Approximating CSPs in more than polynomial time

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ESIGMA Kick-off Meeting

Overview

Things you will hear in this talk:

- Max-CSPs such as Max-SAT, Max-Cut, etc. are Hard!
- . . . even to approximate!
- To solve them we need to:
  - Take into account the input stucture (how?)
  - Invest a little more than polynomial time (how much?)
  - Allow some sub-optimal solutions (but almost optimal!)

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Results based on two papers:

- ”Sub-Exponential Approximation Schemes for CSPs: From Dense to Almost-Sparse.”, Dimitris Fotakis, Michael Lampis, and Vangelis Th. Paschos, STACS ’16.
- ”Complexity and Approximability for Parameterized CSPs.”, Holger Dell, Eunjung Kim, Michael Lampis, Valia Mitsou, Tobias Moemke, IPEC’15 (Algorithmica ’17).

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Definition of “**Solve**”:

- Time-efficiency (polynomial time)
- Optimality
- Generality (handles all instances)

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- Optimality
- Generality (handles all instances)

Main Challenge: Under standard complexity assumptions (P$\neq$NP, ETH), no algorithm achieves all three!

- In fact, tight hardness known for many problems:
- Max-3-SAT cannot be $(7/8-\epsilon)$-approximated, cannot be solved in $2^{o(n)}$.

Research Direction:

- Trade **Time** for **Generality** and/or **Optimality**

Dead on Arrival?

- Better than $3/2$ for TSP, $4/3$ for Max-3-DM, $7/8$ for Max-3-SAT,..., in **sub-exponential time**?

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Probably won't work

(at least for Max-3-SAT)

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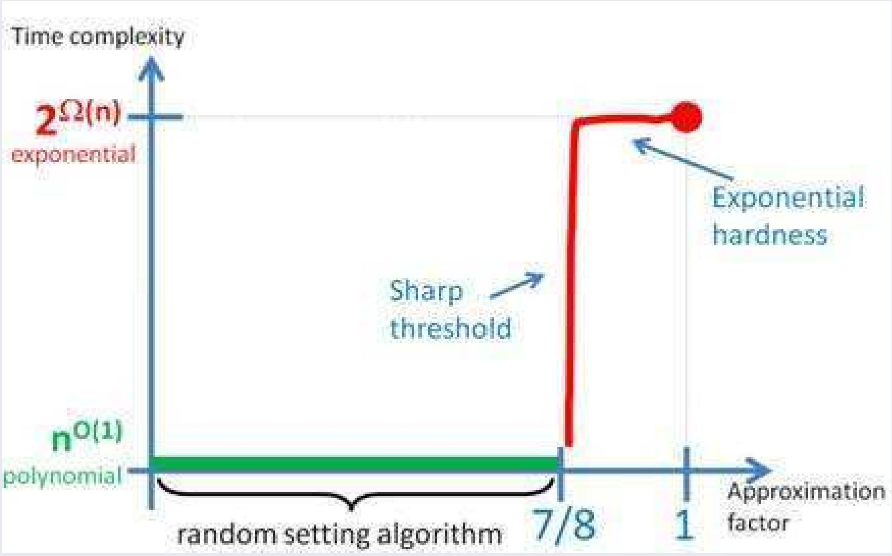

Almost-linear PCPs (Moshkovitz& Raz) and P-time hardness (Håstad) give tight inapproximability for Max-3-SAT even for $2^{n^{1-\epsilon}}$ time.

(Credit: Dana Moshkovitz)

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  If this is the “normal” behavior of APX problems, what’s the point of sub-exponential approximation?

  - Is this the “normal” behavior?
  - **What else can we do?**

Strategy

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An island of tractability:

- Max-k-CSP admits a PTAS (a $(1-\epsilon)$-approximation for all $\epsilon > 0$) for **dense** instances
- (Arora, Karger, Karpinski '99), (Fernandez de la Vega '96)

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Extending the island:

- We give a version of the AKK scheme which **can handle sparser instances**, at the expense of needing **sub-exponential time**.
- Our scheme provides a smooth trade-off
  - For dense instances we get a PTAS
  - As instances gradually get more sparse, we need more time…
  - …until our scheme does not work any more

Summary of results

For any $\epsilon > 0$, $\delta \in [0, 1]$ and fixed $k \geq 2$ we have the following:

- Given a Max-k-CSP instance with $n^{k-1+\delta}$ constraints
- We can produce a $(1 - \epsilon)$-approximate solution
- In time $2^{O(n^{1-\delta} \ln n / \epsilon^3)}$

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- Advantage: we provide a smooth trade-off from the "easy case" (dense instances) to more general cases

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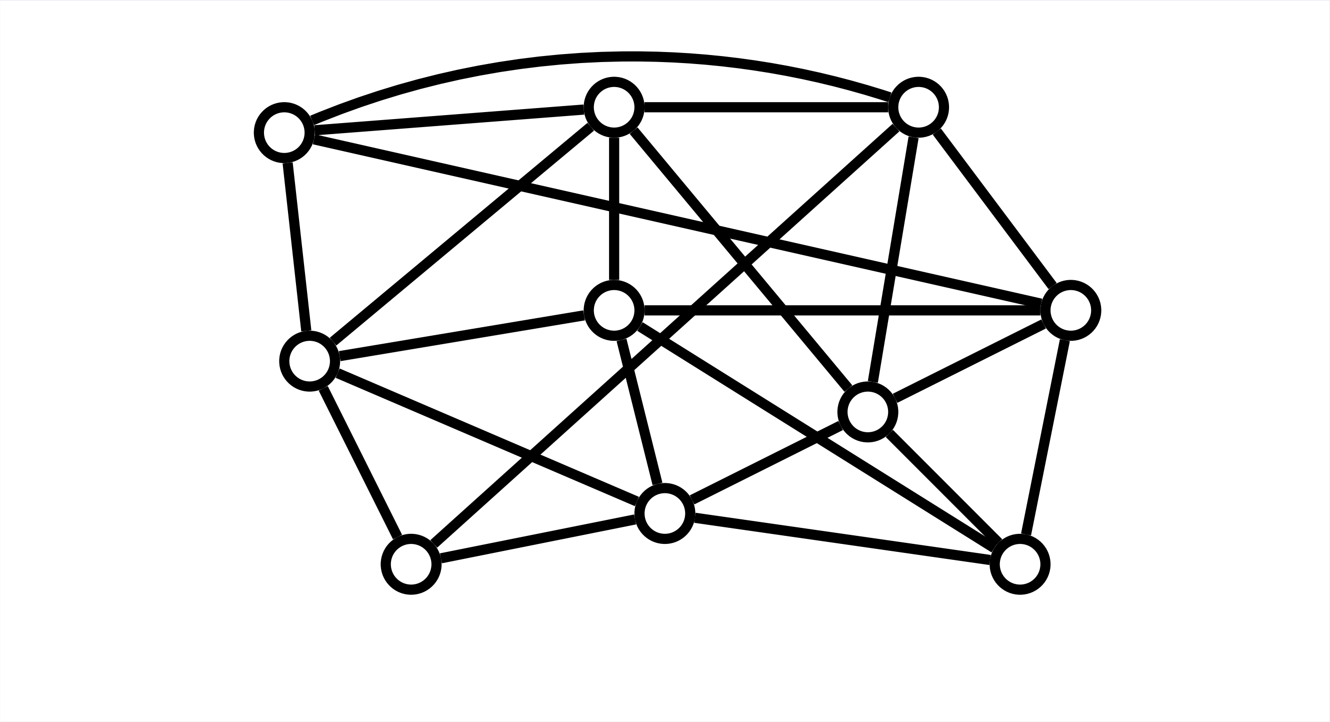
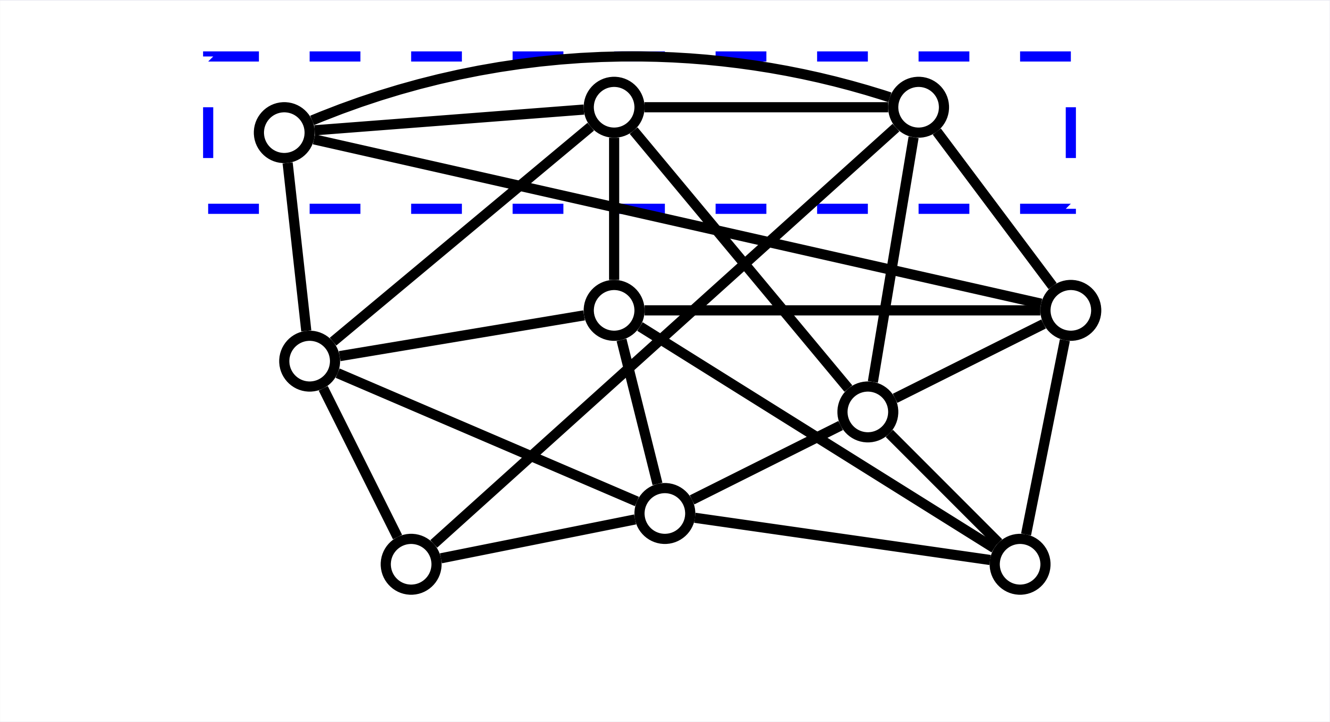
- We will also give some "tight" bounds, ruling out natural possible improvements.

Basic scheme (Max Cut)



We are given a dense graph for which we want to find a large cut



Randomly select a “sample” of its vertices

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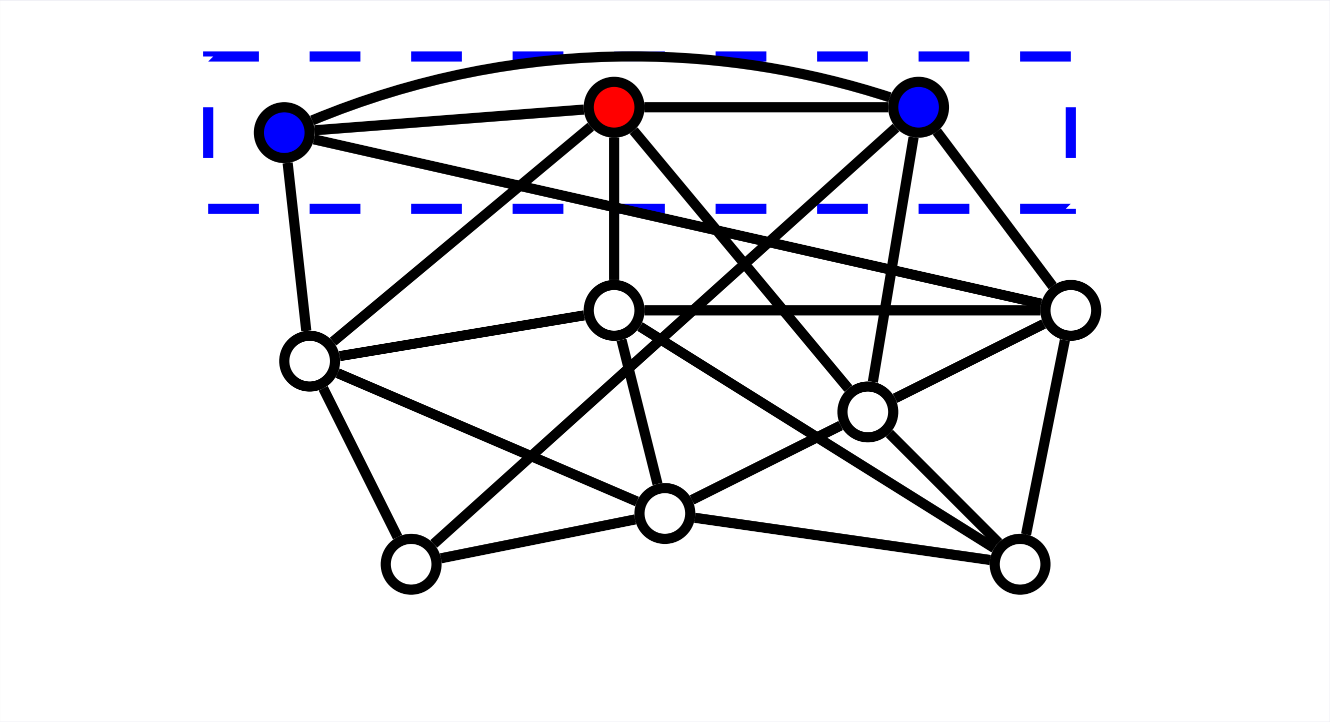

Guess their correct partition

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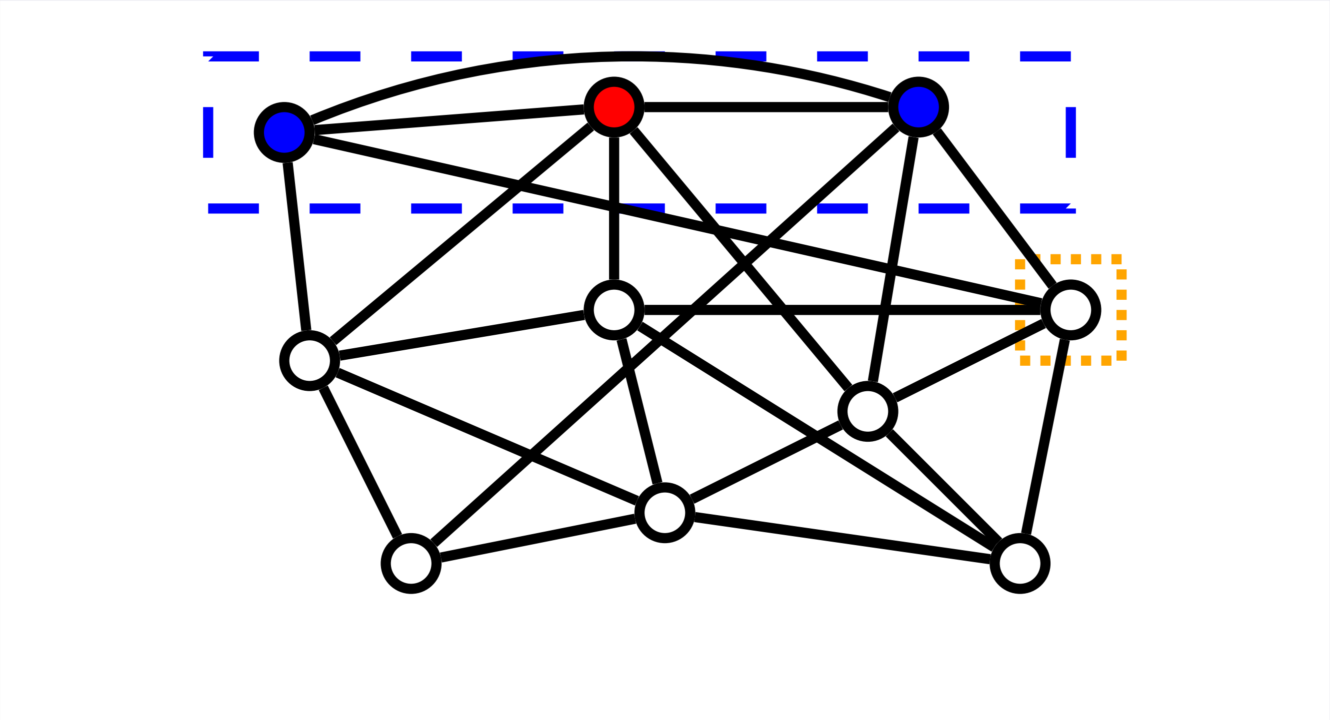

For every vertex outside the sample, examine its neighbors in the sample

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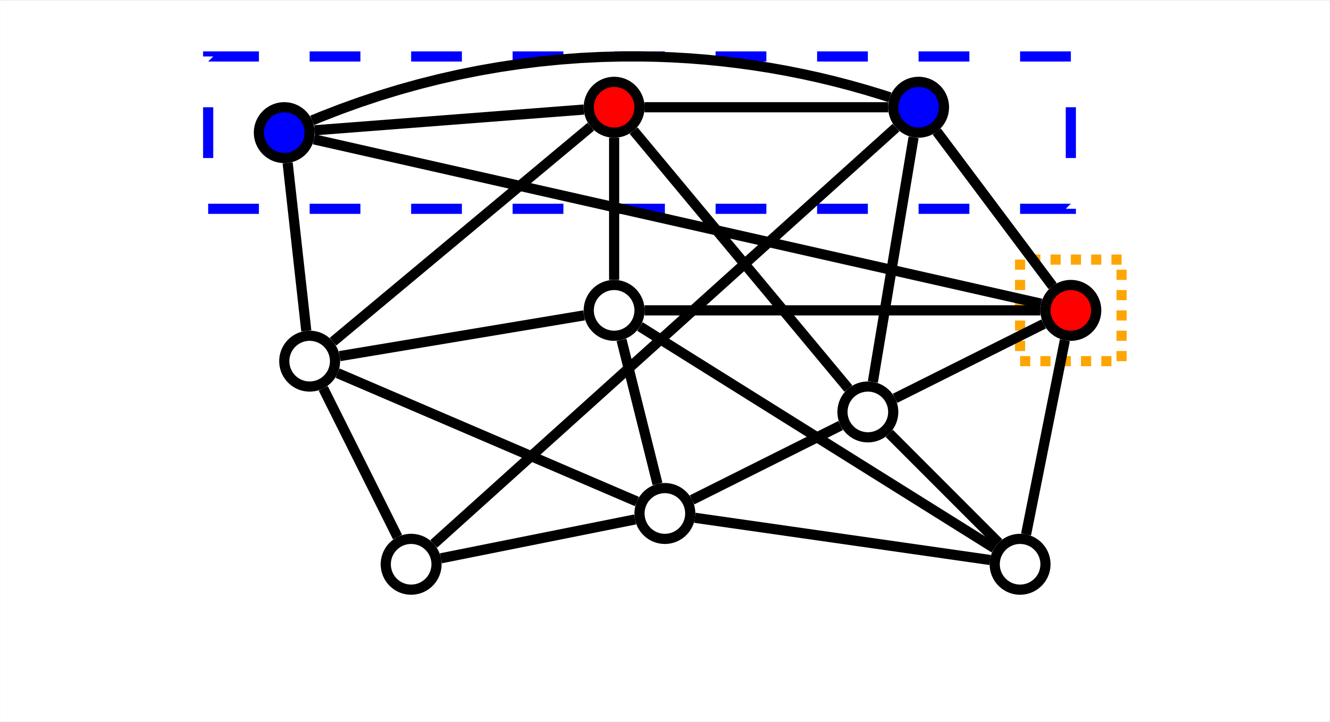

Greedily set its value depending on this neighborhood

Basic scheme (Max Cut)

- The sample we select has size $O(\log n)$ (hidden constants depend on degree and ϵ)
- $\rightarrow$ running time $n^{O(1)}$ (will try all partitions of sample)

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Why this works (intuitively):

- Because graph is dense $\rightarrow$ every vertex outside sample S has many neighbors in S
- $\rightarrow$ examining $N(u) \cap S$ is (whp) a good representation of $N(u)$ in the optimal solution
- If a vertex in $V \setminus S$ has $>> 50\%$ of its neighbors on one side in the optimal solution, it will (whp) have $>> 50\%$ of its neighbors on that side in S

(Fernandez de la Vega '96)

General scheme (Max-k-CSP)

Max Cut:

$$\max \sum_{(i,j)\in E} x_i(1-x_j) + x_j(1-x_i)$$

General scheme (Max-k-CSP)

Max-2-SAT:

$$\max \sum_{(i,j)\in C} x_i(1-x_j) + x_j(1-x_i) + x_i x_j$$

General scheme (Max-k-CSP)

Max-3-SAT:

$$\max \sum_{(i,j,k)\in C} x_i(1-x_j)(1-x_k) + (1-x_i)x_j(1-x_k) + \ldots + x_i x_j x_k$$

General scheme (Max-k-CSP)

Max-k-CSP:

$$\max p(\vec{x})$$

where $p()$ is a degree k polynomial.

The AKK scheme offers a PTAS that finds an assignment almost maximizing p when the polynomial has at least $\Omega(n^k)$ terms.

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Max Cut:

$$\max \sum_{(i,j)} c_{ij} x_i x_j + \sum_i c_i x_i + C$$

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$$\max \sum_i x_i r_i$$

where $r_i(\vec{x} - x_i)$ is the (linear) polynomial of the remaining variables I obtain if I factor out x_i.

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Main idea: Estimate the values of the r_i's using brute force on a small sample.

General scheme (Max-k-CSP)

Max Cut:

$$\max \sum_i x_i r_i$$

s.t.

$$\hat{r}_i - \epsilon n \leq \quad \textstyle\sum_{j \in N(i)} c_{ij} x_j \quad \leq \hat{r}_i + \epsilon n$$

where $\hat{r}_i$ is the estimate I have for r_i.

This is now a **linear** program.

General scheme (Max-k-CSP)

Max Cut:

$$\max \sum_i x_i r_i$$

Summary of algorithm:

- Estimate the r_i values using a sample
  - Need large enough sample to guarantee $\hat{r}_i \approx r_i$
  - This turns QIP $\rightarrow$ ILP
- Solve fractional relaxation of ILP
- Round solution

Sub-exponential Extension (Max Cut)

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Main idea: **Use larger sample**

- Suppose graph has average degree $\Delta = n^\delta$
- We sample $\frac{n \log n}{\Delta} = n^{1-\delta} \log n$ vertices
- $\rightarrow$ whp $\hat{r}_i \approx r_i$.

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Main idea: **Use larger sample**
We are almost done!

- Must prove sample size enough for $\hat{r}_i$
  - Pitfall: Additive error ϵn no longer negligible!
- Must prove rounding step still works

Don't worry, it all works!

General scheme $k \geq 3$

Summary so far $k = 2$:

- AKK: Average degree $\Omega(n)$, sample of $O(\log n)$ vertices
- Extension: Average degree n^δ, sample of $n^{1-\delta} \log n$
- $\rightarrow$ in time $2^{\sqrt{n}}$ can "solve" Max-Cut for $|E| \geq n^{1.5}$

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How about Max-3-SAT?

- In poly time can solve instances with n^3 clauses
- In $2^{\sqrt{n}}$ time can solve instances with $\ldots$ clauses?

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AKK scheme for $k \geq 3$

- Write $p(\vec{x})$ as $\sum_i x_i r_i$
- Each r_i has degree $k-1$
- Write $r_i = \sum_j x_j r_{ij}$
- Each r_{ij} has degree $k-2$
- $\ldots$
- Until we get to linear $\rightarrow$ write ILP

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Note: In order for this to work, all $r_{ij\ldots}$ polynomials must be **dense**

- This is true if original polynomial was dense.

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- In our scheme, if p has $n^{k-1+\delta}$ terms
- r_i has $n^{k-2+\delta}$ terms
- r_{ij} has $n^{k-3+\delta}$ terms
- ...

It seems that the “right” density to require is $n^{k-1+\delta}$?

General scheme – summary

- Input: Max-k-CSP instance with $n^{k-1+\delta}$ constraints
- Algorithm:
  - Sample $2^{n^{1-\delta}} \log n/\epsilon^3$ variables, guess their value
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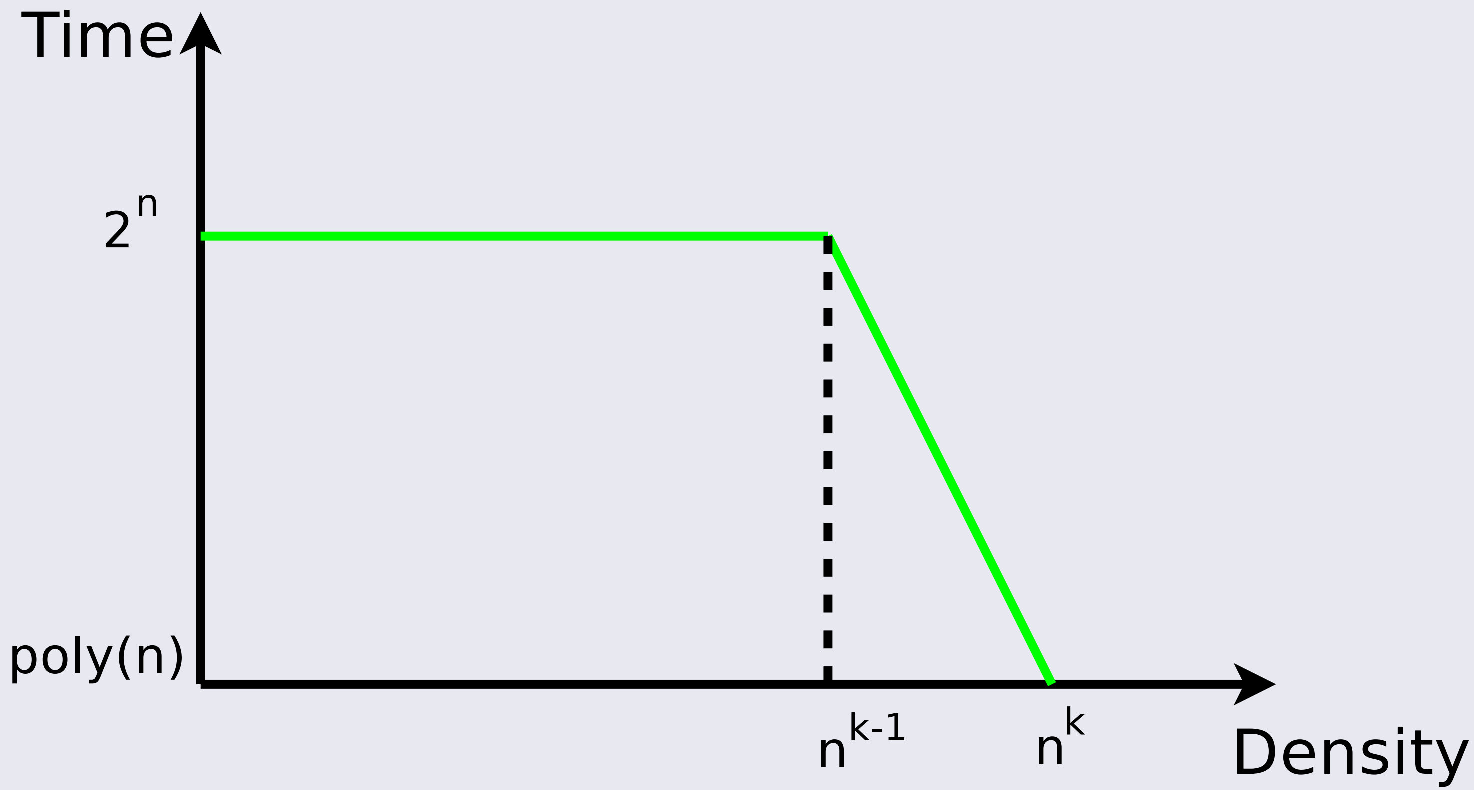
Can we do better?

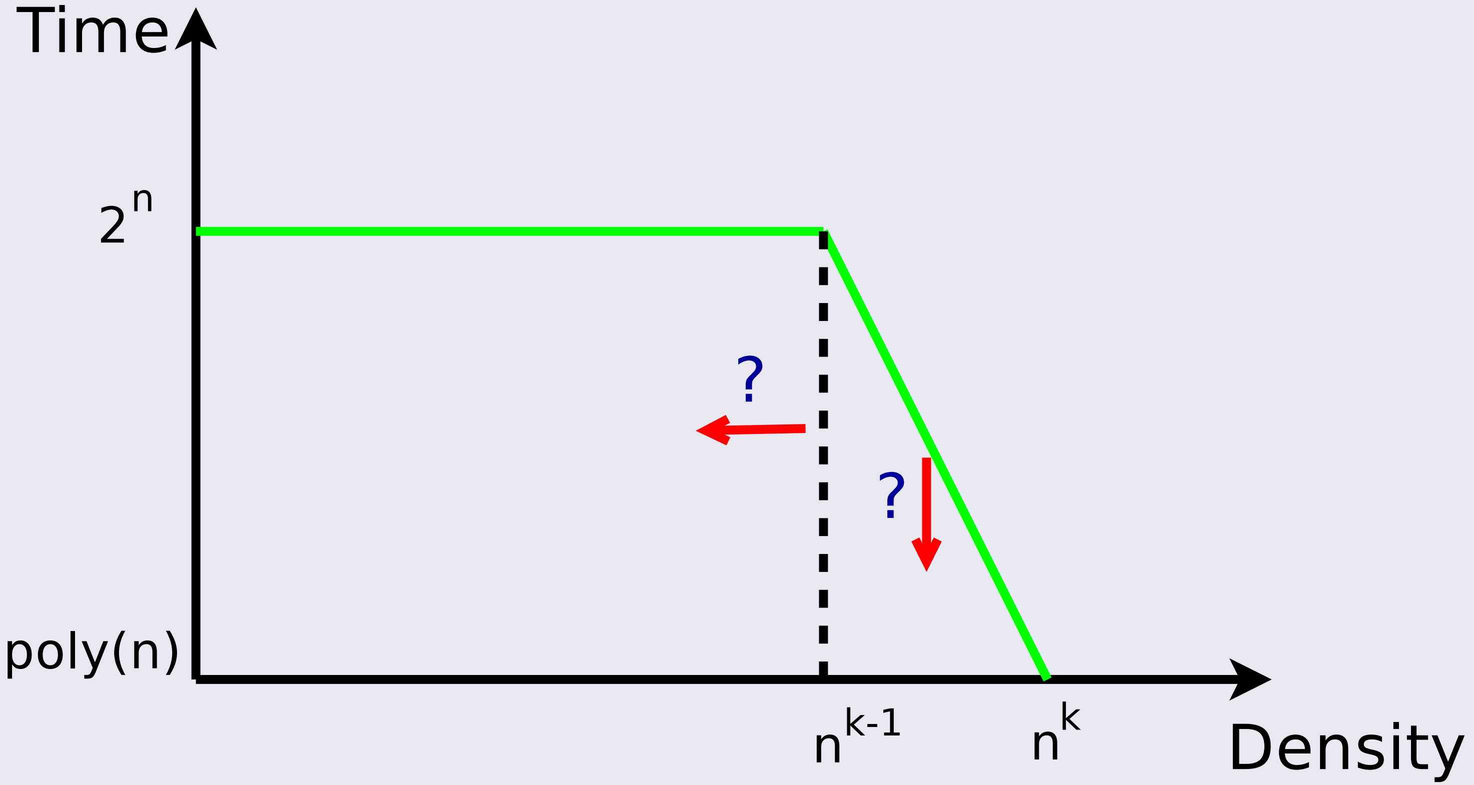
- Smaller sample/faster running time?
- Handle $k \geq 3$ better?

Summary – with a picture



Complexity/Density trade-off for Max-k-CSP

Summary – with a picture



Possible Improvements? Faster? More general?

Density lower bound

Theorem: Assuming ETH, for all $k \geq 3$, $\exists r < 1$ s.t. $\forall \epsilon > 0$ no algorithm can r-approximate Max-k-SAT with $m \leq n^{k-1}$ in time $2^{n^{1-\epsilon}}$

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In English: For density less than n^{k-1} we need exponential time to get $(1-\epsilon)$-approximation.

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Starting Point: Max-2-SAT is “APX-ETH”-hard on instances with $|V| = n$ and $m = O(|V|)$.

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- Add n new variables $y_1, \ldots, y_n$
- For each clause $(x_i \vee x_j)$, for each $k \in \{1, \ldots, n\}$ we construct the clauses $(x_i \vee x_j \vee y_k)$ and $(x_i \vee x_j \vee \neg y_k)$
- Gap remains!
- Number of clauses $\approx n^2$

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Reduction similar for $k > 3$

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In English: Our sample size is optimal. For density n^{δ} we need time $2^{n^{1-\delta}}$.

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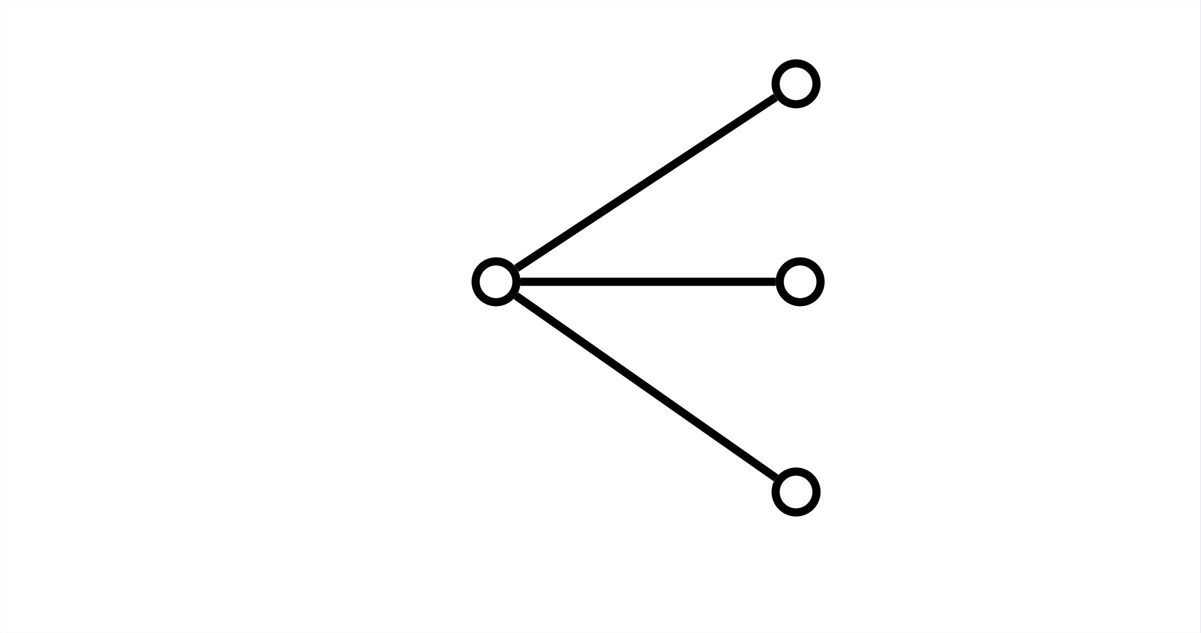

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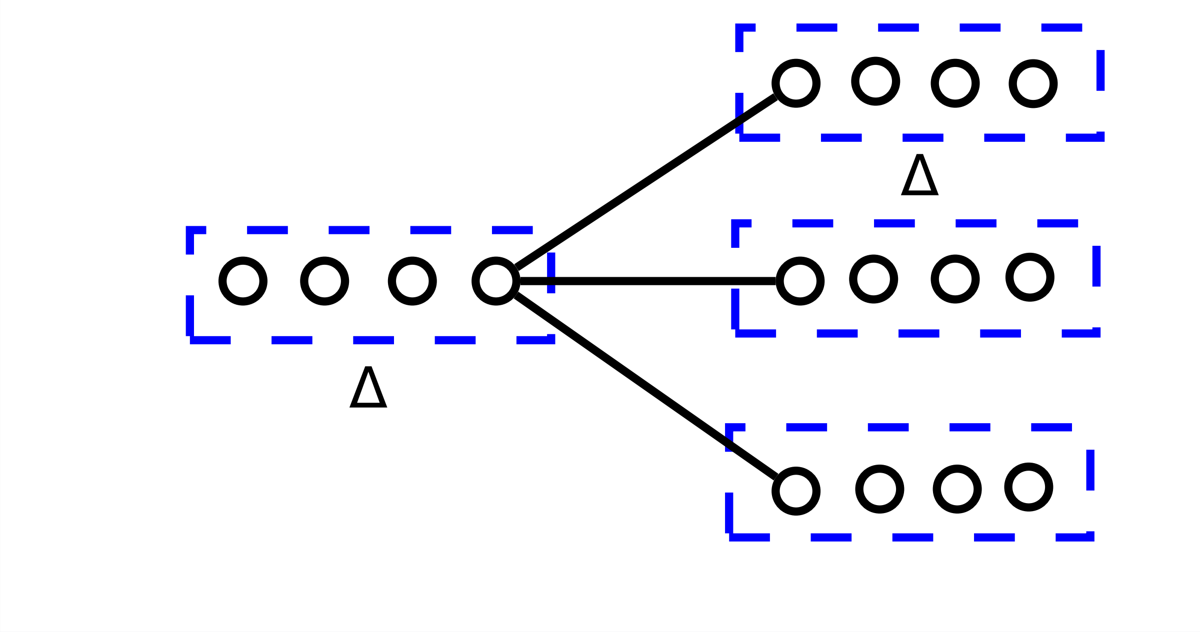

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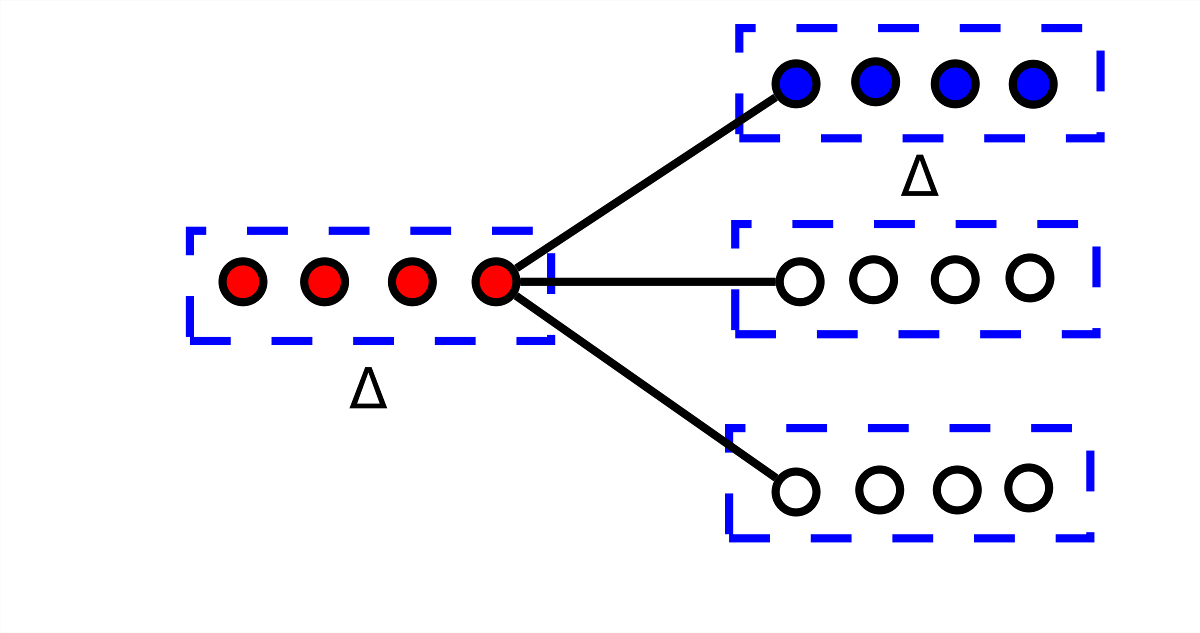

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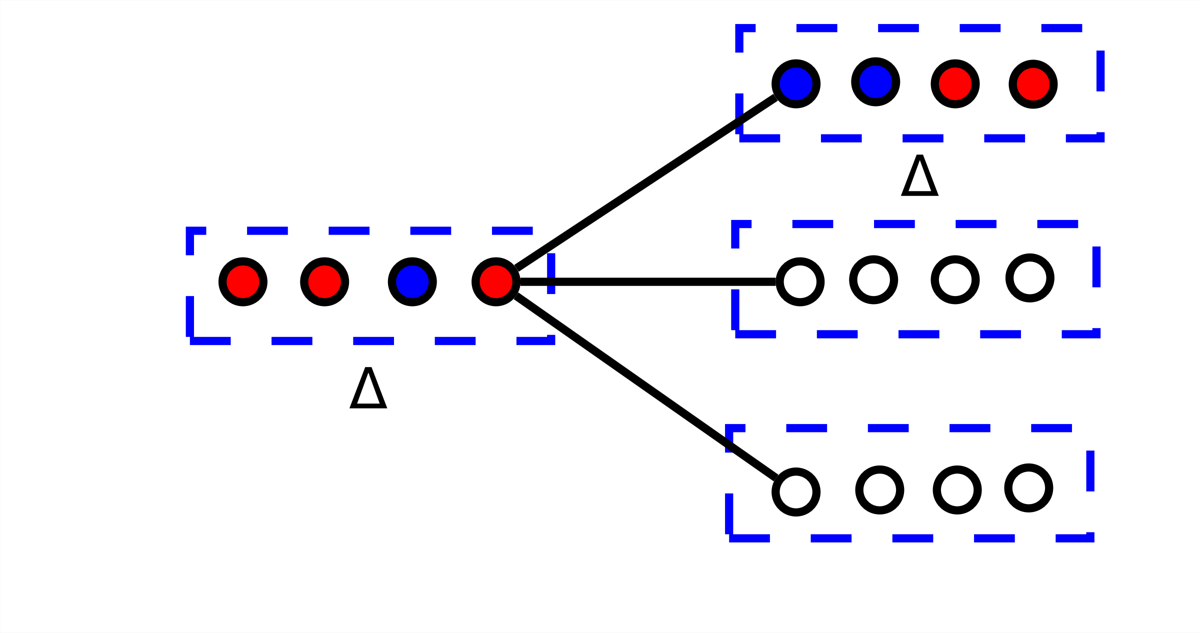

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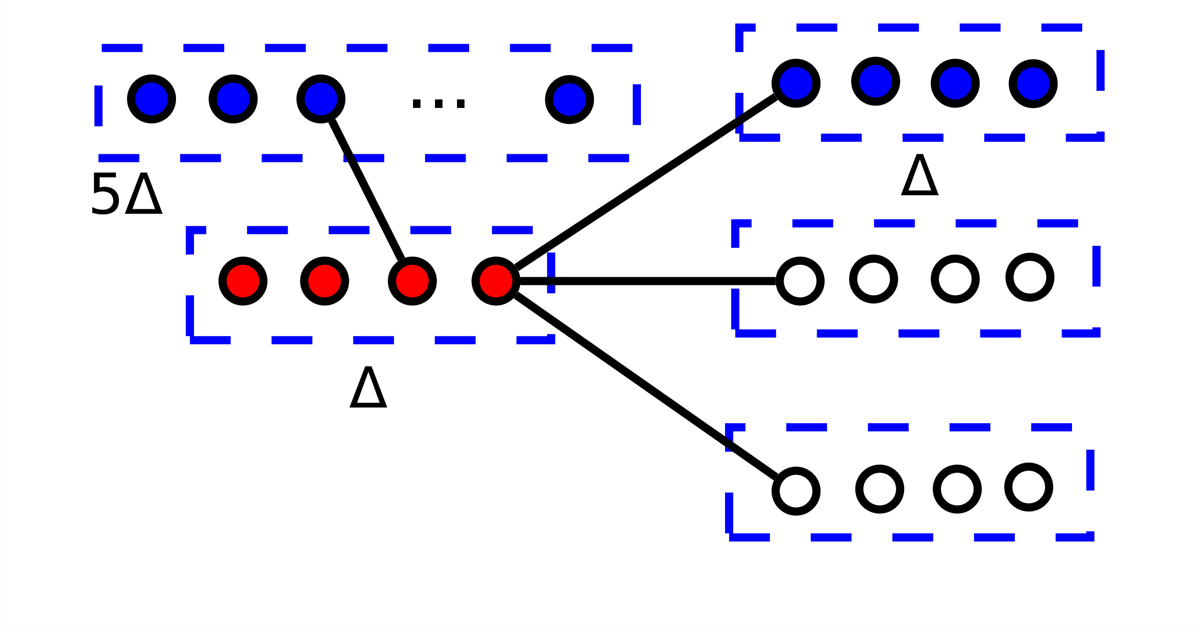

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Starting Point: Max Cut is “APX-ETH”-hard on 5-regular instances with $|V| = n$.

- A constant gap remains for any Δ
- $|V'| = n\Delta$, $|E| = n\Delta^2$, Avg. degree $= \Delta$
- If we cound do better than $2^{|V'|/\Delta}$ then $\neg$ETH

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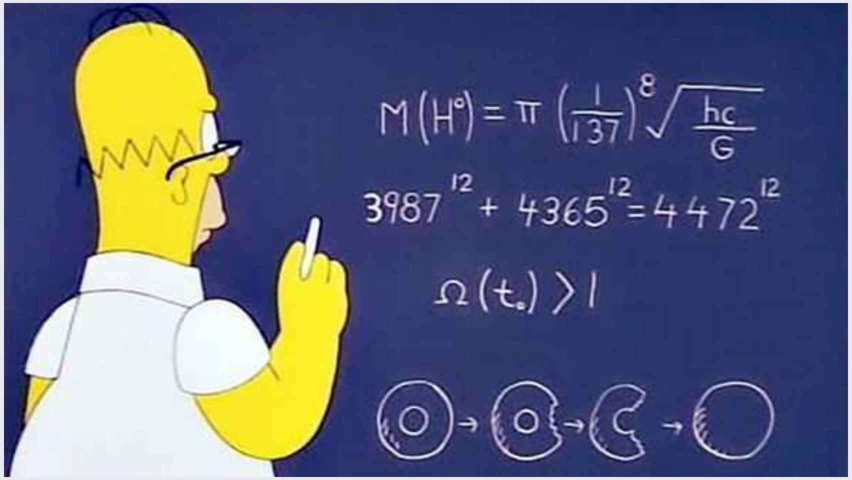
- **Bonus:** The two reductions compose! Optimal running times everywhere!

Conclusions

- Density is a crucial parameter for approximating Max-k-CSP
  - Especially useful in sub-exponential setting
  - Smooth trade-off between performance and generality
  - “Tight” bounds
- Lesson: Don’t forget to take into account input structure!



Leveraging Structure to Solve CSP

- Must take into account that input may have some useful properties (otherwise problem too hard!)
- So far, we have used simple properties (density)
- Time to measure structure in a more sophisticated way!
    - **Parameterized Complexity** == Trading Time for Generality
    - Define some distance k from a tractable case (distance from triviality)
    - Try to produce algorithm whose performance **slowly degenerates** as k increases.
- Use tools from Graph Theory to describe input structure.

Structural CSP

- Define some graph structure of ϕ.
- Study CSPs for special graph classes.

Structural CSP

- Define some graph structure of ϕ.

Incidence graph representation of a CSP

- (Unsigned) variables and constraints are represented by vertices;
- a constraint vertex is connected to a variable vertex iff the corresponding constraint involves the corresponding variable.

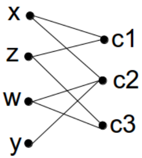


Figure: The incidence graph representation of the previous formula $(\neg x \vee z) \wedge (x \vee y \vee \neg w) \wedge (\neg z \vee w)$.

Structural CSP

- Study CSPs for special graph classes.

Example classes

- Low degree: Bounding degree of incidence graph doesn't help (3CNFSAT where every variable appears at most 3 times is NP-complete).

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Example classes

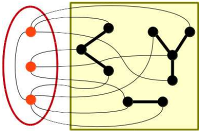
- Low degree: Bounding degree of incidence graph doesn't help (3CNFSAT where every variable appears at most 3 times is NP-complete).
- Acyclicity: Start from the leaves and work your way up (poly-time).

Distance from being acyclic

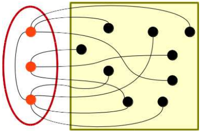
- *Treewidth*

Distance from being acyclic

- *Treewidth*
- *Feedback Vertex Set*: Set of vertices whose removal leaves the graph acyclic.



- *Treewidth*
- *Feedback Vertex Set*: Set of vertices whose removal leaves the graph acyclic.
- *Vertex Cover*: Set of vertices whose removal leaves an independent set.



Distance from being acyclic

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$fvs \leq vc$: Independent set is acyclic.

Distance from being acyclic

- *Treewidth*
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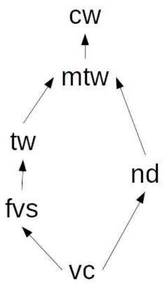
- *Vertex Cover*: Set of vertices whose removal leaves an independent set.

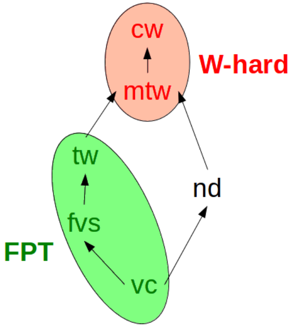
$fvs \leq vc$: Independent set is acyclic.
$tw \leq fvs + 1$:

- Make a tree-decomposition of the forest;
- Put the *fvs* in all bags;

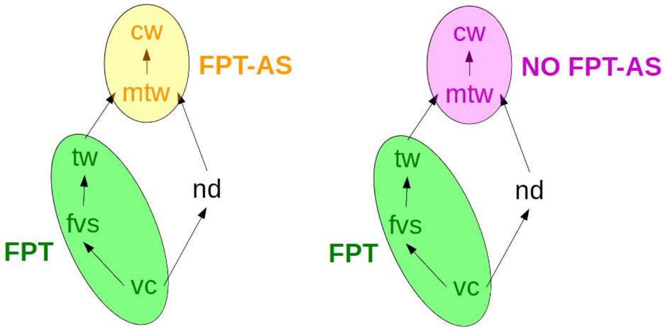


Parameter map: $q \leftarrow p$ (which reads 'q *dominates* p') between two parameters means that q is bounded when p is bounded.

Structural Parameterizations

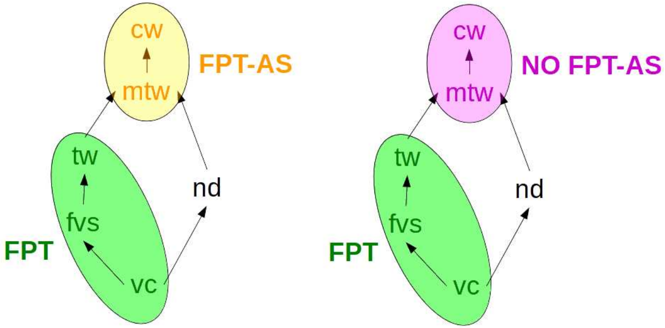


Goal: design algorithms for most dominant parameter (hold downward) and hardness for least dominant (hold upward).



New approach: study FPT approximations to evade hardness. In this talk we examine the existence of FPT Approximation Schemes.

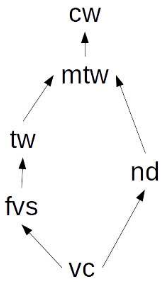
Structural Parameterizations



Definition

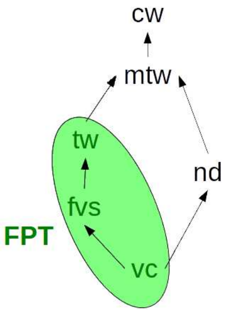
FPT Approximation Scheme (FPT-AS): $\forall \epsilon > 0$ there is an $(1-\epsilon)$-approximation algorithm running in time $O(f(\epsilon, k) \cdot \text{poly}(n))$.

CNFSAT and MAXCNFSAT



Theorem [Szeider 2004]

MAXCNFSAT parameterized by incidence treewidth (tw^*) is FPT.



CNFSAT and MAXCNFSAT

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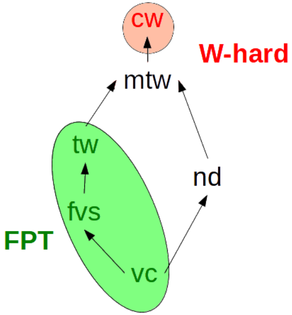
Theorem [Ordyniak, Paulusma, Szeider 2013]

CNFSAT parameterized by cw^* is W[1]-hard.



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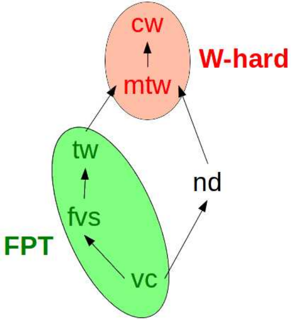
Hardness even holds for a more restricted parameter *modular treewidth* [Paulusma, Slivovsky, Szeider 2013].



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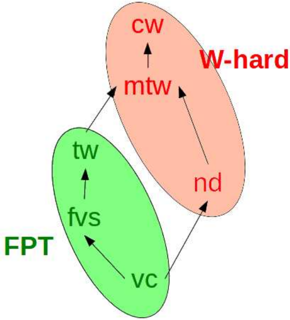
$\rightarrow$ **We extend W[1]-hardness to incidence neighborhood diversity (nd^*).**



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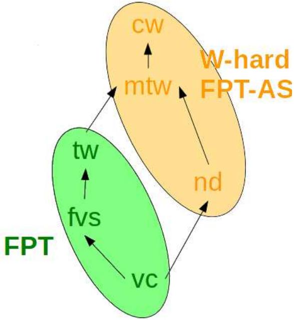
Hardness even holds for a more restricted parameter *modular treewidth* [Paulusma, Slivovsky, Szeider 2013].

$\rightarrow$ **We extend W[1]-hardness to incidence neighborhood diversity (nd^*).**

$\rightarrow$ **We also present an FPT-AS for cw^*.**



On the positive side...

Theorem

MaxCNFSat *parameterized by* cw^* *admits an FPT-AS.*

On the positive side...

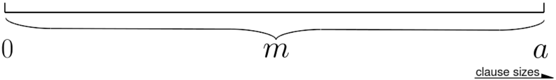
Theorem

MaxCNFSat *parameterized by* cw^* *admits an FPT-AS.*

Reminder

FPT-AS (FPT Approximation Scheme) for a maximization problem parameterized by k: $\forall \epsilon > 0$ there exists an $(1-\epsilon)$-approximation algorithm running in $O(f(\epsilon, k) \cdot \text{poly}(n))$.

An FPT-AS for MaxCNFSat parameterized by cw*

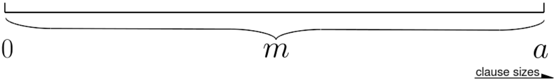


Arrange the clauses in increasing order of arity (0 to a).

An FPT-AS for MAXCNFSAT parameterized by cw*

B: Clauses of arity $\geq D = g(\epsilon)$.
S: Clauses of arity $\leq d = g'(\epsilon)$.

$D = d \cdot \epsilon^4$



Arrange the clauses in increasing order of arity (0 to a).
Split them into big (arity at least $g(\epsilon)$), small (arity at most $g'(\epsilon)$, and medium.
Consider the following cases:

B: Clauses of arity $\geq D = g(\epsilon)$.
S: Clauses of arity $\leq d = g'(\epsilon)$.

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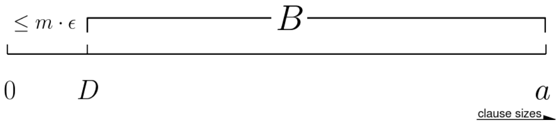

(Almost) all clauses are big:

B: Clauses of arity $\geq D = g(\epsilon)$.
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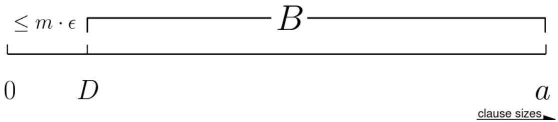

(Almost) all clauses are big:

- ignore small clauses;

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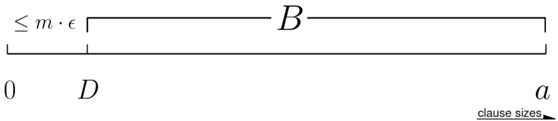

(Almost) all clauses are big:

- ignore small clauses;
- a random assignment satisfies $\geq (1-\epsilon)(1-2^{-g(\epsilon)}) \cdot m$ clauses (with high probability).

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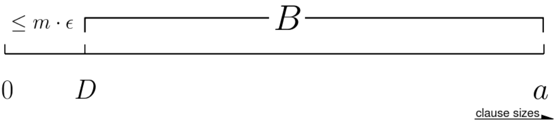

(Almost) all clauses are big:

- ignore small clauses;
- a random assignment satisfies $\geq (1-\epsilon)(1-2^{-g(\epsilon)}) \cdot m$ clauses (with high probability).
- Since $m \geq OPT$, $SOL \geq (1-\epsilon')OPT$, for some ϵ' depending on ϵ.

An FPT-AS for MaxCNFSat parameterized by cw*

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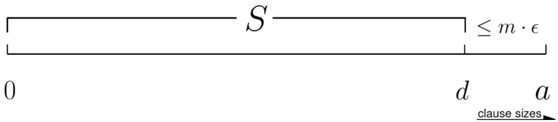

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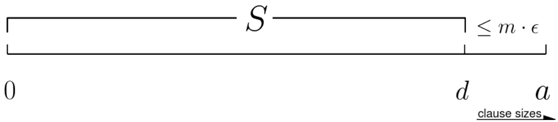

(Almost) all clauses are small:

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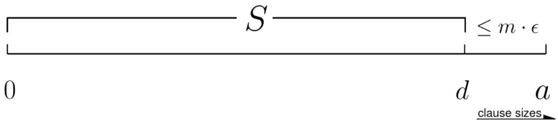

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- degree on one side of the incidence graph is bounded $\rightarrow$ no large biclique subgraphs;

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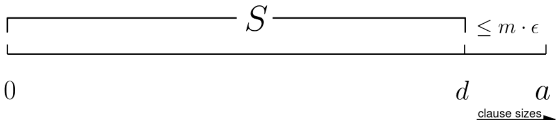

(Almost) all clauses are small:

- ignore large clauses;
- degree on one side of the incidence graph is bounded $\rightarrow$ no large biclique subgraphs;
- By [Gurski, Wanke 2000], the incidence graph has bounded treewidth $\rightarrow$ solve optimally the remaining small clauses;

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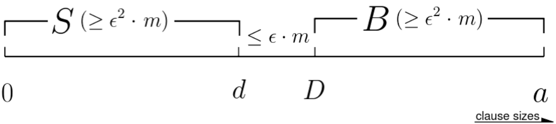

(Almost) no medium-size clauses and B, S are balanced:

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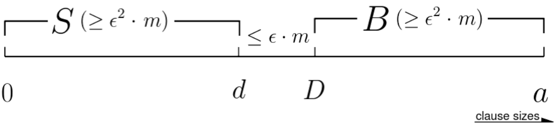

(Almost) no medium-size clauses and B, S are balanced:

- variable occurences$(B) \geq |B| \cdot D = \frac{m \cdot d}{\epsilon^2}$;
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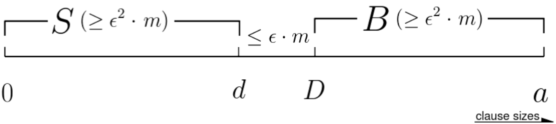

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$\rightarrow \exists y \in V$ that appears $1/\epsilon^2$ more times in B than in S.

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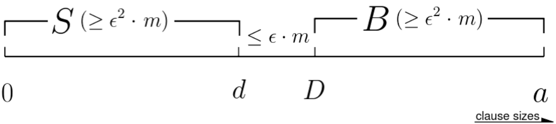

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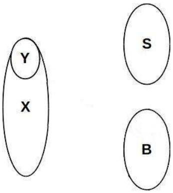
From the previous observation, we iteratively create a set of variables Y with the following properties:



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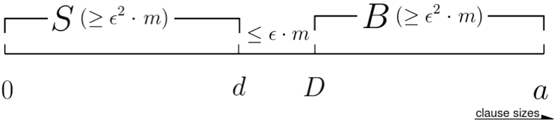

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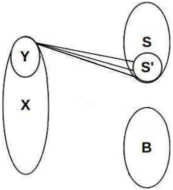
- Y *hits* few clauses of S (call this set S');



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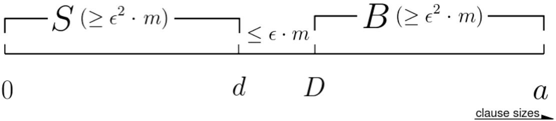
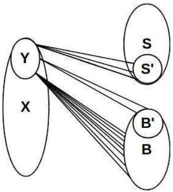


(Almost) no medium-size clauses and B, S are balanced:
From the previous observation, we iteratively create a set of variables Y with the following properties:

- Y *hits* few clauses of S (call this set S');
- at most ϵ^2 clauses of B have $\leq 1/\epsilon$ neighbors in Y (call this set B').



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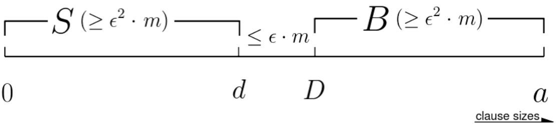

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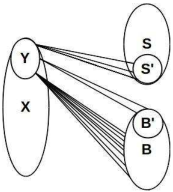

Randomly assigning Y should satisfy whp $\geq (1-\epsilon^2)\cdot(1-2^{-1/\epsilon})$ of $B \setminus B'$, while $S \setminus S'$ can be solved optimally.

An FPT-AS for MaxCNFSat parameterized by cw^*

Lemma

We can always find a small set M ($|M| \leq \epsilon \cdot m$) of medium-size clauses (arities $d \sim D$).

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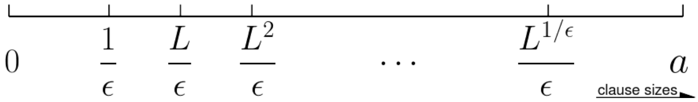
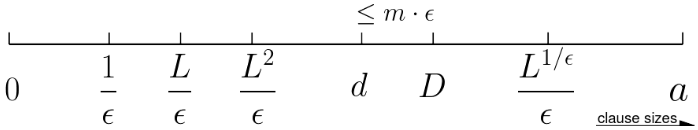


Define $1/\epsilon + 1$ independent intervals of medium-arity clauses (right-left bounds are an $L(= \epsilon^{-4})$-factor apart).

An FPT-AS for MAXCNFSAT parameterized by cw*

Lemma

We can always find a small set M ($|M| \leq \epsilon \cdot m$) of medium-size clauses (arities $d \sim D$).



There should be at least one interval $[d, D]$ ($D = L \cdot d$) containing $\leq \epsilon \cdot m$ clauses.

Lemma

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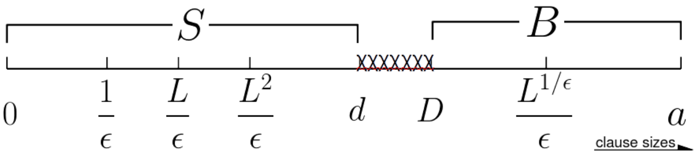

Removing them divides the clauses into small (S) and big (B).

The algorithm

- Find interval [d,D] of at most $\epsilon \cdot m$ clauses of medium arities as in the previous Lemma and ignore them.

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- Find interval [d,D] of at most $\epsilon \cdot m$ clauses of medium arities as in the previous Lemma and ignore them.
- Split remaining clauses into S (arity $< d$) and B (arity $> D$).
- If $|S| \leq \epsilon^2 \cdot m$
  - Ignore S;

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- Find interval [d,D] of at most $\epsilon \cdot m$ clauses of medium arities as in the previous Lemma and ignore them.
- Split remaining clauses into S (arity $< d$) and B (arity $> D$).
- If $|S| \leq \epsilon^2 \cdot m$
  - Ignore S;
  - Randomly assign variables to satisfy most of B.

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- If $|S| \leq \epsilon^2 \cdot m$
  - Ignore S;
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- If at most $|B| \leq \epsilon^2 \cdot m$
  - Ignore B;

The algorithm

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- Split remaining clauses into S (arity $< d$) and B (arity $> D$).
- If $|S| \leq \epsilon^2 \cdot m$
  - Ignore S;
  - Randomly assign variables to satisfy most of B.
- If at most $|B| \leq \epsilon^2 \cdot m$
  - Ignore B;
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  - Find set of variables Y as in the last case and set it randomly to satisfy most of B.
  - Ignore part of S that contains variables from Y and solve the rest optimally.

Conclusions

- Trading **Time-Generality-Approximation**
- Crucial: Take into account input structure!
- Long-term Goal: Map out complete trade-offs
  - For each desired approximation ratio, for each class of inputs (defined by k), what is the correct running time?

A concrete problem for ESIGMA

- Use these techniques for **approximate formula representation** (aka formula learning/knowledge compilation).
- What are the key measures of input structure?

Thank you!
Questions?