Connectivity degeneracy measures

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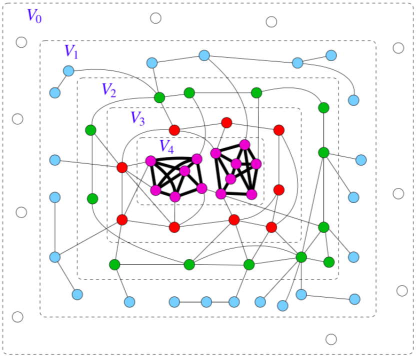
k-core(G) =maximum subgraph H where $\delta(H) \geq k$.
Core decomposition: $V_0, V_1, \ldots, V_k$,
$V_i = i$-core(G) $\setminus$ $(i-1)$-core(G)



Graph degeneracy: $\delta^*(G) = \max\{k \mid k\text{-core}(G) \text{ is non-trivial}\}$

Core decomposition: graph hierarchization

Graph degeneracy: $\delta^*(G) = \max\{k \mid k\text{-core}(G) \text{ is non-trivial}\}$

[P.Erdős. On the structure of linear graphs. Israel J. Math., 1:156–160, 1963.]

[S.Seidman. Network structure and minimum degree. Social Networks, 5(3):269–287, 1983.]

[R. Anderson & E. Mayr. Parallelism and greedy algorithms, Adv. Comput. Res., 4 (1987) pp. 17-38;]

Good news: $O(kn)$ algorithm to compute the core decomposition:

- Remove vertices of minimum degree... the degrees determine the core levels.

Graph degeneracy as a measure of collaboration:

just being a *hub* is not enough in order to be "collaborative"

Other variants of (degree) degeneracy:

Directed graphs:

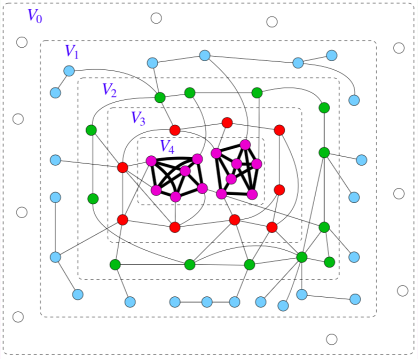
[C. Giatsidis, D.M. Thilikos, M. Vazirgiannis: D-cores: measuring collaboration of directed graphs based on degeneracy. Knowl. Inf. Syst. 35(2): 311-343 (2013)]

Signed graphs:

[C. Giatsidis, B. Cautis, S. Maniu, D.M. Thilikos, M.Vazirgiannis: Quantifying trust dynamics in signed graphs, the S-Cores approach. SDM 2014: 668-676]

Clustering:

[C. Giatsidis, F. D. Malliaros, D. M. Thilikos, M. Vazirgiannis: CoreCluster: A Degeneracy Based Graph Clustering Framework. AAAI 2014: 44-50]



Back to the definition:

k-core(G) =maximum subgraph H where $\delta(H) \geq k$.

Where $\delta(H) = \min\{\mathbf{deg}_H(v) \mid v \in V(H)\}$

Other definitions of $\mathbf{deg}_H(v)$?

For instance: $\mathbf{deg}_H(v) = \#$ triangles containing v:
(i.e., $|E(G[N_H(v)])|$)

[F.D. Malliaros, M.G. Rossi, and Michalis Vazirgiannis. Locating influential nodes in complex networks. Scientific reports, Nature Publishing Group, 2016.]

Another example: $\mathbf{deg}_H(v) = \#$ edge-disjoint triangles on v:
(i.e., max matching in $G[N_H(v)]$). Better? ... still polynomial!

Which “degree” function $\mathbf{deg}_H(v)$ define good “core decompositions”?

- Combinatorially Good: i-core$\subseteq (i-1)$-core. (hierarchization)
- Algorithmically Good: polynomially computable.

(*polynomially?*)

A major advance:

[V. Batagelj, M. Zaversnik: Fast algorithms for determining (generalized) core groups in social networks. Adv. Data Analysis and Classification 5(2): 129-145 (2011)]

Let $H \subseteq G$

$\mathbf{deg}_H(v)$ is *monotone* if:

$H_1 \subseteq H_2 \Rightarrow \forall v \in V : \mathbf{deg}_{H_1}(v) \leq \mathbf{deg}_{H_2}(v)$

$\mathbf{deg}_H(v)$ is *local* if: $\mathbf{deg}_H(v) = \mathbf{deg}_{N_H[v]}(v)$

Theorem

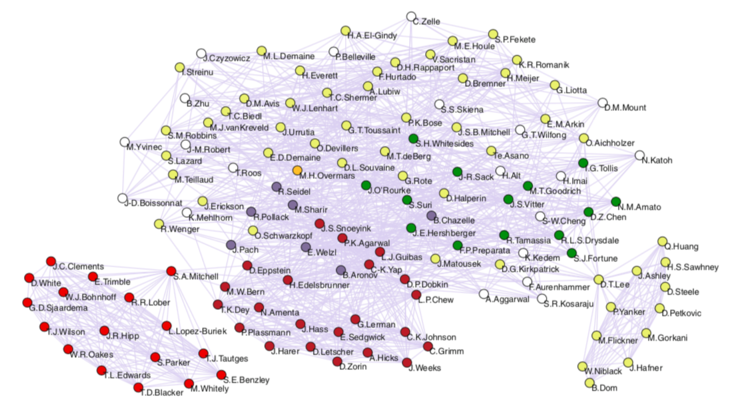
If $\mathbf{deg}_H(v)$ *is* monotone *then* δ^* *is hierarchical*

Theorem

If $\mathbf{deg}_H(v)$ *is* local *then* δ^* *can be computed in* $O(m \cdot \max(\Delta, \log n))$ *steps*

not just polynomial!

Some reflection on (degree) cores...



The 10-core of the "Computational geometry" network

- It might be quite *disconnected*! Degree Locality is not enough!

Connectivity degeneracy

$S \subseteq H \subseteq G$

edge connectivity of a vertex set S of a subgraph H:
$\lambda_H(S) = \#$cut-edges in H between S & $H \setminus S$

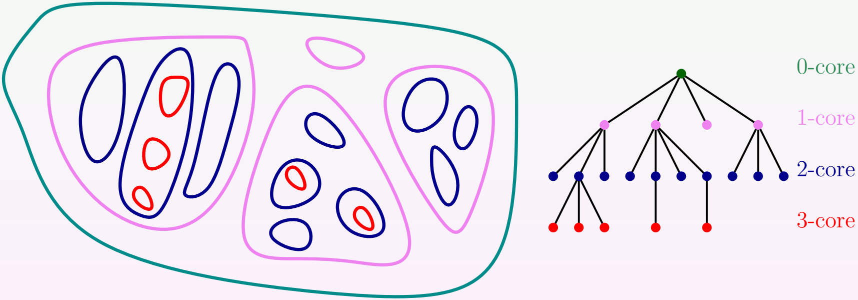
$H \subseteq G$

edge connectivity of a subgraph H:
$\lambda(H) = \min\{\lambda_H(S) \mid \emptyset \subsetneq S \subsetneq H\}$

(edge) connectivity degeneracy of G:
$\lambda^*(G) = \max\{\lambda(H) \mid H \subseteq G\}$

What is a k-λ-core?

A partition of G so that each part induces a maximal k-edge-connected subgraph.



Connectivity degeneracy vs degree degeneracy:

$\delta^*(G) \leq \lambda^*(G) \leq 2 \cdot \delta^*(G)$

[W. Mader, Existenz n-fach zusammenhängender Teilgraphen in Graphen genügend großer Kantendichte, Abh. Math. Sem Univ. Ham- burg 37, 86–97 (1972).]

A "polynomial" algorithm (also parallel approximation):

[L. Kirousis, M. Serna, P. Spirakis, The parallel complexity ofthe subgraph connectivity problem, SIAM J. Comput., 22 (1993), pp. 573-586.]

More on connectivity degeneracy in the next talk...

Which "cut" functions $\lambda_H(S)$ define good "core decompositions"?

- Combinatorially Good:

i-core is a refinement of $(i-1)$-core. (hierarchization)

- Algorithmically Good: polynomially computable.

(*polynomially?*)

Consider a universe U and a function $\lambda : U \to \mathbb{N}$.
We say that λ is a *connectivity function on U* if the following hold:

- $\lambda(\emptyset) = 0$.
- $\forall X \subseteq U,\ \lambda(X) = \lambda(U \setminus X)$ (symmetry).
- $\forall X, Y \subseteq U,\ \lambda(X \cup Y) + \lambda(X \cap Y) \leq \lambda(X) + \lambda(Y)$ (submodularity).

We say a family $\{\lambda_S : S \subseteq U\}$ of functions $\lambda_S : 2^S \to \mathbb{N}$ is *subset monotone* if $\lambda_{S'}(X) \leq \lambda_S(X)$ whenever $S' \subseteq S$.

An abstract notion of connectivity degeneracy

Let U be a finite set (a "universe").
Given a family $\mathcal{F} = \{\lambda_S : S \subseteq U\}$ of functions $\lambda_S : 2^S \to \mathbb{N}$, we define

$$\lambda^*_{\mathcal{F}}(U) = \max\{\min\{\lambda_S(X) \mid \emptyset \subsetneq X \subsetneq S\} \mid S \subseteq U\}$$

Our results:

Theorem
If $\mathcal{F}$ is a subset-monotone family, then $\lambda^*_{\mathcal{F}}$ is hierarchical.

Theorem
If $\mathcal{F}$ is a family of connectivity functions, then $\lambda^*_{\mathcal{F}}$ can be computed in polynomial time.

Conclusions and research directions

- We set up a theoretical framework for connectivity degeneracy.
- The theorems of [Batagelj & Zaversnik] follow as the degree analogues our our results.
- We consider several connectivity functions/measures fitting in the above framework.
- We study $\lambda_H(S) =$ max # edge-disjoint paths between S and $H \setminus S$.
- We work on an analogous framework on vertex-conectivity (clusters may overlap).
- We work on optimizing the running times of the polynomial algorithms for certain interesting instantiations of λ
- We work on clustering problems under alternative connectivity measures.

Ongoing project Involving:

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- Stratis Limnios

and hopefully more!

Merci!