decomposition

A list of pairs $(\mathfrak{A}_1, \mathfrak{p}_1), \ldots, (\mathfrak{A}_k, \mathfrak{p}_k)$, where $\mathfrak{A}_i \subset D_n(\Bbbk)$ is a finite subset, is a **differential primary decomposition for I** if

$$I \;=\; \{f \in R \mid \delta(f) \in \mathfrak{p}_i \text{ for all } \delta \in \mathfrak{A}_i \text{ and } i = 1, 2, \ldots, k\}.$$

Remark

A differential primary decomposition always exists. Take a primary decomposition $I = Q_1 \cap \cdots \cap Q_k$ and set $\mathfrak{A}_i$ to be a set of Noetherian operators for the $\mathfrak{p}_i$-primary ideal Q_i.

Example

Let $I = (x_1^3, x_1^2 x_2^2) \subset \Bbbk[x_1, x_2]$ with primary decomposition $I = (x_1^2) \cap (x_1^3, x_2^2)$.

We have Noetherian operators for the primary components:

$$(x_1^2) = \{f \in R \mid \delta(f) \in (x_1) \text{ for all } \delta \in \mathfrak{B}_1\},$$

$$(x_1^3, x_2^2) = \{f \in R \mid \delta(f) \in (x_1, x_2) \text{ for all } \delta \in \mathfrak{B}_2\},$$

where $\mathfrak{B}_1 = \{1, \partial_{x_1}\}$ and $\mathfrak{B}_2 = \{1, \partial_{x_1}, \partial_{x_2}, \partial_{x_1}\partial_{x_2}, \partial_{x_1}^2, \partial_{x_1}^2\partial_{x_2}\}$; this would yield a total of 8 differential operators.

But, we can describe I with just 6 differential operators:

$$I = \{f \in R \mid \delta(f) \in \mathfrak{p}_i \text{ for all } \delta \in \mathfrak{A}_i \text{ and } i = 1, 2\}$$

where $\mathfrak{A}_1 = \{1, \partial_{x_1}\}$ and $\mathfrak{A}_2 = \{\partial_{x_2}, \partial_{x_1}\partial_{x_2}, \partial_{x_1}^2, \partial_{x_1}^2\partial_{x_2}\}$.

Important question

What is the minimal size of a differential primary decomposition?

Defintion (arithmetic multiplicity)

$$\mathrm{amult}(I) = \mathrm{mult}_I(\mathfrak{p}_1) + \mathrm{mult}_I(\mathfrak{p}_2) + \cdots + \mathrm{mult}_I(\mathfrak{p}_k)$$

where $\mathrm{mult}_I(\mathfrak{p}_i) = \lambda_{R_{\mathfrak{p}_i}}(\mathrm{H}^0_{\mathfrak{p}_i}(R_{\mathfrak{p}_i}/IR_{\mathfrak{p}_i}))$. In `Macaulay2` we can compute

$$\mathrm{mult}_I(\mathfrak{p}_i) = \texttt{degree}\big(\texttt{saturate}(I, \mathfrak{p}_i)\big)/\texttt{degree}(\mathfrak{p}_i).$$

Remark: the arithmetic multiplicity is **not** the sum of the multiplicities coming from a primary decomposition of I.

Defintion (arithmetic multiplicity)

$$\text{amult}(I) = \text{mult}_I(\mathfrak{p}_1) + \text{mult}_I(\mathfrak{p}_2) + \cdots + \text{mult}_I(\mathfrak{p}_k)$$

where $\text{mult}_I(\mathfrak{p}_i) = \lambda_{R_{\mathfrak{p}_i}}(\text{H}^0_{\mathfrak{p}_i}(R_{\mathfrak{p}_i}/IR_{\mathfrak{p}_i}))$. In `Macaulay2` we can compute

$$\text{mult}_I(\mathfrak{p}_i) = \texttt{degree}(\texttt{saturate}(I, \mathfrak{p}_i))/\texttt{degree}(\mathfrak{p}_i).$$

Remark: the arithmetic multiplicity is **not** the sum of the multiplicities coming from a primary decomposition of I.

Theorem (—, Sturmfels)

Let $I \subset R$ and $\text{Ass}(I) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_k\}$. The size of any differential primary decomposition is at least $\text{amult}(I)$ and this result is sharp. More precisely:

1. I has a differential primary decomposition with $|\mathfrak{A}_i| = \text{mult}_I(\mathfrak{p}_i)$.
2. A differential primary decomposition for I satisfies $|\mathfrak{A}_i| \geq \text{mult}_I(\mathfrak{p}_i)$.

GHENT UNIVERSITY

Thanks!

Merci!