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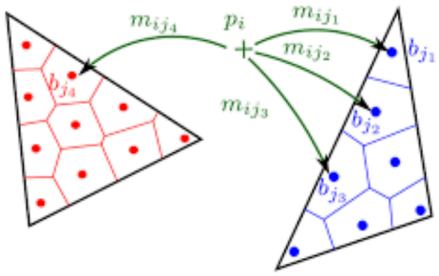

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Quantitative Behavioural Reasoning for Higher-order Effectful Programs: Applicative Distances

Keywords applicative distance, applicative similarity, applicative bisimilarity, Howe's method, algebraic effects, Fuzz, relator

1 Introduction

Program preorders and equivalences are fundamental concepts in the theory of programming languages since the very birth of the discipline. Such notions are usually defined by means of relations between program phrases aimed to order or identify programs according to their observable *behaviours*, the latter being usually defined by means of a primitive notion of observation such as termination to a given value. We refer to such relations as *behavioural relations*. Well-known behavioural relations for higher-order functional languages include the *contextual preorder* and *contextual equivalence* [38], *applicative (bi)similarity* [1], and *logical relations* [42].

Instead of asking when two programs e and e' are behaviourally similar or equal, a more informative question may be asked, namely how much (behaviourally) different e and e' are. That means that instead of looking at relations relating programs with similar or equal behaviours we look at relations assigning programs a numerical value representing their *behavioural distance*, i.e. a numerical value quantifying the observable differences between their behaviours. The question of quantifying observable differences between programs turned out to be particularly interesting (and challenging) for effectful higher-order languages, where ordinary qualitative (i.e. boolean-valued) equivalences and preorders are too strong. This is witnessed by recent results on behavioural pseudometrics for probabilistic λ-calculi [12, 13] as well as results on semantics of higher-order languages for differential privacy [17, 41]. In the first case one soon realises that programs exhibiting a different behaviour only with probability close to zero are fully discriminated by ordinary behavioural relations, whereas in the second case relational reasoning does not provide any information on how much behavioural differences between inputs affect behavioural differences between outputs.

These problems can be naturally addressed by working with quantitative relations capturing weakened notions of metric such as *generalised metrics* [32] and *pseudometrics* [47]. It is then natural to ask whether and to what extent ordinary behavioural relations can be refined into quantitative relations still preserving their nice properties. Although easy to formulate, answering such question is far from trivial and requires major improvements in the current theory of behavioural reasoning about programs.

This paper contributes to answering the above question, and it does so by studying the quantitative refinement of Abramsky's *applicative similarity* and *bisimilarity* [1] for higher-order languages enriched with algebraic effects. Applicative similarity (resp. bisimilarity) is a coinductively defined preorder (resp. equivalence) relating programs that exhibit similar (resp. equal) extensional behaviours. Due to its coinductive nature and to its nice properties, applicative (bi)similarity has been studied for a variety of calculi, both pure and effectful. Notable examples are extensions to nondeterministic [31] and probabilistic [11, 15] λ-calculi, and its more recent extension [14] to λ-calculi with algebraic effects *à la* Plotkin and Power [40]. In [14] an abstract notion of applicative similarity is studied for an untyped λ-calculus enriched with a signature of effect-triggering operation symbols. Operation symbols are interpreted as algebraic operations with respect to a monad T encapsulating the kind of effect such operations produce. Examples are probabilistic choices with the (sub)distribution monad, and nondeterministic choices with the powerset monad. The main ingredient used to extend Abramsky's applicative similarity is the concept of a *relator* [6, 48] for a monad T, i.e. an abstraction meant to capture the possible ways a relation on a set X can be turned into a relation on TX. That allows to define an abstract notion of *effectful* applicative similarity parametric in a relator, and to prove an abstract precongruence theorem stating the resulting notion of applicative similarity is a compatible preorder.

The present work originated from the idea of generalising the theory developed in [14] to relations taking values over arbitrary quantitative domains (such as the real extended half-line $[0,\infty]$ or the unit interval $[0,1]$). Such generalisation requires three major improvements in the current theory of effectful applicative (bi)similarity:

1. The first improvement is to move from boolean-valued relations to relations taking values on quantitative domains such as $[0,\infty]$ or $[0,1]$ in such a way that restricting these domains to the two element set $\{0,1\}$ (or $\{\mathsf{false},\mathsf{true}\}$) makes the theory collapse to the usual theory of applicative (bi)similarity. For that we rely on Lawvere's analysis [32] of generalised metric spaces and preordered sets as enriched categories. Accordingly, we replace boolean-valued relations with relations taking values over quantales [43] $(\mathsf{V},\leq,\otimes,k)$, i.e. algebraic structures (notably complete lattices equipped with a monoid structure) that play the role of sets of abstract quantities. Examples of quantales include the extended real half-line $([0,\infty],\geq,0,+)$ ordered by the "greater or equal" relation $\geq$ and with monoid structure given by addition (and its restriction to the unit interval $[0,1]$), and the extended real half-line $([0,\infty],\geq,0,\max)$ with monoid structure given by binary maximum (in place of addition), as well as any complete Boolean and Heyting algebra. This allows to develop an algebra of quantale-valued relations, V-relations for short, which provides a general framework for studying both behavioural relations and behavioural distances (for instance, an equivalence V-relation instantiates to an ordinary equivalence relation on the boolean quantale $(\{\mathsf{false},\mathsf{true}\},\leq,\wedge,\mathsf{true})$, and to a pseudometric on the quantale $([0,\infty],\geq,0,+)$).
2. The second improvement is the generalisation of the notion of relator to quantale-valued relators, i.e. relators acting on relations taking values over quantales. Perhaps surprisingly, such generalisation is at the heart of the filed of *monoidal topology* [25], a subfield of categorical topology aiming to unify ordered,

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metric, and topological spaces in categorical terms. Central to the development of monoidal topology is the notion of V-relator or V-lax extension of a monad T which, analogously to the notion of relator, is a construction lifting V-relations on a set X to V-relations on TX. Notable examples of V-relators are obtained from the Hausdorff distance (for the powerset monad) and from the Wasserstein-Kantorovich distance [49] (for the distribution monad).

3. The third improvement (on which we will expand more in the next paragraph) is the development of a *compositional* theory of behavioural V-relations (and thus of behavioural distances). As we are going to see, ensuring compositionality in an higher-order setting is particularly challenging due to the ability of higher-order programs to copy their input several times, a feature that allows them to amplify distances between their inputs *ad libitum*.

The result is an abstract theory of behavioural V-relations that allows to define notions of quantale-valued applicative similarity and bisimilarity parametric in a quantale-valued relator. The notions obtained generalise the existing notions of real-valued applicative (bi)similarity and can be instantiated to concrete calculi to provide new notions of applicative (bisimilarity) distance. A remarkable example is the case of probabilistic λ-calculi, where to the best of the author's knowledge a (non-trivial) applicative distance for a universal (i.e. Turing complete) probabilistic λ-calculus is still lacking in the literature (but see Section 9).

The main theorem of this paper states that under suitable conditions on monads and quantale-valued relators the abstract notion of quantale-valued applicative similarity is a compatible—i.e. compositional—reflexive and transitive V-relation. Under mild conditions such result extends to quantale-valued applicative bisimilarity, which is thus proved to be a compatible, reflexive, symmetric, and transitive V-relation (i.e. a compatible pseudometric).

In addition to the concrete results obtained for quantale-valued applicative (bi)similarity, the contribution of the present work also relies on introducing and combining several notions and results developed in different fields (such as monoidal topology, coalgebra, and programming language theory) to build an abstract framework for studying quantitative refinements of behavioural relations for higher-order languages whose applications go beyond the present study of applicative (bi)similarity.

Compositionality, distance amplification, and linear types Once we have understood what is the behavioural distance $\delta(e, e')$ (which, for the sake of this argument, we assume to be a non-negative real number) between two programs e and e', it is natural to ask if and how much such distance is modified when e and e' are used inside a bigger program—i.e. a context—$C[-]$. Indeed we would like to reason about the distance $\delta(C[e], C[e'])$ *compositionally*, i.e. in terms of the distance $\delta(e, e')$.

Compositionality is at the heart of relational reasoning about program behaviours. Informally, compositionality states that observational indistinguishability is preserved by language constructors; formally, a relation is compositional if it is *compatible* with all language constructors, meaning that whenever two programs e and e' are related, then so are the bigger programs $C[e]$ and $C[e']$.

Analogous to the idea that compatible relations are preserved by language constructors, we are tempted to define as compatible those distances that are not increased by language constructors. That is, we would like to say that a behavioural distance δ is compatible if the distance $\delta(C[e], C[e'])$ between $C[e]$ and $C[e']$ is always bounded by the distance $\delta(e, e')$, no matter how $C[-]$ uses e and e'. However, we soon realise that such proposal cannot work: not only how $C[-]$ uses e and e' matters, but also *how much* it uses them does. This phenomenon, called *distance amplification* [13], can be easily observed when dealing with probabilistic languages. Consider the following example for a probabilistic untyped λ-calculus [15] taken from [13]. Let I be the identity combinator and $I \oplus \Omega$ be the program evaluating to I with probability $\frac{1}{2}$, and diverging with probability $\frac{1}{2}$. Assuming we observe the probability of convergence of a program, it speaks by itself that we would expect the behavioural distance $\delta(I, I \oplus \Omega)$ between I and $I \oplus \Omega$ to be $\frac{1}{2}$. However, it is sufficient to consider a family $\{C_n[-]\}_{n \geq 0}$ of contexts that duplicate their input n-times[1] to see that any such context amplifies the observable distance between I and $I \oplus \Omega$: as n grows, the probability of convergence of $C[I \oplus \Omega]$ tends to zero, whereas the one of $C[I]$ remains always equal to one. During its evaluation, every time the context C_n evaluates its inputs the detected distance between the latter is somehow accumulated to the distances previously observed, thus exploiting the *linear*—in opposition to classical—nature of the act of measuring. Such linearity naturally reflects the monoidal closed structure of categories of metric spaces, in opposition with the cartesian closed structure characterising 'classical' (i.e. boolean-valued) observations.

The above example shows that if we want to reason compositionally about behavioural distances, then we have to accept that contexts can amplify distances, and thus we should take into account the number of times a program accesses its input. More concretely, our notion of compatibility allows a context $C[-]$ using its input s times to increase the distance $\delta(e, e')$ between e and e', but of a factor *at most* s. That is, the distance $\delta(C[e], C[e'])$ should be bounded by $s \cdot \delta(e, e')$. Our main result states that quantale-valued applicative (bi)similarity is compatible in this sense. This result allows us to reason about behavioural distances compositionally, so that we can e.g. conclude that the distance between I and $I \oplus \Omega$ is indeed $\frac{1}{2}$ (Example 14).

Reasoning about the number of times programs use (or test) their inputs requires a shift from ordinary languages to refined languages tracking information about the so-called *program sensitivity* [17, 41]. The sensitivity of a program is the 'law' describing how much behavioural differences in outputs are affected by behavioural differences in inputs, and thus provides the abstraction needed to handle distance amplification.

Our refined language is a generalisation of the language Fuzz [17, 41], which we call V-Fuzz. Fuzz is a PCF-like language refining standard λ-calculi by means of a powerful linear type system enriched with sensitivity-indexed 'bang types' that allow to track program sensitivity. Despite being parametric with respect to an arbitrary quantale, the main difference between V-Fuzz and Fuzz is that the former is an effectful calculus parametric with respect to a signature of (algebraic) operation symbols. This allows to consider imperative, nondeterministic, and probabilistic versions of Fuzz, as well as combinations thereof.

Structure of the work After having recalled some necessary mathematical preliminaries, we introduce V-Fuzz and its monadic

[1] For instance $\{(\lambda x.\underbrace{(xI)\ldots(xI)}_{n})(\lambda y.[-])\}_{n \geq 0}$.

operational semantics (Section 3). We then introduce (Section 4) the machinery of V-relators showing how it can be successfully instantiated on several examples. In Section 5 we define applicative Γ-similarity, a V-relation generalising effectful applicative similarity parametric with respect to aV-relator Γ, and prove it is a reflexive and transitive V-relation whose kernel induces an abstract notion of applicative similarity. Our main theorem states that under suitable conditions on the V-relator Γ, applicative Γ-similarity is compatible. Finally, in Section 7 we define the notion of applicative Γ-bisimilarity and prove that under mild conditions such notion is a compatible equivalence V-relation (viz. a compatible pseudometric).

2 Preliminaries

In this section we recall some basic definitions and results needed in the rest of the paper. Unfortunately, there is no hope to be comprehensive, and thus we assume the reader to be familiar with basic domain theory [2] (in particular we assume the notions of ω-complete (pointed) partial order, ω-cppo for short, monotone, and continuous functions), basic order theory [16], and basic category theory [35]. In particular, for a monoidal category $\langle \mathbb{C}, I, \otimes \rangle$ we assume the reader to be familiar with the notion of *strong Kleisli triple* [28, 35] $\mathbb{T} = \langle T, \eta, -^* \rangle$. We use the notation $f^* : Z \otimes TX \to TY$ for the strong Kleisli extension of $f : Z \otimes X \to TY$ (and use the same notation for the ordinary Kleisli lifting of $f : X \to TY$, the latter being essentially the subcase of $-^*$ for $Z = I$) and reserve the letter η to denote the unit of $\mathbb{T}$. Oftentimes, we refer to a (strong) Kleisli triples as a (strong) monad. We denote by $\mathbb{C}_\mathbb{T}$ the Kleisli category of $\mathbb{T}$. Finally, we recall that every monad on **Set**, the category of sets and functions, is strong (with respect to the cartesian structure).

We also try to follow the notation used in the just mentioned references. As a small difference, we denote by $g \cdot f$ the composition of g with f rather than by $g \circ f$.

2.1 Monads and Algebraic Effects

Following [40] we consider algebraic operations as sources of side effects. Syntactically, algebraic operations are given via a signature Σ consisting of a set of operation symbols (uninterpreted operations) together with their arity (i.e. their number of operands). Semantically, operation symbols are interpreted as algebraic operations on strong monads on **Set**. To any n-ary operation symbol $\mathbf{op} \in \Sigma$ and any set X we associate a map $op_X : (TX)^n \to TX$ (so that we equip TX with a Σ-algebra structure) such that f^* is a parametrised Σ-algebra (homo)morphis, for any $f : Z \times X \to TY$. Concretely, we require $op_Y(f^*(z, x_1), \dots, f^*(z, x_1)) = f^*(z, op_X(x_1, \dots, x_n))$ to hold for all $z \in Z, x_i \in TY$.

We also use monads to give operational semantics to V-Fuzz [14]. Intuitively, a program e evaluates to a *monadic value* $v \in T\mathcal{V}$, where $\mathcal{V}$ denotes the set of values. For instance, a nondeterministic program evaluates to a *set* of values, whereas a probabilistic program evaluates to a *(sub)distribution* of values. Due to the presence of non-terminating programs the evaluation of a term is defined as the limit of its "finite evaluations", and thus we need monads to carry a suitable domain structure. Recall that any category $\mathbb{C}$ is ω-cppo-enriched if the hom-set $\mathbb{C}(X, Y)$ carries an ω-cppo-structure, for all objects X, Y, and composition is continuous. A (strong) monad $\mathbb{T}$ is ω-cppo-enriched if $\mathbb{C}_\mathbb{T}$ is. In particular, in **Set** that means that we have an ω-cppo $\langle TX, \sqsubseteq_X, \bot_X \rangle$ for any set X. In particular, ω-cppo-enrichment of $\mathbb{T}$ gives the following equalities for $g, g_n : X \to TY$ and $f, f_n : Y \to TZ$ arrows in $\mathbb{C}$:

$$(\bigsqcup_{n<\omega} f_n)^* \cdot g = \bigsqcup_{n<\omega} f_n^* \cdot g,$$
$$f^* \cdot (\bigsqcup_{n<\omega} g_n) = \bigsqcup_{n<\omega} (f^* \cdot g_n).$$

Since V-Fuzz is a call-by-value language, we also require the equality $f^*(z, \bot_X) = \bot_Y$, for $f : Z \otimes X \to TY$.

Finally, we say that $\mathbb{T}$ is Σ-*continuous* if satisfies the above conditions and operations $op_X : (TX)^n \to TX$ are continuous, meaning that for all ω-chains $c_1, \dots, c_n$ in TX we have:

$$op_X(\bigsqcup c_1, \dots, \bigsqcup c_n) = \bigsqcup op_X(c_1, \dots, c_n).$$

The reader can consult[14, 40] for more details.

Example 1. The following are Σ-continuous monads:

1. The partiality monad $(-)_\bot$ mapping a set X to $X_\bot \triangleq X + \{\bot_X\}$. We give $X_\bot$ an ω-cppo structure via $\sqsubseteq_X$ defined by $x \sqsubseteq_X y$ if and only if $x = \bot_X$ or $x = y$. We equip the function space $X \to Y_\bot$ with the pointwise order induced by $\sqsubseteq$.
2. The powerset monad mapping a set to its powerset. The unit maps an element x to $\{x\}$, whereas $f^* : Z \times \mathcal{P}(X) \to \mathcal{P}(Y)$ is defined by $f^*(z, \mathcal{X}) \triangleq \bigcup_{x \in \mathcal{X}} f(z, x)$, for $f : Z \times X \to \mathcal{P}(Y)$, $\mathcal{X} \subseteq X$, and $z \in Z$. We give $\mathcal{P}(X)$ an ω-cppo structure via subset inclusion $\subseteq$ and order the function space $X \to \mathcal{P}(Y)$ with the pointwise order induced by $\subseteq$. Finally, we consider the signature $\Sigma = \{\oplus\}$ consisting of a single binary operation symbol for pure nondeterministic choice and interpret it as set-theoretic union.
3. The discrete *subdistribution* monad $\mathcal{D}_{\leq 1}$ mapping a set X to $\mathcal{D}(X_\bot)$, where $\mathcal{D}$ denotes the discrete *full* distribution monad. The unit of $\mathcal{D}$ maps an element x to the Dirac distribution $|x\rangle$ on it, whereas the strong Kleisli extension $f^* : Z \times \mathcal{D}X \to \mathcal{D}Y$ of $f : Z \times X \to \mathcal{D}Y$ is defined by $f^*(z, \mu)(y) \triangleq \sum_{x \in X} \mu(x) \cdot f(z, x)(y)$. On $\mathcal{D}(X_\bot)$, define the order $\sqsubseteq_X$ by $\mu \sqsubseteq_X \nu$ if and only if $\forall x \in X.\ \mu(x) \leq \nu(x)$ holds. The pair $(\mathcal{D}(X_\bot), \sqsubseteq_X)$ forms an ω-cppo, with bottom element given by the Dirac distribution on $\bot_X$ (the distribution modelling the always zero subdistribution). The ω-cppo structure lifts to function spaces pointwisely. Finally, consider the signature $\Sigma \triangleq \{\oplus_p \mid p \in \mathbb{Q},\ 0 < p < 1\}$ whose interpretation on the subdistribution monad is defined by $(\mu \oplus_p \nu)(x) \triangleq p \cdot \mu(x) + (1 - p) \cdot \nu(x)$. Restricting to $p \triangleq \frac{1}{2}$ we obtain fair probabilistic choice $\oplus$.
4. The partial global state monad $\mathcal{G}_\bot$ is obtained from the partiality monad and the global state monad; it maps a set X to $(S \times X)^X_\bot$. The global state monad $\mathcal{G}$ maps a set X to $(S \times X)^S$. Since ultimately a location stores a bit we take $S \triangleq \{0, 1\}^\mathcal{L}$, where $\mathcal{L}$ is a set of (public) location names. We can give an ω-cppo structure to $\mathcal{G}_\bot X$ by extending the order of point 1 pointwise. We consider the signature $\Sigma_\mathcal{L} \triangleq \{\mathbf{get}, \mathbf{set}_{\ell:=0}, \mathbf{set}_{\ell:=1} \mid \ell \in \mathcal{L}\}$ and interpret operations in $\Sigma_\mathcal{L}$ on $\mathcal{G}$ as follows:

$$\mathbf{set}_{\ell:=0}(f)(b) \triangleq f(b[\ell := 0]),$$
$$\mathbf{set}_{\ell:=1}(f)(b) \triangleq f(b[\ell := 1]),$$
$$\mathbf{get}(f, g)(b) \triangleq \begin{cases} f(b) & \text{if } b = 0, \\ g(b) & \text{if } b = 1, \end{cases}$$

where for $b \in S$, $b[\ell := x](\ell) \triangleq x$ and $b[\ell := x](\ell') \triangleq b(\ell')$, for $\ell' \neq \ell$.

2.2 Relations, Metrics, and Quantales

We now recall basic notions on quantales [43] and quantale-valued relations (V-relations) along the lines of [32]. The reader is referred to the monograph [25] for an introduction.

Definition 1. *A (unital) quantale* $(\mathsf{V}, \leq, \otimes, k)$, V *for short, consists of a monoid* $(\mathsf{V}, \otimes, k)$ *and a sup-lattice* $(\mathsf{V}, \leq)$ *satisfying the following distributivity laws:*

$$b \otimes \bigvee_{i \in I} a_i = \bigvee_{i \in I} (b \otimes a_i), \qquad (\bigvee_{i \in I} a_i) \otimes b = \bigvee_{i \in I} (a_i \otimes b).$$

The element k is called unit, whereas $\otimes$ is called multiplication of the quantale. Given quantales V, W, *a* quantale lax morphism *is a* monotone *map $h : \mathsf{V} \to \mathsf{W}$ satisfying the following inequalities:*

$$\ell \leq h(k), \qquad h(a) \otimes h(b) \leq h(a \otimes b),$$

where ℓ is the unit of W.

It is easy to see that $\otimes$ is monotone in both arguments. We denote top and bottom elements of a quantale by $\top$ and $\bot$, respectively. Moreover, we say that a quantale is commutative if its underlying monoid is, and it is non-trivial if $k \neq \bot$. Finally, we observe that for any $a \in \mathsf{V}$, the map $a \otimes (-) : \mathsf{V} \to \mathsf{V}$ has a right adjoint $a \multimap (-) : \mathsf{V} \to \mathsf{V}$ which is uniquely determined by:

$$a \otimes b \leq c \iff b \leq a \multimap c.$$

From now on we tacitly assume quantales to be commutative and non-trivial.

Example 2. The following are examples of quantales:

1. The *boolean quantale* $(2, \leq, \wedge, \mathsf{true})$ where $2 = \{\mathsf{true}, \mathsf{false}\}$ and $\mathsf{false} \leq \mathsf{true}$.
2. The extended real half-line $([0, \infty], \geq, +, 0)$ ordered by the "greater or equal" relation $\geq$ and extended[2] addition as monoid multiplication. We refer to such quantale as the *Lawvere quantale*. Note that in the Lawvere quantale the bottom element is ∞, the top element is 0, whereas infimum and supremum are defined as sup and inf, respectively. Notice also that $\multimap$ is truncated subtraction.
3. Replacing addition with maximum in the Lawevere quantale we obtain the *ultrametric Lawvere quantale* $([0, \infty], \geq, \max, 0)$, which has been used to study generalised ultrametric spaces [44] (note that in the ultrametric Lawvere quantale monoid multiplication and binary meet coincide).
4. Restricting the Lawvere quantale to the unit interval we obtain the *unit interval quantale* $([0, 1], \geq, +, 0)$, where $+$ stands for truncated addition.
5. A left continuous *triangular norm* (t-norm for short) is a binary operator $* : [0, 1] \times [0, 1] \to [0, 1]$ that induces a quantale structure over the complete lattice $([0, 1], \leq)$ in such a way that the quantale is commutative. Examples t-norms are:
   a. The *product t-norm*: $x *_p y \triangleq x \cdot y$.
   b. The *Łukasiewicz t-norm*: $x *_l y \triangleq \max\{x + y - 1, 0\}$.
   c. The *Gödel t-norm*: $x *_g y \triangleq \min\{x, y\}$.

In all quantales of Example 2 the unit k coincide the top element (i.e. $k = \top$). Quantales with such property are called *integral quantales*, and are particularly well-behaved. For instance, in an integral quantale $a \otimes b$ is a lower bound of a and b (and thus $a \otimes \bot = \bot$, for any $a \in \mathsf{V}$). From now on we tacitly assume quantales to be integral.

[2] We extend ordinary as follows: $x + \infty \triangleq \infty \triangleq \infty + x$.

V-relations The notion of V-relation, for a quantale V, provides an abstraction of the notion relation that subsumes both the qualitative—boolean valued—and the quantitative—real valued—notion of relation, as well as the associated notions of equivalence and (pseudo)metric. Moreover, sets and V-relations form a category which, thanks to the quantale structure of V, behaves essentially like **Rel**, the category of sets and relations. That allows to develop an algebra of V-relations on the same line of the usual algebra of relations.

Formally, for a quantale V, a V-relation $\alpha : X \nrightarrow Y$ between sets X and Y is a function $\alpha : X \times Y \to \mathsf{V}$. For any set X we can define the identity V-relation $id_X : X \nrightarrow X$ mapping diagonal elements (x, x) to k, and all other elements to $\bot$. Moreover, for V-relations $\alpha : X \nrightarrow Y$ and $\beta : Y \nrightarrow Z$, we can define the composition $\beta \cdot \alpha : X \nrightarrow Z$ by the so-called 'matrix multiplication formula':

$$(\beta \cdot \alpha)(x, z) \triangleq \bigvee_{y \in Y} \alpha(x, y) \otimes \beta(y, z).$$

Composition of V-relations is associative, and id is the unit of composition. As a consequence, we have that sets and V-relations form a category, called V-**Rel**. V-**Rel** is a monoidal category with unit given by the one-element set and tensor product given by cartesian product of sets with $\alpha \otimes \beta : X \times Y \nrightarrow X' \times Y'$ defined pointwise, for $\alpha : X \nrightarrow X'$ and $\beta : Y \to Y'$. Moreover, for all sets X, Y, the hom-set V-**Rel**(X, Y) inherits a complete lattice structure from V according to the pointwise order. Actually, the whole quantale structure of V is inherited, in the sense that V-**Rel** is a *quantaloid* [25]. In particular, for all V-relations $\alpha : X \nrightarrow Y$, $\beta_i : Y \nrightarrow Z$ $(i \in I)$, and $\gamma : Z \nrightarrow W$ we have the following distributivity laws:

$$\gamma \cdot (\bigvee_{i \in I} \beta_i) = \bigvee_{i \in I} (\gamma \cdot \beta_i), \qquad (\bigvee_{i \in I} \beta_i) \cdot \alpha = \bigvee_{i \in I} (\beta_i \cdot \alpha).$$

There is a bijection $-^\circ : \mathsf{V}\text{-}\mathbf{Rel}(X, Y) \to \mathsf{V}\text{-}\mathbf{Rel}(Y, X)$ that maps each V-relation α to its dual α° defined by $\alpha^\circ(y, x) \triangleq \alpha(x, y)$. It is straightforward to see that $-^\circ$ is monotone (i.e. $\alpha \leq \beta$ implies $\alpha^\circ \leq \beta^\circ$), idempotent (i.e. $(\alpha^\circ)^\circ = \alpha$), and preserves the identity relation (i.e. $id^\circ = id$). Moreover, since V is commutative we also have the equality $(\beta \cdot \alpha)^\circ = \alpha^\circ \cdot \beta^\circ$.

Finally, we define the graph functor $\mathcal{G}$ from **Set** to V-**Rel** acting as the identity on sets and mapping each function f to its graph (so that $\mathcal{G}(f)(x, y)$ is equal to k if $y = f(x)$, and $\bot$ otherwise). It is easy to see that since V is non-trivial $\mathcal{G}$ is faithful. In light of this observation we will use the notation $f : X \to Y$ in place of $\mathcal{G}(f) : X \nrightarrow Y$ in V-**Rel**.

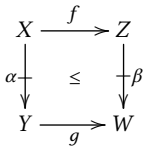
A direct application of the definition of composition gives the equality:

$$(g^\circ \cdot \alpha \cdot f)(x, w) = \alpha(f(x), g(w))$$

for $f : X \to Y$, $\alpha : Y \nrightarrow Z$, and $g : W \to Z$. Moreover, it is useful to keep in mind the following adjunction rules [25] (for α, β, γ V-relations, and f, g functions with appropriate source and target):

$$g \cdot \alpha \leq \beta \iff \alpha \leq g^\circ \cdot \beta,$$
$$\beta \cdot f^\circ \leq \gamma \iff \beta \leq \gamma \cdot f.$$

The above inequalities turned out to be useful in making point-free calculations with V-relations. In particular, we can use *lax* commutative diagrams of the form



as diagrammatic representation for the inequation $g \cdot \alpha \leq \beta \cdot f$. By adjunction rules, the latter is equivalent to $\alpha \leq g^\circ \cdot \beta \cdot f$, which pointwisely gives the following generalised non-expansiveness condition[3]: $\forall (x, y) \in X \times Y.\ \alpha(x, y) \leq \beta(f(x), g(y))$.

Among V-relations we are interested in those generalising equivalences and pseudometrics.

Definition 2. *A* V*-relation* $\alpha : X \nrightarrow X$ *is* reflexive *if* $id_X \leq \alpha$, transitive *if* $\alpha \cdot \alpha \leq \alpha$, *and* symmetric *if* $\alpha \leq \alpha^\circ$.

Pointwisely, reflexivity, transitivity, and symmetry give the following inequalities:

$$k \leq \alpha(x, x), \quad \alpha(x, y) \otimes \alpha(y, z) \leq \alpha(x, z), \quad \alpha(x, y) \leq \alpha(y, x),$$

for all $x, y, z \in X$. We call a reflexive and transitive V-relation a V*-preorder* or *generalised metric* [7, 32], and a reflexive, symmetric, and transitive V-relation a V*-equivalence* or *pseudometric*.

Example 3. 1. We see that 2-**Rel** is the ordinary category **Rel** of sets and relations. Moreover, instantiating reflexivity and transitivity on the boolean quantale, we recover the usual notion of preorder. If we additionally require symmetry, then we obtain the usual notion of equivalence relation.
2. On the Lawvere quantale transitivity gives:
$$\inf_y \alpha(x, y) + \alpha(y, z) \geq \alpha(x, z),$$
which means $\alpha(x, z) \leq \alpha(x, y) + \alpha(y, z)$, for any $y \in X$. That is, in the Lawvere quantale transitivity gives exactly the triangle inequality. Similarly, reflexivity gives $0 \geq \alpha(x, x)$, i.e. $\alpha(x, x) = 0$. If additionally α is symmetric, then we recover the usual notion of pseudometric [47].
3. Analogously to point 2, if we consider the ultrametric Lawvere quantale, we recover the ultrametric variants of the above notions.

Digression 1 (V-categories)**.** Lawvere introduced generalised metric spaces in his seminal paper [32] as pairs (X, α) consisting of a set X and a generalised metric $\alpha : X \nrightarrow X$ over the Lawvere quantale. Generalising from the Lawvere quantale to an arbitrary quantale V we obtain the so-called V-categories [25]. In fact, a V-category (X, α) is nothing but a category enriched over V regarded as a bicomplete monoidal category. The notion of V-enriched functor precisely instantiates as non-expansive map between V-categories, so that one can consider the category V-Cat of V-categories and V-functors. The category V-Cat has a rich structure. In particular, it is monoidal closed category. Given V-categories $(X, \alpha), (Y, \beta)$, their exponential $(Y^X, [\alpha, \beta])$ is defined by

$$[\alpha, \beta](f, g) \triangleq \bigwedge_{x \in X} \beta(f(x), g(x))$$

(cf. with the usual, real-valued, sup-metric on function spaces), whereas their tensor product $(X \times Y, \alpha \otimes \beta)$ is defined pointwise.

[3]Taking $f = g$ generalised non-expansiveness expresses monotonicity of f in the boolean quantale, and non-expansiveness of f in the Lawvere quantale and its variants (recall that when we instantiate V as e.g. the Lawvere quantale we have to *reverse* inequalities).

Although in this work we will not work with V-categories (we will essentially work in V-**Rel**), it is sometimes useful to think in terms of V-categories for 'semantical intuitions'.

Operations For a signature Σ, we need to specify how operations in Σ interact with V-relations (e.g. how they modify distances), and thus how they interact with quantales.

Definition 3. *Let* Σ *be a signature. A* Σ*-quantale is a quantale* V *equipped with* monotone *operations* $op_\mathsf{V} : \mathsf{V}^n \to \mathsf{V}$*, for each n-ary operation* $\mathbf{op} \in \Sigma$*, satisfying the following inequalities:*

$$k \leq op_\mathsf{V}(k, \ldots, k),$$
$$op_\mathsf{V}(a_1, \ldots, a_n) \otimes op_\mathsf{V}(b_1, \ldots, b_n) \leq op_\mathsf{V}(a_1 \otimes b_1, \ldots, a_n \otimes b_n).$$

Example 4. Both in the Lawvere quantale and in the unit interval quantale we can interpret operations $\oplus_p$ from Example 1 as probabilistic choices: $x \oplus_p y \triangleq p \cdot x + (1 - p) \cdot y$. In general, for a quantale V we can interpret $op_\mathsf{V}(a_1, \ldots, a_n)$ both as $a_1 \otimes \ldots \otimes a_n$ and $a_1 \wedge \ldots \wedge a_n$.

Change of Base Functors We model sensitivity of a program as a function giving the 'law' describing how distances between inputs are modified by the program. The notion of *change of base functor* provides a mathematical abstraction to model the concept of sensitivity with respect to an arbitrary quantale.

Definition 4. *A change of base functor [25], CBF for short, between quantales* V, W *is a lax quantale morphism* $h : \mathsf{V} \to \mathsf{W}$ *(see Definition 1). If* $\mathsf{V} = \mathsf{W}$ *we speak of change of base* endofunctors *(CBEs, for short), and denote them by* $s, r \ldots$*. Clearly, every CBE* s *is also a CBF.*

The action $h \circ \alpha$ of a CBF $h : \mathsf{V} \to \mathsf{W}$ on a V-relation $\alpha : X \nrightarrow Y$ is defined by $h \circ \alpha(x, y) \triangleq h(\alpha(x, y))$ (to improve readability we omit brackets). Note that since V is integral, CBFs preserve the unit.

Example 5. 1. Extended[4] real-valued multiplication $c \cdot -$, for $c \in [0, \infty]$, is a CBE on the Lawvere quantale. Functions $c \cdot -$ act as CBEs also on the unit interval quantale (where multiplication is meant to be truncated).
2. Both in the Lawvere quantale and in the unit interval quantale, polynomials P such that $P(0) = 0$ are CBEs.
3. Define CBEs $n, \infty : \mathsf{V} \to \mathsf{V}$, for $n < \omega$ by $0(a) \triangleq k$, $(n + 1)(a) \triangleq a \otimes n(a)$, and $\infty(a) \triangleq \mathbb{1}$. Note that 1 acts as the identity function.

Finally, we observe that the action of CBFs on a V-relation obeys the following laws:

$$(h \cdot h')(\alpha) = h \circ (h' \circ \alpha),$$
$$(h \circ \alpha) \cdot (h \circ \beta) \leq h \circ (\alpha \cdot \beta).$$

Digression 2. We saw that V-categories generalise the notions of metric space and ordered set, and that the notion of V-functor generalises the notions of monotone and non-expansive function. However, when dealing with metric spaces besides non-expansive functions, a prominent role is played by *Lipshitz continuous* functions. Given metric spaces (X, d_X) and (Y, d_Y), a function $f : X \to Y$ is called *c-continuous*, for $c \in \mathbb{R}_{\geq 0}$ if the inequation $c \cdot d_X(x, x') \geq d_Y(f(x), f(x'))$ holds, for all $x, x' \in X$. Example 5 shows that multiplication $c \cdot -$ by a real number c is a change of base endofunctor on the Lawvere quantale, meaning that using CBEs we can generalise the notion of Lipshitz-continuity to V-categories. In fact, easy calculations show that for any V-category (X, α) and any CBE s on V,

[4]We extend real-valued multiplication by: $0 \cdot \infty \triangleq 0 \triangleq \infty \cdot 0$, $\infty \cdot x \triangleq \infty \triangleq x \cdot \infty$.

$(X, s \circ \alpha)$ is a V-category. In particular, we can define s-continuous functions from (X, α) to (Y, β) as V-functors from $(X, s \circ \alpha)$ to (Y, β). That is, we say that a function $f : X \to Y$ is s-continuous if $s \circ \alpha(x, x') \leq \beta(f(x), f(x'))$ holds, for all $x, x' \in X$.

We conclude this section with the following result on the algebra of CBEs.

Lemma 1. *Let* V *be a* Σ*-quantale. CBEs are closed under the following operations (where* $\mathbf{op} \in \Sigma$*):*

$$(s \otimes r)(a) \triangleq s(a) \otimes r(a),$$
$$(r \cdot s)(a) \triangleq r(s(a)),$$
$$(s \wedge r)(a) = s(a) \wedge s(b),$$
$$op_{\mathsf{V}}(s_1, \ldots, s_n)(a) \triangleq op_{\mathsf{V}}(s_1(a), \ldots, s_n(a)).$$

3 The V-fuzz Language

As already observed in the introduction, when dealing with behavioural V-relations a crucial parameter in amplification phenomena is program sensitivity. To deal with such parameter we introduce V-fuzz, a higher-order effectful language generalising Fuzz [17]. As Fuzz, V-Fuzz is characterised by a powerful type system inspired by *bounded linear logic* [21] giving syntactic information on program sensitivity.

Syntax V-fuzz is a *fine-grained call-by-value* [34] *linear* λ-calculus with finite sum and recursive types. In particular, we make a formal distinction between values and computations (which we simply refer to as terms), and use syntactic primitives to returning values (**val**) and sequentially compose computations (via a **let**-**in** constructor). The syntax of V-Fuzz is parametrised over a signature Σ of operation symbols, a Σ-quantale V, and a family Π of CBEs. From now on we assume Σ, V, and Π to be fixed. Moreover, we assume Π to contain at least CBEs n, ∞ in Example 5 and to be closed under operations in Lemma 1. Types, values, and terms of V-Fuzz are defined in Figure 1, where t denotes a type variable, I is a *finite* set (whose elements are denoted by $\hat{\imath}, \hat{\jmath}, \ldots$), and s is in Π.

$$\begin{aligned}
\sigma &::= t \mid \sum_{i \in I} \sigma_i \mid \sigma \multimap \sigma \mid \mu t.\sigma \mid !_s\sigma. \\
v &::= x \mid \lambda x.e \mid \langle \hat{\imath}, v \rangle \mid \mathbf{fold}\ v \mid !v. \\
e &::= \mathbf{val}\ v \mid vv \mid \mathbf{case}\ v\ \mathbf{of}\ \{\langle i, x\rangle \to e_i\} \mid \mathbf{let}\ x = e\ \mathbf{in}\ e \\
&\quad \mid \mathbf{case}\ v\ \mathbf{of}\ \{!x \to e\} \mid \mathbf{case}\ v\ \mathbf{of}\ \{\mathbf{fold}\ x \to e\} \mid \mathbf{op}(e, \ldots, e).
\end{aligned}$$

Figure 1. Types, values, and terms of V-Fuzz.

Free and bound variables in terms and values are defined as usual. We work with equivalence classes of terms modulo renaming and tacitly assume conventions on bindings. Moreover, we denote by $w[v/x]$ and $e[x := v]$ the *value* and *term* obtained by capture-avoiding substitution of the *value* v for x in w and e, respectively (see [14] for details).

Similar conventions hold for types. In particular, we denote by $\sigma[\tau/t]$ the result of capture-avoiding substitution of type τ for the type variable t in σ. Finally, we write $\mathbf{0}$ for the empty sum type, $\mathbf{1}$ for $\mathbf{0} \multimap \mathbf{0}$, and nat for $\mu t.\mathbf{1} + t$. We denote the numeral n by $\underline{n}$.

V-Fuzz type system is essentially based on judgments of the form $x_1 :_{s_1} \sigma_1, \ldots, x_n :_{s_n} \sigma_n \vdash e : \sigma$, where $s_1, \ldots, s_n$ are CBEs.

$$\frac{s \leq 1}{\Gamma, x :_s \sigma \vdash^{\mathsf{v}} x : \sigma} \qquad \frac{\Gamma_1 \vdash e_1 : \sigma \quad \cdots \quad \Gamma_n \vdash e_n : \sigma}{op_{\mathsf{V}}(\Gamma_1, \ldots, \Gamma_n) \vdash \mathbf{op}(e_1, \ldots, e_n) : \sigma}$$

$$\frac{\Gamma, x :_1 \sigma \vdash e : \tau}{\Gamma \vdash^{\mathsf{v}} \lambda x.e : \sigma \multimap \tau} \qquad \frac{\Gamma \vdash^{\mathsf{v}} v : \sigma \multimap \tau \quad \Delta \vdash^{\mathsf{v}} w : \sigma}{\Gamma \otimes \Delta \vdash vw : \tau}$$

$$\frac{\Gamma \vdash^{\mathsf{v}} v : \sigma_{\hat{\imath}}}{\Gamma \vdash^{\mathsf{v}} \langle \hat{\imath}, v \rangle : \sum_{i \in I} \sigma_i} \qquad \frac{\Gamma \vdash^{\mathsf{v}} v : \sum_{i \in I} \sigma_i \quad \Delta, x :_s \sigma_i \vdash e_i : \tau \quad (\forall i \in I)}{s \cdot \Gamma \otimes \Delta \vdash \mathbf{case}\ v\ \mathbf{of}\ \{\langle i, x\rangle \to e_i\} : \tau}$$

$$\frac{\Gamma \vdash^{\mathsf{v}} v : \sigma}{\Gamma \vdash \mathbf{val}\ v : \sigma} \qquad \frac{\Gamma \vdash e : \sigma \quad \Delta, x :_s \sigma \vdash f : \tau}{(s \wedge 1) \cdot \Gamma \otimes \Delta \vdash \mathbf{let}\ x = e\ \mathbf{in}\ f : \tau}$$

$$\frac{\Gamma \vdash^{\mathsf{v}} v : \sigma}{s \cdot \Gamma \vdash^{\mathsf{v}} !v : !_s\sigma} \qquad \frac{\Gamma \vdash^{\mathsf{v}} v : !_r\sigma \quad \Delta, x :_{s \cdot r} \sigma \vdash e : \tau}{s \cdot \Gamma \otimes \Delta \vdash \mathbf{case}\ v\ \mathbf{of}\ \{!x \to e\} : \tau}$$

$$\frac{\Gamma \vdash^{\mathsf{v}} v : \sigma[\mu t.\sigma/t]}{\Gamma \vdash^{\mathsf{v}} \mathbf{fold}\ v : \mu t.\sigma} \qquad \frac{\Gamma \vdash^{\mathsf{v}} v : \mu t.\sigma \quad \Delta, x :_s \sigma[\mu t.\sigma/t] \vdash e : \tau}{s \cdot \Gamma \otimes \Delta \vdash \mathbf{case}\ v\ \mathbf{of}\ \{\mathbf{fold}\ x \to e\} : \tau}$$

Figure 2. Typing rules.

The informal meaning of such judgment is that on input x_i ($i \leq n$), the term e has sensitivity s_i. That is, e amplifies the (behavioural) distance between two input values v_i, w_i of *at most* a factor s_i; symbolically, $s_i \circ \alpha(v_i, w_i) \leq \alpha(e[x_i := v_i], e[x_i := w_i])$

An *environment* Γ is a sequence $x_1 :_{s_1} \sigma_1, \ldots, x_n :_{s_n} \sigma_n$ of distinct identifiers with associated closed types and CBEs (we denote the empty environment by $\emptyset$). We can lift operations on CBEs in Lemma 1 to environments as follows:

$$r \cdot \Gamma = x_1 :_{r \cdot s_1} \sigma_1, \ldots, x_n :_{r \cdot s_n} \sigma_n,$$
$$\Gamma \otimes \Delta = x_1 :_{s_1 \otimes r_1} \sigma_1, \ldots, x_n :_{s_n \otimes r_n} \sigma_n,$$
$$op_{\mathsf{V}}(\Gamma^1, \ldots, \Gamma^m) = x_1 :_{op_{\mathsf{V}}(s_1^1, \ldots, s_1^m)} \sigma_1, \ldots, x_n :_{op_{\mathsf{V}}(s_n^1, \ldots, s_n^m)} \sigma_n,$$

for $\Gamma = x_1 :_{s_1} \sigma_1, \ldots, x_n :_{s_n} \sigma_n$, $\Delta = x_1 :_{r_1} \sigma_1, \ldots, x_n :_{r_n} \sigma_n$, and $\Gamma^i = x_1 :_{s_1^i} \sigma_1, \ldots, x_n :_{s_n^i} \sigma_n$. Note that the above operations are defined for environments having the same structure (i.e. differing only on CBEs). This is not a real restriction since we can always add the missing identifiers $y :_k \sigma$, where k is the constant function returning the unit of the quantale (but see [41]).

The type system for V-Fuzz is defined in Figure 2. The system is based on two kinds of judgment (exploiting the fine-grained style of the calculus): judgments of the form $\Gamma \vdash^{\mathsf{v}} v : \sigma$ for values and judgments of the form $\Gamma \vdash e : \sigma$ for terms. We denote by $\mathcal{V}_\sigma$ and Λ_σ for the set of closed values and terms of type σ, respectively. Sometimes we also use the notation $\Lambda_{\Gamma \vdash \sigma}$ for the set $\{e \in \Lambda \mid \Gamma \vdash e : \sigma\}$ (and similarity for values).

Example 6. 1. Instantiating V-Fuzz with $\Sigma \triangleq \emptyset$, the Lawvere quantale, and CBEs $\Pi = \{c \cdot - \mid c \in [0, \infty]\}$ we obtain the original Fuzz [41] (provided we add a basic type for real numbers). We can also add nondeterminism via a binary nondeterminism choice operation $\oplus$.

2. We define the language P-Fuzz as the instantiation of V-Fuzz with a fair probabilistic choice operation $\oplus$, the unit interval quantale $([0, 1], \geq, +, 0)$, and CBEs $\Pi = \{c \cdot - \mid c \in [0, \infty]\}$ (as usual we are actually referring to truncated multiplication). We interpret $\oplus$ in $[0, 1]$ as in Example 4.
3. We can add global states to P-Fuzz enriching P-Fuzz's signature with operations in $\Sigma_{\mathcal{L}}$ from Example 1.

Typing rules for V-Fuzz are similar to those of Fuzz (e.g. in the variable rule we require $s \leq 1$, meaning that the open value x can access x at least once) with the exception of the rule for sequencing where we apply sensitivity $s \wedge 1$ to the environment Γ even if the sensitivity of x in f is s. Consider the following instance of the sequencing rule on the Lawvere quantale:

$$\frac{x :_1 \sigma \vdash e : \sigma \quad y :_0 \sigma \vdash f : \tau}{x :_{\max(0,1)\cdot 1} \sigma \vdash \mathbf{let}\ y = e\ \mathbf{in}\ f : \tau}$$

where f is a closed term of type τ and thus we can assume it to have sensitivity 0 on all variables. According to our informal intuition, e has sensitivity 1 on input x, meaning that (i) e can possibly detect (behavioural) differences between input values v, w, and (ii) e cannot amplify their behavioural distance of a factor bigger than 1. Formally, point (ii) states that we have the inequality $\alpha(v, w) \geq \alpha(e[x := v], e[x := w])$, where α denotes a suitable behavioural $[0, 1]$-relation. On the contrary, f is closed term and thus has sensitivity 0 on any input, meaning that it cannot detect any observable difference between input values. In particular, for all values v, w we have $\alpha(f[y := v], f[y := w]) = \alpha(f, f) = 0$ (provided that α is reflexive). Replacing $\max(0, 1)$ with 0 in the above rule (i.e. $s \wedge 1$ with s in the general case) would allow to infer the judgment $x :_0 \sigma \vdash \mathbf{let}\ y = e\ \mathbf{in}\ f : \tau$, and thus to conclude $\alpha(\mathbf{let}\ y = e[x := v]\ \mathbf{in}\ f, \mathbf{let}\ y = e[x := w]\ \mathbf{in}\ f) = 0$. The latter equality is unsound as evaluating $\mathbf{let}\ y = e[x := v]\ \mathbf{in}\ f$ (resp. $\mathbf{let}\ y = e[x := w]\ \mathbf{in}\ f$) requires to *first* evaluate $e[x := v]$ (resp. $e[x := w]$) thus making observable differences between v and w detectable (see also Section 5 for a formal explanation).

Example 7. For every type σ we have the term $I \triangleq \mathbf{val}\ (\lambda x.\mathbf{val}\ x)$ of type $\sigma \multimap \sigma$ as well as the purely divergent divergent term $\Omega \triangleq \omega!(\mathbf{fold}\ \omega)$ of type σ, where $\omega \in \Lambda_{!_\infty(\mu t.!_\infty t \multimap \sigma) \multimap \sigma}$ is defined by: $\omega \triangleq \lambda x.\mathbf{case}\ x\ \mathbf{of}\ \{!y \to \mathbf{case}\ y\ \mathbf{of}\ \{\mathbf{fold}\ z \to z!(\mathbf{fold}\ z)\}\}$.

Before moving to the operational semantics of V-Fuzz, we remark that the syntactic distinction between terms and values gives the following equalities.

Lemma 2. *The following equalities hold:*

$$\mathcal{V}_{\sigma \multimap \tau} = \{\lambda x.e \mid x :_1 \sigma \vdash e : \tau\},$$
$$\mathcal{V}_{\sum_{i \in I} \sigma_i} = \bigcup_{i \in I} \{\langle i, v\rangle \mid v \in \mathcal{V}_{\sigma_i}\},$$
$$\mathcal{V}_{!_s \sigma} = \{!v \mid v \in \mathcal{V}_\sigma\}.$$

Operational Semantics We give V-Fuzz monadic operational (notably evaluation) semantics in the style of [14]. Let $\mathbb{T} = \langle T, \eta, -^*\rangle$ be a Σ-continuous monad. Operational semantics is defined by means of an evaluation function $|-|^\sigma$ indexed over closed types, associating to any term in Λ_σ a monadic value in $T\mathcal{V}_\sigma$. The evaluation function $|-|^\sigma$ is itself defined by means of the family of functions $\{|-|^\sigma_n\}_{n<\omega}$ defined in Figure 3. Indeed $|-|^\sigma_n$ is a function from $\mathcal{V}_\sigma$ to $T\mathcal{V}_\sigma$.

Let us expand on the definition of $|\mathbf{let}\ x = e\ \mathbf{in}\ f|^\sigma_{n+1}$. Since $\mathbf{let}\ x = e\ \mathbf{in}\ f \in \Lambda_\sigma$, there must be derivable judgments $\emptyset \vdash e : \tau$ and $x :_s \tau \vdash f : \sigma$. As a consequence, for any $v \in \mathcal{V}_\tau$, we have $|f[x := v]|^\sigma_n \in T\mathcal{V}_\sigma$. This induces a function $|f[x := -]|^{\tau,\sigma}_n$ from $\mathcal{V}_\tau$ to $T\mathcal{V}_\sigma$ whose Kleisli extension can be applied to $|e|^\tau_n \in T\mathcal{V}_\tau$.

Finally, it is easy to see that $(|e|_n)_{n<\omega}$ forms an ω-chain in $T\mathcal{V}_\sigma$ (see Appendix A.1 for a proof of the following result).

Lemma 3. *For any $e \in \Lambda_\sigma$, we have $|e|^\sigma_n \sqsubseteq_{\mathcal{V}_\sigma} |e|^\sigma_{n+1}$, for any $n \geq 0$.*

$$\begin{aligned}
|e|^\sigma_0 &\triangleq \bot_{\mathcal{V}_\sigma}\\
|\mathbf{val}\ v|^\sigma_{n+1} &\triangleq \eta_{\mathcal{V}_\sigma}(v)\\
|(\lambda x.e)v|^\sigma_{n+1} &\triangleq |e[x := v]|^\sigma_n\\
|\mathbf{case}\ \langle i, v\rangle\ \mathbf{of}\ \{\langle i, x\rangle \to e_i\}|^\sigma_{n+1} &\triangleq |e_i[x := v]|^\sigma_n\\
|\mathbf{case}\ (\mathbf{fold}\ v)\ \mathbf{of}\ \{\mathbf{fold}\ x \to e\}|^\sigma_{n+1} &\triangleq |e[x := v]|^\sigma_n\\
|\mathbf{case}\ !v\ \mathbf{of}\ \{!x \to e\}|^\sigma_{n+1} &\triangleq |e[x := v]|^\sigma_n\\
|\mathbf{let}\ x = e\ \mathbf{in}\ f|^\sigma_{n+1} &\triangleq (|f[x := -]|^{\tau,\sigma}_n)^* |e|^\tau_n\\
|\mathbf{op}(e_1, \ldots, e_k)|^\sigma_{n+1} &\triangleq op_{\mathcal{V}_\sigma}(|e_1|^\sigma_n, \ldots, |e_k|^\sigma_n)
\end{aligned}$$

Figure 3. Approximation evaluation semantics.

As a consequence, we can define $|-|^\sigma : \Lambda_\sigma \to T\mathcal{V}_\sigma$ by

$$|e|^\sigma \triangleq \bigsqcup_{n<\omega} |e|^\sigma_n.$$

In order to improve readability we oftentimes omit type superscripts in $|e|^\sigma$. We also notice that because op is continuous and $\mathbb{T}$ is ω-cppo-enriched, $|-|^\sigma$ is itself continuous.

Proposition 1. *The following equations hold:*

$$\begin{aligned}
|\mathbf{val}\ v| &= \eta(v),\\
|(\lambda x.e)v| &= |e[x := v]|,\\
|\mathbf{case}\ \langle i, v\rangle\ \mathbf{of}\ \{\langle i, x\rangle \to e_i\}| &= |e_i[x := v]|,\\
|\mathbf{case}\ (\mathbf{fold}\ v)\ \mathbf{of}\ \{\mathbf{fold}\ x \to e\}| &= |e[x := v]|,\\
|\mathbf{case}\ !v\ \mathbf{of}\ \{!x \to e\}| &= |e[x := v]|,\\
|\mathbf{let}\ x = e\ \mathbf{in}\ f| &= |f[x := -]|^*(|e|),\\
|\mathbf{op}(e_1, \ldots, e_k)| &= op_{\mathcal{V}_\sigma}(|e_1|, \ldots, |e_k|).
\end{aligned}$$

4 V-relators and V-relation Lifting

In [14] the abstract theory of relators [6, 48] has been used to define notions of applicative (bi)similarity for an untyped λ-calculus enriched with algebraic operations. Intuitively, a relator Γ for a set endofunctor T is an abstraction meant to capture the possible ways a relation on a set X can be turned (or lifted) into a relation on TX. Relators allow to abstractly express the idea that bisimilar programs, when executed, exhibit the same observable behaviour (i.e. they produce the same effects) and evaluate to bisimilar values. In particular, whenever two programs e and e' are related by a (bi)simulation $\mathcal{R}$, then the results $|e|$ and $|e'|$ of their evaluation must be related by $\Gamma\mathcal{R}$. The latter relation ranging over monadic values, it takes into account the visible effects of executing e and e', such effects being encapsulated via T.

The notion of V-relator [25] is somehow the 'quantitative' generalisation of the concept of a relator. Analogously to ordinary relators, V-relators for a set endofunctor T are abstractions meant to capture the possible ways a V-relation on a set X can be (nicely) turned into a V-relation on TX, and thus provide ways to lift a behavioural distance between programs to a (behavioural) distance between monadic values. On a formal level, we say that a V-relator extends T from **Set** to V-**Rel**, laxly[5].

[5] Relators are also known as *lax extensions* [23, 25].

Definition 5. *For a set endofucunctor T a* V-relator *for T is a mapping $(\alpha : X \nrightarrow Y) \mapsto (\Gamma\alpha : TX \nrightarrow TY)$ satisfying conditions* (V-rel 1)-(V-rel 4). *We say that Γ is conversive if it additionally satisfies condition* (V-rel 5).

$$1_{TX} \leq \Gamma(1_X), \tag{V-rel 1}$$
$$\Gamma\beta \cdot \Gamma\alpha \leq \Gamma(\beta \cdot \alpha), \tag{V-rel 2}$$
$$Tf \leq \Gamma f, \ \ (Tf)^\circ \leq \Gamma f^\circ, \tag{V-rel 3}$$
$$\alpha \leq \beta \implies \Gamma\alpha \leq \Gamma\beta, \tag{V-rel 4}$$
$$\Gamma(\alpha^\circ) = (\Gamma\alpha)^\circ. \tag{V-rel 5}$$

Conditions (V-rel 1), (V-rel 2), and (V-rel 4) are rather standard. Condition (V-rel 3), which actually consists of two conditions, states that V-relators behave in the expected way on functions. It is immediate to see that when instantiated with $\mathsf{V} = 2$, the above definition gives the usual notion of relator, with some minor differences. In [14] and [33] a kernel preservation condition is required in place of (V-rel 3). Such condition is also known as stability in [27]. Stability requires the equality

$$\Gamma(g^\circ \cdot \alpha \cdot f) = (Tg)^\circ \cdot \Gamma\alpha \cdot Tf$$

to hold. It is easy to see that a V-relator always satisfies stability. Notice also that stability gives the following implication:

$$\alpha \leq g^\circ \cdot \beta \cdot f \implies \Gamma\alpha \leq (Tg)^\circ \cdot \Gamma\beta \cdot Tf,$$

which can be diagrammatically expressed as:

$$\begin{array}{ccc} X & \xrightarrow{f} & Z \\ {\scriptstyle\alpha}\downarrow & \leq & \downarrow{\scriptstyle\beta} \\ Y & \xrightarrow[g]{} & W \end{array} \implies \begin{array}{ccc} TX & \xrightarrow{Tf} & TZ \\ {\scriptstyle\Gamma\alpha}\downarrow & \leq & \downarrow{\scriptstyle\Gamma\beta} \\ TY & \xrightarrow[Tg]{} & TW \end{array}.$$

Finally, we observe that any V-relator Γ for T induces an endomap T_Γ on V-**Rel** that acts as T on sets and as Γ as V-relation. It is easy to check that conditions in Definition 5 makes T_Γ a *lax endofunctor*.

Before giving examples of V-relators it is useful to observe that the collection V-relators is closed under specific operations.

Proposition 2. *Let T, U be set endofunctors. Then:*

1. *If Γ and Δ are V-relators for T and U, respectively, then $\Delta \cdot \Gamma$ defined by $(\Delta \cdot \Gamma)\alpha \triangleq \Delta\Gamma\alpha$ is a V-relator for UT.*
2. *If $\{\Gamma\}_{i\in I}$ is a family of V-relators for T, then $\bigwedge_{i\in I}\Gamma_i$ defined by $(\bigwedge_{i\in I}\Gamma_i)\alpha \triangleq \bigwedge_{i\in I}\Gamma_i\alpha$ is a V-relator for T.*
3. *If Γ is a V-relator for T, then Γ° defined by $\Gamma^\circ\alpha \triangleq (\Gamma\alpha^\circ)^\circ$ is a V-relator for T.*
4. *For any V-relator Γ, $\Gamma \wedge \Gamma^\circ$ is the greatest conversive V-relator smaller than Γ.*

Proof. See Appendix A.2. □

Example 8. Let us consider the monads in Example 1 regarded as functors.

1. For the partiality functor $(-)_\perp$ define the V-relator $(-)_\perp$ by:

   $$\alpha_\perp(x,y) \triangleq \alpha(x,y), \quad \alpha_\perp(\perp_X, y) \triangleq k, \quad \alpha_\perp(x, \perp_Y) = \perp\!\!\!\perp,$$

   where $x \in X, y \in Y, y \in Y_\perp$, and $\alpha : X \nrightarrow Y$. The V-relation $\alpha_\perp$ generalises the usual notion of *simulation* for partial computations. Similarly, $\alpha_{\perp\perp} \triangleq \alpha_\perp \wedge ((\alpha^\circ)_\perp)^\circ$ generalises the usual notion of *bisimulation* for partial computation.
2. For the powerset functor $\mathcal{P}$ define the V-relator H (called Hausdorff lifting) and its conversive counterpart $H^s \triangleq H \wedge H^\circ$ by $H\alpha(\mathcal{X}, \mathcal{Y}) \triangleq \bigwedge_{x\in\mathcal{X}} \bigvee_{y\in\mathcal{Y}} \alpha(x,y)$. If we instantiate V as the Lawvere quantale, then H^s gives the usual Hausdorff lifting of distances on a set X to distances on $\mathcal{P}X$, whereas for $\mathsf{V} = 2$ we recover the usual notion of *(bi)simulation* for unlabelled transition systems.
3. For the full distribution functor $\mathcal{D}$ we define a $[0,1]$-relator (with respect to the unit interval quantale) using the so-called *Wasserstein-Kantorovich lifting* [49]. For $\mu \in \mathcal{D}(X), \nu \in \mathcal{D}(Y)$, the set $\Omega(\mu,\nu)$ of *couplings* of μ and ν is the set of joint distributions $\omega \in \mathcal{D}(X \times Y)$ such that $\mu = \sum_{y\in Y} \omega(-, y)$ and $\nu = \sum_{x\in X} \omega(x, -)$. For a $[0,1]$-relation $\alpha : X \nrightarrow Y$ define:

   $$W\alpha(\mu,\nu) \triangleq \inf_{\omega\in\Omega(\mu,\nu)} \sum_{x,y} \alpha(x,y) \cdot \omega(x,y).$$

   $W\alpha(\mu,\nu)$ attains its infimum and has a dual characterisation.

   Proposition 3. *Let $\mu \in \mathcal{D}(X), \nu \in \mathcal{D}(Y)$ be countable distributions and $\alpha : X \nrightarrow Y$ be a $[0,1]$-relation. Then:*

   $$\begin{aligned} W\alpha(\mu,\nu) &= \min\{\textstyle\sum_{x,y} \alpha(x,y)\cdot\omega(x,y) \mid \omega \in \Omega(\mu,\nu)\} \\ &= \max\{\textstyle\sum_x a_x \cdot \mu(x) + \sum_y b_y \cdot \nu(y) \\ &\qquad \mid a_x + b_y \leq \alpha(x,y), a_x, b_y \text{ bounded}\}, \end{aligned}$$

   where a_x, b_y bounded means that there exist $\bar{a}, \bar{b} \in \mathbb{R}$ such that $\forall x.\ a_x \leq \bar{a}$, and $\forall y.\ b_y \leq \bar{b}$.

   The above proposition (see Appendix A.3 for a proof) is a direct consequence of the Duality Theorem for *countable* transportation problems [29] (Theorem 2.1 and 2.2). Using Proposition 3 we can show that W indeed defines a $[0,1]$-relator (but see Digression 3). Finally, we can compose the Wasserstein lifting W with the V-relator $(-)_\perp$ of point 1 obtaining the (non-conversive) $[0,1]$-relator $W_\perp$ for the countable subdistribution functor $\mathcal{D}_{\leq 1}$.

Digression 3 (Building V-relators)**.** Most of the V-relators in Example 8 can be obtained using a general abstract construction refining the so-called *Barr extension* of a functor [30]. Recall that any relation $\mathcal{R} : X \nrightarrow Y$ (i.e. a 2-relation $\mathcal{R} : X \times Y \to 2$) can be equivalently presented as a subset of $X \times Y$ via its graph $G_\mathcal{R}$. This allows to express $\mathcal{R}$ as $\pi_2 \cdot \pi_1^\circ$ (in **Rel**), where $\pi_1 : G_\mathcal{R} \to X$, $\pi_2 : G_\mathcal{R} \to Y$ are the usual projection functions.

Definition 6. *Let T be an endofunctor on* **Set** *and $\mathcal{R} : X \nrightarrow Y$ be a a relation. The* Barr extension $\overline{T}$ *of T to* **Rel** *is defined by:*

$$\overline{T}\mathcal{R} \triangleq T\pi_2 \cdot (T\pi_1)^\circ,$$

where $\mathcal{R} = \pi_2 \cdot \pi_1^\circ$. Pointwise, $\overline{T}$ is defined by:

$$\chi\ \overline{T}\mathcal{R}\ y \iff \exists w \in TG_\mathcal{R}.\ (T\pi_1(w) = \chi,\ T\pi_2(w) = y),$$

where $\chi \in TX$ and $y = TY$

In general, $\overline{T}$ is not a 2-relator, but it is so if T preserves weak pullback diagrams [30] (or, equivalently, if T satisfies the Beck-Chevalley condition [25]). Such condition is satisfied by all functors we have considered so far in our examples.

Definition 6 crucially relies on the double nature of a relation, which can be viewed both as an arrow in **Rel** and as an object in **Set**. This is no longer the case for a V-relation, and thus it is not clear how to define the Barr extension of a functor T from **Set** to V-**Rel**. However, the Barr extension of T can be characterised

in an alternative way if we assume T to preserves weak pullback diagrams (although the reader can see [24, 36] for more general conditions). Let $\xi : T2 \to 2$ be the map defined by $\xi(\mathcal{X}) = \mathsf{true}$ if and only if $\mathcal{X} \in T\{\mathsf{true}\}$, where $T\{\mathsf{true}\}$ is the image of the map $T\iota$ for the inclusion $\iota : \{\mathsf{true}\} \to 2$. That is, $\xi(\mathcal{X}) = \mathsf{true}$ if and only if there exists an element $y \in T\{\mathsf{true}\}$ such that $T\iota(y) = \mathcal{X}$. Note that this makes sense since T preserves monomorphisms (recall that we can describe monomorphism as weak pullbacks) and thus $T\iota : T\{\mathsf{true}\} \to T2$ is a monomorphism. We can now characterise $\overline{T}\mathcal{R}$ without mentioning the graph of $\mathcal{R}$:

$$\overline{T}\mathcal{R}(\mathcal{X}, y) = \mathsf{true} \iff \exists w \in T(X \times Y).\ \begin{cases} T\pi_1(w) & = \mathcal{X}, \\ T\pi_2(w) & = y, \\ \xi \cdot T\mathcal{R}(w) & = \mathsf{true}. \end{cases}$$

Since the existential quantification is nothing but the joint of the boolean quantale 2, the above characterisation of $\overline{T}$ can be turned into a definition of an extension of T to V-**Rel** parametric with respect to a map $\xi : T\mathsf{V} \to \mathsf{V}$.

Definition 7. *For a set endofunctor T and a map $\xi : T\mathsf{V} \to \mathsf{V}$ define the* V-Barr extension $\overline{T}_\xi$ *of T to* V-**Rel** *with respect to ξ as follows:*

$$\overline{T}_\xi \alpha(\mathcal{X}, y) \triangleq \bigvee_{w \in \Omega(\mathcal{X}, y)} \xi \cdot T\alpha(w),$$

for $\mathcal{X} \in TX, y \in TY$, where the set $\Omega(\mathcal{X}, y)$ of generalised couplings of $\mathcal{X}, y$ is defined by:

$$\Omega(\mathcal{X}, y) \triangleq \{w \in T(X \times Y) \mid T\pi_1(w) = \mathcal{X},\ T\pi_2(w) = y\}.$$

Example 9. 1. Taking $\xi : \mathcal{P}\mathsf{V} \to \mathsf{V}$ defined by $\xi(\mathcal{X}) \triangleq \bigwedge \mathcal{X}$ we recover the Hausdorff lifting H^s.
2. Taking expectation function $\xi : \mathcal{D}[0,1] \to [0,1]$ defined by $\xi(\mu) \triangleq \sum_x x \cdot \mu(x)$ we recover Wasserstein lifting W.

Using the map $\xi : T\mathsf{V} \to \mathsf{V}$ we can define an extension of T to V-**Rel**. However, such extension is in general not a V-relators. Nonetheless, under mild conditions on ξ and assuming T to preserve weak pullback, it is possible to show that $\overline{T}_\xi$ is indeed a V-relator. The following proposition has been proved in [9, 24] (a similar result for real-valued pseudometric spaces has been proved in [4, 5], where an additional extension still parametric over ξ is also studied).

Proposition 4. *Let T be functor preserving weak pullbacks and $\xi : T\mathsf{V} \to \mathsf{V}$ be a map such that:*

1. ξ respect quantale multiplication:

$$\begin{array}{ccc} T(\mathsf{V} \times \mathsf{V}) & \xrightarrow{T\otimes} & T\mathsf{V} \\ {\scriptstyle \langle \xi \cdot T\pi_1, \xi \cdot T\pi_2 \rangle}\downarrow & \leq & \downarrow{\scriptstyle \xi} \\ \mathsf{V} \times \mathsf{V} & \xrightarrow[\otimes]{} & \mathsf{V} \end{array}.$$

2. ξ respects the unit of the quantale:

$$\begin{array}{ccc} T1 & \xrightarrow{Tk} & T\mathsf{V} \\ {\scriptstyle !}\downarrow & \leq & \downarrow{\scriptstyle \xi} \\ 1 & \xrightarrow[k]{} & \mathsf{V} \end{array}.$$

3. ξ respects the order of the quantale. That is, the map $\varphi \mapsto \xi \cdot T\varphi$, for $\varphi : X \to \mathsf{V}$, is monotone.

Then $\overline{T}_\xi$ is a conversive V-relator.

It is straightforward to check that the expectation function in Example 9 satisfies the above three conditions. By Proposition 4 it follows that the Wasserstein lifting gives indeed a $[0,1]$-relator, and thus so does its composition with the $[0,1]$-relator $(-)_\perp$.

The extension $\overline{T}_\xi$ gives a somehow canonical *conversive* V-relator and thus provides a way to build canonical (applicative) V-*bi*simulations. However, $\overline{T}_\xi$ being intrinsically conversive it is not a good candidate to build V-simulations. For most of the examples considered we can get around the problem considering $(\overline{T}_\xi)_\perp$ (as we do with e.g. $W_\perp$). Nonetheless, it is desirable to have a general notion of extension characterising notions of V-simulations. That has been done for ordinary relations in e.g. [27, 33] for functors T inducing a suitable order $\leq_X$ on TX and considering the relator $\overline{T}_\leq \triangleq \leq_- \cdot \overline{T} \cdot \leq_-$. Proving that $\overline{T}_\leq$ gives indeed a relator requires T to satisfy specific conditions. For instance, in [33] it is proved that if T satisfies a suitable form of weak-pullback preservation (which takes into account the order induced by T), then $\overline{T}_\leq$ is indeed a relator. This suggests to consider functors T inducing a suitable V-relation α_X on TX and thus to study if, and under which conditions, $\alpha_- \cdot \overline{T}_\xi \cdot \alpha_-$ is a V-relator. This proposal has not been investigated in the context of the present work but it definitely constitutes a topic for future research.

V-relators for Strong Monads In previous paragraph we saw that a V-relator extends a functor from **Set** to V-**Rel** laxly. Since we model effects through strong monads it seems more natural to require V-relators to extend strong monads from **Set** to V-**Rel** laxly.

The reason behind such requirement can be intuitively understood as follows. Recall that by Proposition 1 we have (for readability we omit types) $|\mathbf{let}\ x = e\ \mathbf{in}\ f| = |f[x := -]|^* |e|$. This operation can be described using the so called bind function

$$\gg\!= : (X \to TY) \times TX \to TY,$$

so that we have $|\mathbf{let}\ x = e\ \mathbf{in}\ f| = |f[x := -]| \gg\!= |e|$. Now, let $f, g : X \to Y$ be functions, $\alpha : X \nrightarrow X, \beta : Y \nrightarrow Y$ be V relations, and Γ be a V-relator for T. Considering the compound V relation $[\alpha, \Gamma\beta] \otimes \Gamma\alpha$ (see Digression 1) and ignoring issues about sensitivity, it is then natural to require $\gg\!=$ to be non-expansive. That is, we require the inequality

$$[\alpha, \Gamma\beta](f, g) \otimes \Gamma\alpha(\mathcal{X}, y) \leq \Gamma\beta(f \gg\!= \mathcal{X}, g \gg\!= y)$$

i.e.

$$\bigwedge_{x \in X} \Gamma\beta(f(x), g(x)) \otimes \Gamma\alpha(\mathcal{X}, y) \leq \Gamma\beta(f \gg\!= \mathcal{X}, g \gg\!= y).$$

Informally, we are requiring the behavioural distance between sequential compositions of programs to be bounded by the behavioural distances between their components (this is of course a too strong requirement, but at this point it should be clear to the reader that it is sufficient to require $\gg\!=$ to be Lipshitz continuous rather than non-expansive). Since $\gg\!=$ is nothing but the strong Kleisli extension apply* of the application function apply : $(X \to TY) \times X \to TY$ defined by $\mathsf{apply}(f, x) \triangleq f(x)$, what we need to do is indeed to extend strong monads from **Set** to V-**Rel** (laxly).

Definition 8. *Let $\mathbb{T} = \langle T, \eta, -^* \rangle$ be a strong monad on* **Set**, *and Γ be a V-relator for T (regarded as a functor). We say that Γ is an L-continuous*[6] *V-relator for $\mathbb{T}$ if it satisfies the following conditions*

[6] Instantiating V as the Lawvere quantale, we see that condition (L-Strong lax bind) is requiring Lipshitz continuity of multiplication and strength of $\mathbb{T}$.

for any CBE $s \leq 1$.

$$\alpha \leq \eta^{\circ}_{Y} \cdot \Gamma\alpha \cdot \eta_{X}, \qquad \text{(Lax unit)}$$

$$\gamma \otimes (s \circ \alpha) \leq g^{\circ} \cdot \Gamma\beta \cdot f \implies \gamma \otimes (s \circ \Gamma\alpha) \leq (g^{*})^{\circ} \cdot \Gamma\beta \cdot f^{*}, \qquad \text{(L-Strong lax bind)}$$

The condition $s \leq 1$ reflects the presence of $s \wedge 1$ in the typing rule for sequencing. Also notice that by taking $s \triangleq 1$, conditions (Lax unit) and (*L*-Strong lax bind) are equivalent to requiring unit, multiplication, and strength of $\mathbb{T}$ to be non-expansive.

Example 10. It is easy to check that V-relators for the partiality and the powerset monads satisfy conditions in Definition 8. Using Proposition 3 it is possible to show that also the Wasserstein lifting(s) W and $W_{\perp}$ do, although this is less trivial (see Appendix A.3).

Finally, if $\mathbb{T}$ is Σ-continuous we require V-relators for $\mathbb{T}$ to be compatible with the Σ-continuous structure.

Definition 9. *Let $\mathbb{T}$ be a Σ-continuous monad, V be a Σ-quantale, and Γ be a V-relator for $\mathbb{T}$. We say that Γ is* Σ-compatible *and* inductive *if the following inequalities hold:*

$$op_{\mathsf{V}}(\Gamma\alpha(u_1, y_1), \dots \Gamma\alpha(u_n, y_n)) \leq \Gamma\alpha(op_X(u_1, \dots, u_n), op_Y(y_1, \dots, y_n)),$$

$$k \leq \Gamma\alpha(\perp_X, y),$$

$$\bigwedge\nolimits_n \Gamma\alpha(x_n, y) \leq \Gamma\alpha(\bigsqcup\nolimits_n x_n, y).$$

for any ω-chain $(x_n)_{n<\omega}$ and elements $u_1, \dots, u_n$ in TX, elements $y, y_1, \dots, y_n \in TY$, n-ary operation symbol $\mathbf{op} \in \Sigma$, and V-relation $\alpha : X \nrightarrow Y$.

In particular, if Γ is inductive and $a \leq \Gamma\alpha(x_n, y))$ holds for any $n < \omega$, then $a \leq \Gamma\alpha(\bigsqcup_{n<\omega} x_n, y)$.

Example 11. Easy calculations show that $(-)_{\perp}$ and H are inductive and Σ-compatible. Using results from [49] and [8] (Lemma 5.2) it is possible to show that $W_{\perp}$ is inductive, the relevant inequality being

$$W_{\perp}\alpha(\sup_n \mu_n, \nu) \leq \sup_n W_{\perp}\alpha(\mu_n, \nu).$$

Proving Σ-compatibility of W and $W_{\perp}$ amounts to prove

$$\Gamma\alpha(\mu_1 \oplus_p \nu_1, \mu_2 \oplus_p \nu_2) \leq \Gamma\alpha(\mu_1, \mu_2) \oplus_p \Gamma\alpha(\nu_1, \nu_2),$$

which is straightforward.

From V-relators to 2-relators Before applying the abstract theory of V-relators to V-Fuzz we show how a V-relator induces a canonical 2-relator (this will be useful in the next section). Consider the maps:

$$\varphi : \mathsf{V} \to 2 \qquad \psi : 2 \to \mathsf{V}$$
$$k \mapsto \mathsf{true},\ a \mapsto \mathsf{false} \qquad \mathsf{true} \to k,\ \mathsf{false} \to \perp\!\!\!\perp$$

We immediately see that φ and ψ are CBFs and that φ is the right adjoint of ψ. We associate to every V-relation α its kernel 2-relation $\varphi \circ \alpha$ and to any 2-relation $\mathcal{R}$ the V-relation $\psi \circ \mathcal{R}$. Similarly, we can associate to each V-relator Γ the 2-relator $\Delta_{\Gamma}\mathcal{R} \triangleq \varphi \circ \Gamma(\psi \circ \mathcal{R})$. Moreover, since φ is the right adjoint of ψ we have the inequalities:

$$\psi \circ \Delta_{\Gamma}\mathcal{R} \leq \Gamma(\psi \circ \mathcal{R})$$
$$\Delta_{\Gamma}(\varphi \circ \alpha) \leq \varphi \circ \Gamma\alpha.$$

Finally, we say that Γ is compatible with φ if $\Delta_{\Gamma}(\varphi \circ \alpha) = \varphi \circ \Gamma\alpha$ holds for any $\alpha : X \nrightarrow Y$.

Example 12. 1. For the V-relator $(-)_{\perp}$ and $\mathcal{R} : X \nrightarrow Y$ we have $\Delta_{\perp}\mathcal{R}(x, y) = \mathsf{true}$ if and only if $x \in X$, $y \in Y$ and $\mathcal{R}(x, y) = \mathsf{true}$, or $x = \perp$. That is, $\Delta_{\perp}$ gives the usual simulation relator for 'effect-free' λ-calculi. An easy calculation shows that $\Delta_{\perp}(\varphi \circ \alpha) = \varphi \circ \alpha_{\perp}$. Replacing $(-)_{\perp}$ with $(-)_{\perp\perp}$ we recover the bisimulation relator for 'effect-free' λ-calculi.

2. For the V-relator H and $\mathcal{R} : X \nrightarrow Y$ we have:

$$\Delta_H\mathcal{R}(\mathcal{X}, \mathcal{Y}) = \mathsf{true} \iff \forall x \in \mathcal{X}.\, \exists y \in \mathcal{Y}.\, \mathcal{R}(x, y) = \mathsf{true}.$$

Therefore, Δ_H gives the usual notion of simulation for nondeterministic systems. Proving compatibility with φ, i.e. $\Delta_H(\varphi \circ \alpha) = \varphi \circ H\alpha$, is straightforward. A similar argument holds for H^s.

3. Consider the Wasserstein lifting W and observe that we have $\Delta_W\mathcal{R}(\mu, \nu) = \mathsf{true}$ if and only if the following holds:

$$\exists \omega \in \Omega(\mu, \nu).\, \forall x, y.\, \omega(x, y) > 0 \implies \mathcal{R}(x, y) = \mathsf{true}.$$

We have thus recovered the usual notion of probabilistic relation lifting via couplings [30]. Moreover, if $\varphi \circ W\alpha(\mu, \nu) = \mathsf{true}$, then $W\alpha(\mu, \nu) = 0$, meaning that there exists a coupling $\omega \in \Omega(\mu, \nu)$ such that $\sum_{x,y} \omega(x, y) \cdot \alpha(x, y) = 0$. In particular, if $\omega(x, y) > 0$, then $\alpha(x, y) = 0$ i.e. $(\varphi \circ \alpha)(x, y) = \mathsf{true}$. That is, W is compatible with φ. From point 1 it follows that $W_{\perp}$ is compatible with φ as well.

We conclude this section with the following auxiliary lemma (whose proof is given in Appendix A.3), which will be useful to prove that the kernel of applicative distances are suitable applicative (bi)simulations.

Lemma 4. *Let Γ be V-relator compatible with φ. Then the following hold:*

$$\begin{array}{ccc} X & \xrightarrow{f} & TZ \\ {\scriptstyle\alpha}\downarrow & \leq & \downarrow{\scriptstyle\Gamma\beta} \\ Y & \xrightarrow[g]{} & TW \end{array} \implies \begin{array}{ccc} X & \xrightarrow{f} & TZ \\ {\scriptstyle\varphi\circ\alpha}\downarrow & \leq & \downarrow{\scriptstyle\Delta_{\Gamma}(\varphi\circ\beta)} \\ Y & \xrightarrow[g]{} & TW \end{array},$$

$$\begin{array}{ccc} X & \xrightarrow{f} & TZ \\ {\scriptstyle\mathcal{R}}\downarrow & \leq & \downarrow{\scriptstyle\Delta_{\Gamma}\mathcal{S}} \\ Y & \xrightarrow[g]{} & TW \end{array} \implies \begin{array}{ccc} X & \xrightarrow{f} & TZ \\ {\scriptstyle\psi\circ\mathcal{R}}\downarrow & \leq & \downarrow{\scriptstyle\Gamma(\psi\circ\mathcal{S})} \\ Y & \xrightarrow[g]{} & TW \end{array}.$$

5 Behavioural V-relations

In this section we extend the relational theory developed in e.g. [22, 31] for higher-order functional languages to V-relations for V-Fuzz. Following [39] we refer to such relations as *λ-term V-relations*. Among such V-relations we define applicative Γ-similarity, the generalisation of Abramsky's applicative similarity to both algebraic effects and V-relations, and prove that under suitable conditions it is compatible generalised metric. We postpone the study of applicative Γ-bisimilarity to Section 7. As usual we assume a signature Σ, a Σ-quantale V, a collection of CBEs Π (according to Section 3), and a Σ-continuous (strong) monad $\mathbb{T}$ to be fixed. We also assume V-relators to satisfy all requirements given in Section 4.

Definition 10. *A closed λ-term V-relation $\alpha = (\alpha^{\Lambda}, \alpha^{\mathcal{V}})$ associates to each closed type σ, binary V-relations $\alpha^{\mathcal{V}}_{\sigma}, \alpha^{\Lambda}_{\sigma}$ on closed values and terms inhabiting it, respectively.*

Since the syntactic shape of expressions determines whether we are dealing with terms or values, oftentimes we will write $\alpha_\sigma(e,f)$ (resp. $\alpha_\sigma(v,w)$) in place of $\alpha^\Lambda_\sigma(e,f)$ (resp. $\alpha^\mathcal{V}_\sigma(v,w)$).

In order to be able to work with open terms we introduce the notion of *open λ-term V-relation.*

Definition 11. *An* open *λ-term V-relation α associates to each (term) sequent $\Gamma \vdash \sigma$ a V-relation $\Gamma \vdash \alpha(-,-) : \sigma$ on terms inhabiting it, and to each value sequent $\Gamma \vdash^\mathsf{v} \sigma$ a V-relation $\Gamma \vdash^\mathsf{v} \alpha(-,-) : \sigma$ on values inhabiting it. We require open λ-term V-relations to be closed under weakening, i.e. for any environment Δ we require:*

$$(\Gamma \vdash \alpha(e,f) : \sigma) \leq (\Gamma \otimes \Delta \vdash \alpha(e,f) : \sigma),$$
$$(\Gamma \vdash^\mathsf{v} \alpha(v,w) : \sigma) \leq (\Gamma \otimes \Delta \vdash^\mathsf{v} \alpha(v,w) : \sigma).$$

As for closed λ-term V-relations, we will often write $\Gamma \vdash \alpha(v,w) : \sigma$ in place of $\Gamma \vdash^\mathsf{v} \alpha(v,w) : \sigma$ and simply refer to open λ-term V-relations as λ-term V-relations (whenever relevant we will explicitly mention whether we are dealing with open or closed λ-term V-relations).

Example 13. Both the discrete and the indiscrete V-relations are open λ-term V-relations. The discrete λ-term V-relation is defined by:

$$\Gamma \vdash \mathsf{disc}(e,e) : \sigma \triangleq k, \qquad \Gamma \vdash \mathsf{disc}(e,f) : \sigma \triangleq \bot,$$

(and similarly for values), whereas the indiscrete λ-term V-relation is defined by

$$\Gamma \vdash \mathsf{indisc}(e,f) : \sigma \triangleq k$$

(and similarly for values).

We notice that the collection of open λ-term V-relations carries a complete lattice structure (with respect to the pointwise order), meaning that we can define λ-term V-relation both inductively and coinductively.

We can always extend a closed λ-term V-relation $\alpha = (\alpha^\Lambda, \alpha^\mathcal{V})$ to an open one.

Definition 12. *Let $\Gamma \triangleq x_1 :_{s_1} \sigma_1, \dots, x_n :_{s_n} \sigma_n$ be an environment. For values $\vec{v} \triangleq v_1, \dots, v_n$ we write $\vec{v} : \Gamma$ if for any $i \leq n$, $\emptyset \vdash^\mathsf{v} v_i : \sigma_i$ holds. Given a closed λ-term V-relation $\alpha = (\alpha^\Lambda, \alpha^\mathcal{V})$ we define its open extension α^o as follows[7]:*

$$\Gamma \vdash \alpha^o(e,f) : \tau \triangleq \bigwedge\nolimits_{\vec{v}:\Gamma} \alpha^\Lambda_\tau(e[\vec{x} := \vec{v}], f[\vec{x} := \vec{v}])$$
$$\Gamma \vdash^\mathsf{v} \alpha^o(v,w) : \tau \triangleq \bigwedge\nolimits_{\vec{u}:\Gamma} \alpha^\mathcal{V}_\tau(v[\vec{u}/\vec{x}], w[\vec{u}/\vec{x}]).$$

We now define *applicative Γ-similarity.*

Definition 13. *Let Γ be a V-relator and $\alpha = (\alpha^\Lambda, \alpha^\mathcal{V})$ be a closed λ-term V-relation. Define the closed λ-term V-relation $[\alpha] = ([\alpha]^\Lambda, [\alpha]^\mathcal{V})$ as follows:*

$$[\alpha]^\Lambda_\sigma(e,f) \triangleq \Gamma\alpha^\mathcal{V}_\sigma(|e|,|f|),$$
$$[\alpha]^\mathcal{V}_{\sigma\multimap\tau}(v,w) \triangleq \bigwedge\nolimits_{u\in\mathcal{V}_\sigma} \alpha^\Lambda_\tau(vu, wu),$$
$$[\alpha]^\mathcal{V}_{\sum_{i\in I}\sigma_i}(\langle \hat{\imath}, v\rangle, \langle \hat{\imath}, w\rangle) \triangleq \alpha^\mathcal{V}_{\sigma_i}(v,w),$$
$$[\alpha]^\mathcal{V}_{\sum_{i\in I}\sigma_i}(\langle \hat{\imath}, v\rangle, \langle \hat{\jmath}, w\rangle) \triangleq \perp\!\!\!\perp,$$
$$[\alpha]_{\mu t.\sigma}(\mathbf{fold}\ v, \mathbf{fold}\ w) \triangleq \alpha_{\sigma[\mu t.\sigma/t]}(v,w),$$
$$[\alpha]_{!_s\sigma}(!v, !w) \triangleq (s \circ \alpha_\sigma)(v,w).$$

[7] The superscript is the letter 'o' (for open), and should not be confused with $\circ$ which we use for the map $-^\circ$ sending a V-relation to its dual.

(notice that the definition of $[\alpha]^\mathcal{V}$ is by case analysis on $\emptyset \vdash^\mathsf{v} v, w : \sigma$). A λ-term V-relation α is an applicative Γ-simulation if $\alpha \leq [\alpha]$.

The clause for $\sigma \multimap \tau$ generalises the usual applicative clause, whereas the clause for $!_s\sigma$ 'scale' $\alpha^\mathcal{V}_\sigma$ by s. It is easy to see that the above definition induces a map $\alpha \mapsto [\alpha]$ on the complete lattice of closed λ-term V-relations. Moreover, such map is monotone since both Γ and CBEs are.

Definition 14. *Define applicative Γ-similarity δ as the greatest fixed point of $\alpha \mapsto [\alpha]$. That is, δ is the greatest (closed) λ-term V-relation satisfying the equation $\alpha = [\alpha]$ (such greatest solution exists by the Knaster-Tarski Theorem).*

Applicative Γ-similarity comes with an associated coinduction principle: for any closed λ-term V-relation α, if $\alpha \leq [\alpha]$, then $\alpha \leq \delta$.

Example 14. Instantiating Definition 14 with the Wasserstein lifting $W_\perp$ we obtain the quantitative analogue of *probabilistic applicative similarity* [15] for P-Fuzz. In particular, for two terms $e, f \in \Lambda_\sigma$, $\delta(e,f)$ is (for readability we omit subscripts):

$$\min_{\omega\in\Omega(|e|,|f|)} \sum_{v,w\in\mathcal{V}} \omega(v,w)\cdot\delta^\mathcal{V}(v,w) + \sum_{v\in\mathcal{V}} \omega(v,\bot)\cdot\delta^\mathcal{V}_\bot(v,\bot)$$
$$+ \sum_{w\in\mathcal{V}} \omega(\bot,w)\cdot\delta^\mathcal{V}_\bot(\bot,w) + \omega(\bot,\bot)\cdot\delta^\mathcal{V}_\bot(\bot,\bot).$$

The above formula can be simplified observing that we have $\delta^\mathcal{V}_\bot(\bot,\bot) = 0$, $\delta^\mathcal{V}_\bot(v,\bot) = 1$, and $\delta^\mathcal{V}_\bot(\bot,w) = 0$ by very definition of $\delta_\bot$. We immediately notice that δ is adequate in the following sense: for all terms $e, f \in \Lambda_\sigma$ we have the inequality

$$\sum |e| - \sum |f| \leq \delta^\Lambda(e,f),$$

where $\sum |e|$ is the probability of convergence of e, i.e. $\sum_{v\in\mathcal{V}} |e|(v)$, and subtraction is actually truncated subtraction.

Let us now consider terms $I, \Omega \in \Lambda_{\sigma\multimap\sigma}$ of Example 7. We claim that $\delta^\Lambda(I, I\oplus\Omega) = \frac{1}{2}$. By adequacy we immediately see that $\frac{1}{2} \leq \delta^\Lambda(I, I\oplus\Omega)$. We prove $\delta^\Lambda(I, I\oplus\Omega) \leq \frac{1}{2}$. Let $v \triangleq \lambda x.\mathbf{val}\ x$ and consider the coupling ω defined by:

$$\omega(v,v) = \frac{1}{2}, \quad \omega(v,\bot) = \frac{1}{2}$$

and zero for the rest. Indeed ω is a coupling of $|I|$ and $|I\oplus\Omega|$. Moreover, by very definition of δ and $W_\perp$ we have:

$$\delta^\Lambda(I, I\oplus\Omega) \leq \omega(v,v)\cdot\delta^\mathcal{V}(v,v) + \omega(v,\bot).$$

The right hand side of the above inequality gives exactly $\frac{1}{2}$, provided that $\delta^\mathcal{V}(v,v) = 0$. This indeed holds in full generality.

Proposition 5. *Applicative Γ-similarity δ is a reflexive and transitive λ-term V-relation.*

Proof sketch. The proof is by coinduction, showing that both the identity λ-term V-relation and $\delta\cdot\delta$ are applicative Γ-simulations. A formal proof is given in Appendix A.3. □

In light of Example 12 we can look at the kernel of δ and recover well-known notions of (relational) applicative similarity (properly generalised to V-Fuzz).

Proposition 6. *Define applicative Δ_Γ-similarity $\preceq$ by instantiating Definition 13 with the 2-relator Δ_Γ and replacing the clause for types of the form $!_s\sigma$ as follows: $!v\ \mathcal{R}_{!_s\sigma}\ !w$ implies $(\varphi\cdot s\cdot\psi)\circ\mathcal{R}_\sigma(v,w)$. Then the kernel $\varphi\circ\delta$ of δ coincide with $\preceq$.*

Proof sketch. By coinduction (and using Lemma 4) one shows that $\varphi \circ \delta$ is an applicative Δ_Γ-simulation and that $\psi \circ \preceq$ is an applicative Γ-simulation. A detailed proof is given in Appendix A.3. □

Note that if $\mathcal{R}_\sigma(v, w)$ holds, then so does $(\varphi \cdot s \cdot \psi) \circ \mathcal{R}_\sigma(v, w)$, but the vice-versa does not necessarily hold. For instance, taking $s \triangleq 0$ we see that

$$(\varphi \cdot 0 \cdot \psi) \circ \mathcal{R}_\sigma(v, w) = \varphi(0(\psi(\text{false}))) = \varphi(0 \cdot \infty) = \varphi(0) = \text{true},$$

which essentially means we identify distinguishable values if they are not used. Nonetheless, the reader should notice that the encoding of a 'standard' λ-calculus Λ in V-Fuzz can be obtained via the usual encoding of Λ in its linear refinement [37] which corresponds to the fragment of V-Fuzz based on CBEs 1 and ∞, thus avoiding the above undesired result.

Finally, we introduce the notion of *compatibility* which captures a form of Lipshitz-continuity with respect to V-Fuzz constructors. It is useful to follow [31] and define compatibility via the notion of *compatible refinement.*

Definition 15. *The* compatible refinement $\hat{\alpha}$ *of an open λ-term V-relation α is defined by:*

$$(\Gamma \vdash \hat{\alpha}(e, f) : \sigma) \triangleq \bigvee \{a \mid \Gamma \models a \leq \hat{\alpha}(e, f) : \sigma\},$$
$$(\Gamma \vdash^{\mathsf{v}} \hat{\alpha}(v, w) : \sigma) \triangleq \bigvee \{a \mid \Gamma \models^{\mathsf{v}} a \leq \hat{\alpha}(v, w) : \sigma\},$$

where judgments $\Gamma \models a \leq \hat{\alpha}(e, f) : \sigma$ and $\Gamma \models^{\mathsf{v}} a \leq \hat{\alpha}(v, w) : \sigma$ are inductively defined for $a \in \mathsf{V}$, $\Gamma \vdash e, f : \sigma$, and $\Gamma \vdash^{\mathsf{v}} v, w : \sigma$ by rules in Figure 4. We say that α is compatible if $\hat{\alpha} \leq \alpha$.

It is easy to see that if α is compatible, then it satisfies inequalities in Figure 5. Actually, α is compatible precisely if it satisfies the inequalities in Figure 5.

Notice that in the clause for sequential composition the presence of $s \wedge 1$, instead of s, ensures that for terms like $e \triangleq \mathbf{let}\ x = I\ \mathbf{in}\ \underline{0}$ and $e' \triangleq \mathbf{let}\ x = \Omega\ \mathbf{in}\ \underline{0}$, the distance $\alpha(e, e')$ is determined *before* sequencing (which captures the idea that although $\underline{0}$ will not 'use' any input, I and Ω will be still evaluated, thus producing observable differences between e and e'). In fact, if we replace $s \wedge 1$ with s, then by taking $s \triangleq 0$ compatibility would imply $\alpha(e, e') = k$, which is clearly unsound.

In order to make applicative Γ-similarity a useful tool, we need it to allow compositional reasoning about programs. Formally, that amount to prove that applicative Γ-similarity is compatible.

6 Howe's Method

To prove compatibility of applicative Γ-similarity we design a generalisition of the so-called Howe's method [26] combining and extending ideas from [12] and [14]. We start by defining the notion of *Howe's extension*, a construction extending a λ-term V-open relation to a compatible and substitutive λ-term V-relation.

Definition 16 (Howe's extension (1))**.** *The* Howe's extension α^H *of an open λ-term V-relation α is defined as the least solution to the equation $\beta = \alpha \cdot \hat{\beta}$.*

It is easy to see that compatible refinement $\hat{-}$ is monotone, and thus so is the map Φ_α defined by $\Phi_\alpha(\beta) \triangleq \alpha \cdot \hat{\beta}$. As a consequence, we can define α^H as the least fixed point of Φ_α. Since open extension $-^o$ is monotone as well, we can define the Howe's extension of a closed λ-term V-relation α as $(\alpha^o)^H$.

It is also useful to spell out the above definition.

Definition 17 (Howe's extension (2))**.** *The* Howe's extension α^H *of an open λ-term V-relation α is defined by:*

$$(\Gamma \vdash \alpha^H(e, f) : \sigma) \triangleq \bigvee \{a \mid \Gamma \models a \leq \alpha^H(e, f) : \sigma\},$$
$$(\Gamma \vdash^{\mathsf{v}} \alpha^H(v, w) : \sigma) \triangleq \bigvee \{a \mid \Gamma \models^{\mathsf{v}} a \leq \alpha^H(v, w) : \sigma\},$$

where judgments $\Gamma \models a \leq \alpha^H(e, f) : \sigma$ and $\Gamma \models^{\mathsf{v}} a \leq \alpha^H(v, w) : \sigma$ are inductively defined for $a \in \mathsf{V}$, $\Gamma \vdash e, f : \sigma$, and $\Gamma \vdash^{\mathsf{v}} v, w : \sigma$ by rules in Figure 6.

The next lemma (whose proof is given in Appendix A.4) is useful for proving properties of Howe's extension. It states that α^H attains its value via the rules in Figure 6.

Lemma 5. *The following hold:*

1. *Given well-typed values $\Gamma \vdash^{\mathsf{v}} v, w : \sigma$, let*
$$A \triangleq \{a \mid \Gamma \models^{\mathsf{v}} a \leq \alpha^H(v, w) : \sigma\}$$
be non-empty. Then $\Gamma \models^{\mathsf{v}} \bigvee A \leq \alpha^H(v, w)$ is derivable.
2. *Given well-typed terms $\Gamma \vdash e, f : \sigma$, let*
$$A \triangleq \{a \mid \Gamma \models^{\mathsf{c}} a \leq \alpha^H(e, f) : \sigma\}$$
be non-empty. Then $\Gamma \models^{\mathsf{c}} \bigvee A \leq \alpha^H(e, f)$ is derivable.

It is easy to see that Definition 16 and 17 gives the same λ-term V-relation. In particular, for an open λ-term V-relation α, α^H is the least compatible open λ-term V-relation satisfying the inequality $\alpha \cdot \beta \leq \beta$.

The following are standard results on Howe's extension. Proofs are straightforward but tedious (they closely resemble their relational counterparts), and thus are omitted.

Lemma 6. *Let α be a reflexive and transitive open λ-term V-relation. Then the following hold:*

1. *α^H is reflexive.*
2. *$\alpha \leq \alpha^H$.*
3. *$\alpha \cdot \alpha^H \leq \alpha^H$.*
4. *α^H is compatible.*

We refer to property 1 as pseudo-transitivity. In particular, by very definition of V-relator we also have $\Gamma\alpha \cdot \Gamma\alpha^H \leq \Gamma\alpha^H$. We refer to the latter property as Γ-pseudo-transitivity. Notice that Proposition 5 implies that $(\delta^o)^H$ is compatible and bigger than δ^o.

Finally, Howe's extension enjoys another remarkable property, namely substitutivity.

Definition 18. *An open λ-term V-relation α is value substitutive if for all well-typed values $\Gamma, x :_s \sigma \vdash^{\mathsf{v}} v, w : \tau$, $\emptyset \vdash^{\mathsf{v}} u : \sigma$, and terms $\Gamma, x :_s \sigma \vdash e, f : \tau$ we have:*

$$(\Gamma, x :_s \sigma \vdash^{\mathsf{v}} \alpha(v, w) : \tau) \leq (\Gamma \vdash \alpha(v[u/x], w[u/x]) : \tau),$$
$$(\Gamma, x :_s \sigma \vdash \alpha(e, f) : \tau) \leq (\Gamma \vdash \alpha(e[x := u], f[x := u]) : \tau).$$

Lemma 7 (Substitutivity)**.** *Let α be a value substitutive λ-term V-preorder. For all values, $\Gamma, x :_s \sigma \vdash^{\mathsf{v}} u, z : \tau$ and $\emptyset \vdash v, w : \sigma$, and terms $\Gamma, x :_s \sigma \vdash e, f : \tau$, let $\underline{a} \triangleq \emptyset \vdash^{\mathsf{v}} \alpha^H(v, w) : \sigma$. Then:*

$$(\Gamma, x :_s \sigma \vdash^{\mathsf{v}} \alpha^H(u, z) : \tau) \otimes s(\underline{a}) \leq \Gamma \vdash^{\mathsf{v}} \alpha^H(u[v/x], z[w/x]) : \tau,$$
$$(\Gamma, x :_s \sigma \vdash \alpha^H(e, f) : \tau) \otimes s(\underline{a}) \leq \Gamma \vdash \alpha^H(e[x := v], f[x := w]) : \tau.$$

Proof. See Appendix A.4. □

$$\frac{}{\Gamma, x :_s \sigma \models k \leq \hat{\alpha}(x,x) : \sigma} \qquad \frac{a_1 \leq \Gamma_1 \vdash \alpha(e_1, f_1) : \sigma \quad \cdots \quad a_n \leq \Gamma_n \vdash \alpha(e_n, f_n) : \sigma}{op_{\mathsf{V}}(\Gamma_1, \ldots, \Gamma_n) \models op_{\mathsf{V}}(a_1, \ldots, a_n) \leq \hat{\alpha}(\mathbf{op}(e_1, \ldots, e_n), \mathbf{op}(e_1, \ldots, e_n)) : \sigma}$$

$$\frac{a \leq \Gamma, x :_1 \sigma \vdash \alpha(e, f) : \tau}{\Gamma \models^{\mathsf{V}} a \leq \hat{\alpha}(\lambda x.e, \lambda x.f) : \sigma \multimap \tau} \qquad \frac{a \leq \Gamma \vdash^{\mathsf{V}} \alpha(v, v') : \sigma \multimap \tau \quad b \leq \Delta \vdash^{\mathsf{V}} \alpha(w, w') : \sigma}{\Gamma \otimes \Delta \models a \otimes b \leq \hat{\alpha}(vw, v'w') : \tau}$$

$$\frac{a \leq \Gamma \vdash^{\mathsf{V}} \alpha(v, w) : \sigma_{\hat{\imath}}}{\Gamma \models^{\mathsf{V}} a \leq \hat{\alpha}(\langle \hat{\imath}, v\rangle, \langle \hat{\imath}, w\rangle) : \sum_{i \in I} \sigma_i} \qquad \frac{a \leq \Gamma \vdash^{\mathsf{V}} \alpha(\langle \hat{\imath}, v\rangle, \langle \hat{\imath}, w\rangle) : \sum_{i \in I} \sigma_i \quad b_i \leq \Delta, x :_{s_i} \sigma_i \vdash \leq \alpha(e_i, f_i) : \tau \quad (\forall i \in I)}{s \cdot \Gamma \otimes \Delta \models s(a) \otimes b_{\hat{\imath}} \leq \hat{\alpha}(\mathbf{case}\ \langle \hat{\imath}, v\rangle\ \mathbf{of}\ \{\langle i, x\rangle \to e_i\}, \mathbf{case}\ \langle \hat{\imath}, w\rangle\ \mathbf{of}\ \{\langle i, x\rangle \to f_i\}) : \tau}$$

$$\frac{a \leq \Gamma \vdash^{\mathsf{V}} \alpha(v, w) : \sigma}{\Gamma \models a \leq \hat{\alpha}(\mathbf{val}\ v, \mathbf{val}\ w) : \sigma} \qquad \frac{a \leq \Gamma \vdash \alpha(e, e') : \sigma \quad b \leq \Delta, x :_s \sigma \vdash \alpha(f, f') : \tau}{(s \wedge 1) \cdot \Gamma \otimes \Delta \models (s \wedge 1)(a) \otimes b \leq \hat{\alpha}(\mathbf{let}\ x = e\ \mathbf{in}\ f, \mathbf{let}\ x = e'\ \mathbf{in}\ f') : \tau}$$

$$\frac{a \leq \Gamma \models \alpha(v, w) : \sigma}{s \cdot \Gamma \models^{\mathsf{V}} s(a) \leq \alpha(!v, !w) : !_s \sigma} \qquad \frac{a \leq \Gamma \vdash^{\mathsf{V}} \alpha(v, w) : !_r \sigma \quad b \leq \Delta, x :_{s \cdot r} \sigma \vdash \alpha(e, f) : \tau}{s \cdot \Gamma \otimes \Delta \models s(a) \otimes b \leq \hat{\alpha}(\mathbf{case}\ v\ \mathbf{of}\ \{!x \to e\}, \mathbf{case}\ w\ \mathbf{of}\ \{!x \to f\}) : \tau}$$

$$\frac{a\Gamma \vdash^{\mathsf{V}} \alpha(v, w) : \sigma[\mu t.\sigma/t]}{\Gamma \models^{\mathsf{V}} a \leq \hat{\alpha}(\mathbf{fold}\ v, \mathbf{fold}\ w) : \mu t.\sigma} \qquad \frac{a \leq \Gamma \vdash^{\mathsf{V}} \alpha(v, w) : \mu t.\sigma \quad b \leq \Delta, x :_s \sigma[\mu t.\sigma/t] \vdash b \leq \alpha(e, f) : \tau}{s \cdot \Gamma \otimes \Delta \models s(a) \otimes b \leq \hat{\alpha}(\mathbf{case}\ v\ \mathbf{of}\ \{\mathbf{fold}\ x \to e\}, \mathbf{case}\ w\ \mathbf{of}\ \{\mathbf{fold}\ x \to f\}) : \tau}$$

Figure 4. Compatible refinement.

$$k \leq (\Gamma \vdash^{\mathsf{V}} \alpha(x, x) : \sigma)$$

$$\Gamma, x :_1 \sigma \vdash \alpha(e, f) : \tau \leq \Gamma \vdash^{\mathsf{V}} \alpha(\lambda x.e, \lambda x.f) : \sigma \multimap \tau$$

$$(\Gamma \vdash^{\mathsf{V}} \alpha(v, v') : \sigma \multimap \tau) \otimes (\Delta \vdash^{\mathsf{V}} \alpha(w, w') : \sigma) \leq (\Gamma \otimes \Delta \vdash \alpha(vw, v'w') : \tau)$$

$$\Gamma \vdash^{\mathsf{V}} \alpha(v, w) : \sigma_{\hat{\imath}} \leq \Gamma \vdash^{\mathsf{V}} \alpha(\langle \hat{\imath}, v\rangle, \langle \hat{\imath}, w\rangle) : \sum_{i \in I} \sigma_i$$

$$s \circ (\Gamma \vdash^{\mathsf{V}} \alpha(\langle \hat{\imath}, v\rangle, \langle \hat{\imath}, w\rangle) : \sum_{i \in I} \sigma_i) \otimes (\Delta, x :_s \sigma \vdash \alpha(e_{\hat{\imath}}, f_{\hat{\imath}}) : \tau) \leq s \cdot \Gamma \otimes \Delta \vdash \alpha(\mathbf{case}\ \langle \hat{\imath}, v\rangle\ \mathbf{of}\ \{\langle i, x\rangle \to e_i\}, \mathbf{case}\ \langle \hat{\imath}, w\rangle\ \mathbf{of}\ \{\langle i, x\rangle \to f_i\}) : \tau$$

$$\Gamma \vdash^{\mathsf{V}} \alpha(v, w) : \sigma \leq \Gamma \vdash \alpha(\mathbf{val}\ v, \mathbf{val}\ w) : \sigma$$

$$(s \wedge 1) \circ (\Gamma \vdash \alpha(e, e') : \sigma) \otimes (\Delta, x :_s \sigma \vdash \alpha(f, f') : \tau) \leq (s \wedge 1) \cdot \Gamma \otimes \Delta \vdash \alpha(\mathbf{let}\ x = e\ \mathbf{in}\ f, \mathbf{let}\ x = e'\ \mathbf{in}\ f') : \tau$$

$$s \circ (\Gamma \vdash^{\mathsf{V}} \alpha(v, w) : \sigma) \leq s \cdot \Gamma \vdash^{\mathsf{V}} \alpha(!v, !w) : !_s \sigma$$

$$s \circ (\Gamma \vdash^{\mathsf{V}} \alpha(v, w) : !_r \sigma) \otimes (\Delta, x :_{s \cdot r} \sigma \vdash \alpha(e, f) : \tau) \leq s \cdot \Gamma \otimes \Delta \vdash \alpha(\mathbf{case}\ v\ \mathbf{of}\ \{!x \to e\}, \mathbf{case}\ w\ \mathbf{of}\ \{!x \to f\}) : \tau$$

$$\Gamma \vdash^{\mathsf{V}} \alpha(v, w) : \sigma[\mu t.\sigma/t] \leq \Gamma \vdash^{\mathsf{V}} \alpha(\mathbf{fold}\ v, \mathbf{fold}\ w) : \mu t.\sigma$$

$$s \circ (\Gamma \vdash^{\mathsf{V}} \alpha(v, w) : \mu t.\sigma) \otimes (\Delta, x :_s \sigma[\mu t.\sigma/t] \vdash \alpha(e, f) : \tau) \leq s \cdot \Gamma \otimes \Delta \vdash \alpha(\mathbf{case}\ v\ \mathbf{of}\ \{\mathbf{fold}\ x \to e\}, \mathbf{case}\ w\ \mathbf{of}\ \{\mathbf{fold}\ x \to f\}) : \tau$$

$$op_{\mathsf{V}}(\Gamma_1 \vdash \alpha(e_1, f_1) : \sigma, \ldots, \Gamma_n \vdash \alpha(e_n, f_n) : \sigma) \leq op_{\mathsf{V}}(\Gamma_1, \ldots, \Gamma_n) \vdash \alpha(\mathbf{op}(e_1, \ldots, e_n), \mathbf{op}(f_1, \ldots, f_n)) : \sigma$$

Figure 5. Compatibility clauses.

Notice that the open extension of any closed λ-term V-relation is value-substitutive. We can prove the main result of the Howe's method, the the so-called *Key Lemma*. The latter states the Howe's extension of applicative Γ-similarity (restricted to closed terms/values) is an applicative Γ-simulation. By coinduction, we can conclude that δ and δ^H (restricted to closed terms/values) coincide, meaning that the former is compatible.

Lemma 8 (Key Lemma)**.** *Let α be a reflexive and transitive applicative Γ-simulation. Then the Howe's extension of α restricted to closed terms/values in an applicative Γ-simulation.*

Proof sketch. The proof is non-trivial and a detailed account is given in Appendix A.4. Let us write α^H for the Howe's extension of α restricted to closed terms/values. By induction on n one shows that for any $n \geq 0$, $(\alpha^H)^{\wedge}_{\sigma}(e, f) \leq \Gamma(\alpha^H)^{\vee}_{\sigma}(|e|_n, |f|)$ holds for all terms $e, f \in \Lambda_\sigma$. Since Γ is inductive, the above inequality indeed gives the thesis. The base case follows again by inductivity of Γ, whereas the inductive step requires a case analysis on the structure of e. The crucial case is sequencing, where we rely on condition (L-Strong lax bind). □

From the Key Lemma it directly follows our main result.

Theorem 9 (Compatibility)**.** *Applicative Γ-similarity is compatible.*

Proof. We have to prove that δ^o is compatible. By Lemma 6 we know that $\delta^o \leq (\delta^o)^H$ and that $(\delta^o)^H$ is compatible. Therefore, to conclude the thesis it is sufficient to prove $(\delta^o)^H \leq \delta^o$. The Key Lemma implies that the restriction on closed terms/values of $(\delta^o)^H$ is an applicative Γ-simulation, and thus smaller or equal than δ. We can thus show that for all $\Gamma \vdash e, e' : \sigma$, the inequality $\Gamma \vdash (\delta^o)^H(e, e') : \sigma \leq \Gamma \vdash \delta^o(e, e') : \sigma$ holds. In fact, since $(\delta^o)^H$ is

$$\frac{a \le \Gamma, x :_s \sigma \vdash^{\vee} \alpha(x,w) : \sigma}{\Gamma, x :_s \sigma \models a \le \alpha^H(x,w) : \sigma} \text{ (H-var)}$$

$$\frac{\Gamma, x :_1 \sigma \models a \le \alpha^H(e,g) : \tau \quad c \le \Gamma \vdash \alpha(\lambda x.g, f) : \sigma \multimap \tau}{\Gamma \models a \otimes c \le \alpha^H(\lambda x.e, f) : \sigma \multimap \tau} \text{ (H-abs)}$$

$$\frac{\Gamma \models a \le \alpha^H(v,v') : \sigma \multimap \tau \quad \Delta \models b \le \alpha^H(w,w') : \sigma \quad c \le \Gamma \otimes \Delta \vdash \alpha(v'w', f) : \tau}{\Gamma \otimes \Delta \models a \otimes b \otimes c \le \alpha^H(vw, f) : \tau} \text{ (H-app)}$$

$$\frac{\Gamma \models^{\vee} a \le \alpha^H(v,w) : \sigma_{\hat{\imath}} \quad b \le \Gamma \vdash^{\vee} \alpha(\langle \hat{\imath}, w\rangle, u) : \sum_{i\in I} \sigma_i}{\Gamma \models^{\vee} a \otimes b \le \alpha(\langle \hat{\imath}, v\rangle, u) : \sum_{i\in I} \sigma_i} \text{ (H-inj)}$$

$$\frac{\Gamma \models^{\vee} a \le \alpha^H(\langle \hat{\imath}, v\rangle, \langle \hat{\imath}, w\rangle) : \sum_{i\in I} \sigma_i \quad \forall i \in I.\, \Delta, x :_s \sigma_i \models b_i \le \alpha^H(e_i, f_i) : \tau \quad c \le s \cdot \Gamma \otimes \Delta \vdash \alpha(\mathbf{case}\ \langle \hat{\imath}, w\rangle\ \mathbf{of}\ \{\langle i, x\rangle \to f_i\}, g) : \tau}{s \cdot \Gamma \otimes \Delta \models s(a) \otimes b_{\hat{\imath}} \otimes c \le \alpha^H(\mathbf{case}\ \langle \hat{\imath}, v\rangle\ \mathbf{of}\ \{\langle i, x\rangle \to e_i\}, g) : \tau} \text{ (H-sum-cases)}$$

$$\frac{\Gamma \models a \le \alpha^H(v,w) : \sigma \quad c \le \Gamma \vdash \alpha(\mathbf{val}\ w, f) : \sigma}{\Gamma \models a \otimes c \le \alpha^H(\mathbf{val}\ v, f) : \sigma} \text{ (H-val)}$$

$$\frac{\Gamma \models a \le \alpha^H(e,g) : \sigma \quad \Delta, x :_s \sigma \models b \le \alpha^H(e',g') : \tau \quad c \le (s \wedge 1) \cdot \Gamma \otimes \Delta \vdash \alpha(\mathbf{let}\ x = g\ \mathbf{in}\ g', f) : \tau}{(s \wedge 1) \cdot \Gamma \otimes \Delta \models (s \wedge 1)(a) \otimes b \otimes c \le \alpha^H(\mathbf{let}\ x = e\ \mathbf{in}\ e', f) : \tau} \text{ (H-let)}$$

$$\frac{\Gamma \models a \le \alpha^H(v,w) : \sigma \quad c \le s \cdot \Gamma \vdash \alpha(!w, z) : !_s\sigma}{s \cdot \Gamma \models s(a) \otimes c \le \alpha^H(!v, z) : !_s\sigma} \text{ (H-bang)}$$

$$\frac{\Gamma \models a \le \alpha^H(v,w) : !_r\sigma \quad \Delta, x :_{s\cdot r} \sigma \models b \le \alpha^H(e,g) : \tau \quad c \le s \cdot \Gamma \otimes \Delta \vdash \alpha(\mathbf{case}\ w\ \mathbf{of}\ \{!x \to g\}, f) : \tau}{s \cdot \Gamma \otimes \Delta \models s(a) \otimes b \otimes c \le \alpha^H(\mathbf{case}\ v\ \mathbf{of}\ \{!x \to e\}, f) : \tau} \text{ (H-bang-cases)}$$

$$\frac{\Gamma \models a \le \alpha^H(v,w) : \mu t.\sigma \quad \Delta, x :_s \sigma[\mu t.\sigma/t] \models b \le \alpha^H(e,g) : \tau \quad c \le s \cdot \Gamma \otimes \Delta \vdash \alpha(\mathbf{case}\ w\ \mathbf{of}\ \{\mathbf{fold}\ x \to g\}, f) : \tau}{s \cdot \Gamma \otimes \Delta \models s(a) \otimes b \otimes c \le \alpha^H(\mathbf{case}\ v\ \mathbf{of}\ \{\mathbf{fold}\ x \to e\}, f) : \tau} \text{ (H-fold-cases)}$$

$$\frac{\Gamma \models a \le \alpha^H(v,w) : \sigma[\mu t.\sigma/t] \quad c \le \Gamma \vdash \alpha(\mathbf{fold}\ w, z) : \mu t.\sigma}{\Gamma \models a \otimes c \le \alpha^H(\mathbf{fold}\ v, z) : \mu t.\sigma} \text{ (H-fold)}$$

$$\frac{\forall i \le n.\, \Gamma_i \models a_i \le \alpha^H(e_i, g_i) : \sigma \quad c \le op_{\mathsf{V}}(\Gamma_1 \ldots, \Gamma_n) \vdash \alpha(\mathbf{op}(g_1, \ldots, g_n), f) : \sigma}{op_{\mathsf{V}}(\Gamma_1, \ldots, \Gamma_n) \models op_{\mathsf{V}}(a_1, \ldots, a_n) \otimes c \le \alpha^H(\mathbf{op}(e_1, \ldots, e_n), f) : \sigma} \text{ (H-op)}$$

Figure 6. Howe's extension.

substitutive and thus value substitutive[8] we have:

$$\begin{aligned}\Gamma \vdash (\delta^o)^H(e,e) : \sigma &\le \bigwedge_{\vec{v}:\Gamma} \emptyset \vdash (\delta^o)^H(e[\vec{x} := \vec{v}], e'[\vec{x} := \vec{v}]) : \sigma \\ &\le \bigwedge_{\vec{v}} \delta^{\wedge}_{\sigma}(e[\vec{x} := \vec{v}], e'[\vec{x} := \vec{v}]) \\ &= \Gamma \vdash \delta^o(e,e') : \sigma.\end{aligned}$$

A similar argument holds for values. □

It is worth noticing that from our results directly follow the following generalisation of Reed's and Pierce's *metric preservation* [17, 41].

Corollary 1 (Metric Preservation (cf. [17]))**.** *For any environment* $\Gamma \triangleq x_1 :_{s_1} \sigma, \ldots, x_n :_{s_n} \sigma$*, values* $\vec{v}, \vec{w} : \Gamma$*, and* $\Gamma \vdash e : \sigma$ *we have:*

$$s_1 \circ \delta^{\mathcal{V}}_{\sigma_1}(v_1, w_1) \otimes \cdots \otimes s_n \circ \delta^{\mathcal{V}}_{\sigma_n}(v_n, w_n) \le \delta^{\wedge}_{\sigma}(e[\vec{x} := \vec{v}], e[\vec{x} := \vec{w}]).$$

Having proved that applicative Γ-similarity is a compatible generalised metric, we now move to applicative Γ-bisimilarity.

[8] Notice that in Definition 18 we substitute *closed* values (in terms and values) meaning that simultaneous substitution and sequential substitution coincide. In particular, value substitution implies e.g.

$$(\Gamma \vdash \alpha(e,f') : \tau) \le \bigwedge_{\vec{v}:\Gamma} \alpha^{\wedge}_{\tau}(e[\vec{x} := \vec{v}], f[\vec{x} := \vec{v}]).$$

.

7 Applicative Γ-bisimilarity

In previous section we proved that applicative Γ-similarity is a compatible generalised metric. However, in the context of programming language semantics it is often desirable to work with equivalence V-relations—i.e. pseudometrics. In this section we discuss two natural behavioural pseudometrics: applicative Γ-*bisimilarity* and two-way applicative Γ-similarity. We prove that under suitable conditions on CBEs (which are met by all examples we have considered so far) both applicative Γ-*bisimilarity* and two-way applicative Γ-similarity are compatible pseudometrics (V-equivalences). Proving compatibility of the latter is straightforward. However, proving compatibility of applicative Γ-bisimilarity is not trivial and requires a variation of the so-called *transitive closure trick* [26, 31, 39] based on ideas in [46].

Before entering formalities, let us remark that so far we have mostly worked with inequation and inequalities. That was fine since we have been interested in non-symmetric V-relations. However, for symmetric V-relations inequalities seem not to be powerful enough, and often plain equalities are needed in order to make proofs work. For that reason in the rest of this section we assume CBFs to be monotone *monoid (homo)morphism*. That is, we modify

Definition 4 requiring the equalities:

$$h(k) = \ell, \qquad h(a \otimes b) = h(a) \otimes h(b).$$

Note that we do not require CBEs to be join-preserving (i.e. continuous). We also require operations op_{V} to be *quantale (homo)morphism*, i.e. to preserves unit, tensor, and joins. It is easy to see that the new requirements are met by all examples considered so far. We start with two-way applicative Γ-similarity.

Proposition 7. *For a V-relator Γ define two-way applicative Γ-similarity as $\delta \otimes \delta^{\circ}$. Then two-way applicative Γ-similarity is a compatible V-equivalence.*

Proof sketch. Clearly $\delta \otimes \delta^{\circ}$ is symmetric. Moreover, since CBEs are monoid (homo)morphism it is also compatible. □

We now move to the more interesting case of applicative Γ-bisimilarity. In light of Example 8 we give the following definition.

Definition 19. *Recall Proposition 2. Define applicative* Γ-bisimilarity *γ as applicative $(\Gamma \wedge \Gamma^{\circ})$-similarity.*

Proposition 5 implies that γ is reflexive and transitive. Moreover, if CBEs preserve binary meet (a condition satisfied by all our examples), i.e. $s(a) \wedge s(b) = s(a \wedge b)$ for any CBE s in Π, then γ is also symmetric, ad thus a pseudometric. Finally we observe that γ is the greatest λ-term V-relation α such that both α and α° are applicative Γ-simulation.

Proving compatibility of γ is not straightforward, and requires a variation of the so-called *transitive closure trick* [39]. First of all we notice that we cannot apply the Key Lemma on γ since $\Gamma \wedge \Gamma^{\circ}$ being conversive is, in general, not inductive. To overcome this problem, we follow [46] and characterise applicative Γ-bisimilarity differently.

Proposition 8. *Let Γ be a V-relator. Define the λ-term V-relation γ' as follows:*

$$\gamma' \triangleq \bigvee \{\alpha \mid \alpha^{\circ} = \alpha,\ \alpha \leq [\alpha]\}.$$

Then:

1. *γ' is a symmetric applicative Γ-simulation, and therefore the largest such λ-term V-relation.*
2. *γ' coincide with applicative $(\Gamma \wedge \Gamma^{\circ})$-similarity γ.*

Proof. See Appendix A.5. □

Lemma 8 allows to apply the Key Lemma on γ, thus showing that γ^H is compatible. However, the Howe's extension is an intrinsically asymmetrical construction (cf. pseudo-transitivity) and there is little hope to prove symmetry of γ^H (which would imply compatibility of γ). Nevertheless, we observe that for a suitable class of CBEs the transitive closure $(\gamma^H)^T$ of γ^H is a symmetric, compatible, Γ-simulation (and thus smaller than γ).

Definition 20. *We say that a CBE s is* finitely continuous, *if $s \neq \infty$ implies $s(\bigvee A) = \bigvee \{s(a) \mid a \in A\}$, for any set $A \subseteq \mathsf{V}$.*

Example 15. All concrete CBEs considered in previous examples are finitely continuous. Moreover, it is easy to prove the all CBEs defined from the CBEs n, ∞ of Example 5 using operations in Lemma 1 are finitely continuous[9] provided that $op_{\mathsf{V}}(a_1, \ldots, \bot, \ldots, a_n) = \bot$ (which is the case for most of the concrete operations we considered).

[9] Recall that since a is integral we have the inequality $a \otimes \bot = \bot$ for any $a \in \mathsf{V}$.

The following is the central result of our argument (see Appendix A.5 for a proof).

Lemma 10. *Assume CBEs in Π to be finitely continuous. Define the transitive closure α^T of a V-relation α as $\alpha^T \triangleq \bigvee_n \alpha^{(n)}$, where $\alpha^{(0)} \triangleq id$, and $\alpha^{(n+1)} \triangleq \alpha^{(n)} \cdot \alpha$.*

1. *Let α be a reflexive and transitive λ-term V-relation. Then $(\alpha^H)^T$ is compatible.*
2. *Let α be an reflexive, symmetric, and transitive open λ-term V-relation. Then $(\alpha^H)^T$ is symmetric.*

Finally, we can prove that applicative Γ-bisimilarity is compatible.

Theorem 11. *If any CBE in Π is finitely continuous, then applicative Γ-bisimilarity is compatible.*

Proof. From Lemma 10 we know that $(\gamma^H)^T$ is compatible. Therefore it is sufficient to prove $((\gamma)^H)^T = \gamma$. One inequality follows from Lemma 6 as follows: $\gamma \leq \gamma^H \leq (\gamma)^T$. For the other inequality we rely on the coinduction proof principle associated with γ. As a consequence, it is sufficient to prove that $((\gamma)^H)^T$ is a symmetric applicative Γ-simulation. Symmetry is given by Lemma 10. From Key Lemma we know that γ^H is an applicative Γ-simulation. Since the identity λ-term V-relation is a applicative Γ-simulation and that the composition ofapplicative Γ-simulations is itself an applicative Γ-simulation (see the proof of Proposition 5) we see that $(\gamma^H)^T$ is itself an applicative Γ-simulation. □

Finally, we notice that all concrete CBEs considered in this work are finitely continuous. We can then rely on Theorem 11 to come up with concrete notions of compatible applicative Γ-bisimilarity. Notably, we obtain compatible pseudometrics for Fuzz[10] and *P*-Fuzz.

8 Further Developments

In Section 6 we proved that applicative Γ-similarity is a compatible V-peorder (i.e. a compatible generalised metric), whereas in Section 7 we proved that applicative Γ-bisimilarity (and two-way similarity) is a compatible V-equivalence (i.e. a compatible pseudometric) In this last section we shortly sketch a couple of further considerations on the results obtained in this work.

Contextual distances An issue that has not been touched concerns the quantitative counterpart of contextual preorder and contextual equivalence. Recently [12, 13] define a contextual distance δ^{ctx} for probabilistic λ-calculi as:

$$\delta^{ctx}(e, f) \triangleq \sup_C |\sum |C[e]| - \sum |C[f]||,$$

for contexts and terms of appropriate types. Taking into account sensitivity, and thus moving to *P*-Fuzz, such distance could be refined as

$$\delta^{ctx}(e, f) \triangleq \sup_C \frac{|\sum |C[e]| - \sum |C[f]||}{n_C},$$

where n_C is the sensitivity of C. Here some design choices are mandatory in order to deal with division by zero and infinity. Two immediate observations are that we would like

$$\frac{|\sum |C[e]| - \sum |C[f]||}{n_C}$$

[10] Formally, we should extend our definitions adding a basic type for real numbers and primitives for arithmetical operations, but that is straightforward.

to be 0 if $n_C = 0$ and that

$$\frac{|\sum |C[e]| - \sum |C[f]||}{n_C} = 0$$

if $n_C = \infty$. That means that we can restrict contexts to range over those with sensitivity different from 0 and ∞. In particular, excluding the latter means that we are considering finitely continuous CBEs. This observation (together with the fact that division is the right adjoint of multiplication) suggests a possible generalisation of the contextual distance to arbitrary quantales.

Informally, fixed a λ-term V-relation (i.e. a ground observation) α_o we can define the contextual distance α_o^{ctx} between two (appropriate) terms e, e' as:

$$\alpha_o^{ctx}(e, e') \triangleq \bigwedge_C s^*(\alpha_o(C[e], C[e'])),$$

where C ranges over contexts[11] with sensitivity s, and the latter is finitely continuous and different from ∞. We should also exclude the constantly k change of base functor. The map s^* is defined as the right adjoint of s which exists since s preserves arbitrary joints (see Proposition 7.34 in [16]).

Another possibility is to define α^{ctx} as the largest compatible and adequate V-relation, where adequacy is defined via the V-relation α_o. However, proving that such V-relation exists in general seems to be far from trivial. These difficulties seem to suggest that contrary to what happens when dealing with ordinary relations, a notion of contextual V-preorder/equivalence appears to be less natural than the notion of applicative Γ-(bi)similarity.

Combining Effects Our last observation concerns the applicability of the framework developed. In fact, all examples considered in this paper deal with calculi with just one kind of effects (e.g. probabilistic nondeterminism). However, we can apply the theory developed to combined effects as well. We illustrate this possibility by sketching how to add global states to P-Fuzz. Recall that the global state monad $\mathcal{G}$ is defined by $\mathcal{G}X \triangleq (S \times X)^S$ where $S = \{0,1\}^{\mathcal{L}}$ for a set of (public) location names $\mathcal{L}$. Such monad comes together with operation symbols for reading and writing locations: $\Sigma = \{\mathbf{get}, \mathbf{set}_{\ell:=0}, \mathbf{set}_{\ell:=1} \mid \ell \in \mathcal{L}\}$. The intended semantics of $\mathbf{get}(e, f)$ is to read the content of ℓ and to continue as e if the content is 0, otherwise continue as f. Dually, $\mathbf{set}_{\ell:=0}(e)$ (resp. $\mathbf{set}_{\ell:=1}(e)$) stores the bit 0 (resp. 1) in the location ℓ and then continues as e (see Example 1).

Our combination of global stores and probabilistic computations is based on the monad $\mathcal{G}_p X = (\mathcal{D}_\perp(S \times X))^S$. The unit η of the monad is defined by $\eta(x)(b) = |\langle b, x\rangle\rangle$, whereas the strong Kleisli extension $h^\sharp$ of $h : Z \times X \to (\mathcal{D}_\perp(S \times Y))^S$ is defined as follows: first we uncurry h (and apply some canonical isomorphisms) to obtain the function

$$h_u : Z \times (S \times X) \to \mathcal{D}_\perp(S \times Y).$$

We then define $h^\sharp$ by

$$h^\sharp(z, m)(b) = h_u^*(z, m(b)),$$

where $h_u^* : Z \times \mathcal{D}_\perp(S \times X) \to \mathcal{D}_\perp(S \times Y)$ is the strong Klesli extension of h_u with respect to $\mathcal{D}_\perp$. Easy calculations show that the triple $\langle \mathcal{G}_p, \eta, -^\sharp\rangle$ is indeed a strong Kleisli triple.

We now define a $[0,1]$-relator Γ for $\mathcal{G}_p$. Given $\alpha : X \twoheadrightarrow Y$, define

$$\Gamma\alpha(m, n) = \sup_{b \in S} W_\perp(id_S + \alpha)(m(b), n(b)).$$

Notice that $(id_S + \alpha)(\langle b, x\rangle, \langle b', x'\rangle) = 1$ if $b \neq b'$ and $\alpha(x, x')$ otherwise. It is relatively easy to prove that Γ satisfies conditions in Section 4. As an illustrative example we prove the following result.

Lemma 12. *The $[0,1]$-relator Γ satisfies condition* (Strong lax bind)*:*

$$\begin{array}{ccc} Z \times X & \xrightarrow{h} & \mathcal{G}_p X \\ {\scriptstyle \gamma+\alpha}\downarrow & \geq & \downarrow{\scriptstyle \Gamma\beta} \\ Z' \times X' & \xrightarrow[h']{} & \mathcal{G}_p Y' \end{array} \implies \begin{array}{ccc} Z \times \mathcal{G}_p X & \xrightarrow{h^\sharp} & \mathcal{G}_p Y \\ {\scriptstyle \gamma+\Gamma\alpha}\downarrow & \geq & \downarrow{\scriptstyle \Gamma\beta}. \\ Z' \times \mathcal{G}_p X' & \xrightarrow[h'^\sharp]{} & \mathcal{G}_p Y' \end{array}$$

Proof. Let us call (1) and (2) the right-hand side and left-hand side of the above implication, respectively. Moreover, we write α_S, β_S for $id_S + \alpha, id_S + \beta$, respectively. Then:

$$(1) \implies \begin{array}{ccc} Z \times (S \times X) & \xrightarrow{f_u} & \mathcal{D}_\perp(S \times Y) \\ {\scriptstyle \gamma+\alpha_S}\downarrow & \geq & \downarrow{\scriptstyle W_\perp\beta_S} \\ W \times (S \times U) & \xrightarrow[g_u]{} & \mathcal{D}_\perp(S \times V) \end{array}$$

$$\implies \begin{array}{ccc} Z \times \mathcal{D}_\perp(S \times X) & \xrightarrow{f_u^*} & \mathcal{D}_\perp(S \times Y) \\ {\scriptstyle \gamma+W_\perp\alpha_S}\downarrow & \geq & \downarrow{\scriptstyle W_\perp\beta_S} \\ W \times \mathcal{D}_\perp(S \times U) & \xrightarrow[g_u^*]{} & \mathcal{D}_\perp(S \times V) \end{array}$$

$$\implies (2).$$

□

By Theorem 9 we thus obtain a notion of applicative Γ-similarity which is a compatible generalised metric. Since CBEs in P-Fuzz are finitely continuous we can also apply results from Section 7 to obtain a compatible pseudometric.

9 Related Work

Several works have been done in the past years on quantitative (metric) reasoning in the context of programming language semantics. In particular, several authors have used (cartesian) categories of *ultrametric spaces* as a foundation for denotational semantics of both concurrent [3, 18] and sequential programming languages

[11] Give a formal definition of V-Fuzz/ requires some (tedious) work. In fact, contexts should be terms with a hole $[-]$ to be filled in with another *term* of appropriate type. However, due to the fine-grained nature of V-Fuzz, we defined substitution of values only. Therefore, what we should do is to define a grammar and a notion of substitution for contexts. Moreover, we should also design a type system for contexts keeping track of sensitivities (see e.g. [10] for the relational case). This is a tedious exercise but can be done without difficulties. Here we simply notice that it is possible to 'simulate' contexts as follows. Let $\emptyset \vdash^{\mathsf{v}} * : \mathsf{unit}$ be the unit value. Suppose we want to come up with a (closed) context $C[-]$ of type τ and sensitivity s taking as input terms of type σ. For that we consider the term (for readability we annotate the lambda):

$$\lambda y : !_s\mathsf{unit} \multimap \sigma.\mathbf{case}\ y\ \mathbf{of}\ \{!x \to C[y*]\}$$

where y is a fresh variable. To substitute a term e of type σ in C we first thunk it to $\lambda.e \in \mathsf{unit} \multimap \sigma$ and then consider:

$$(\lambda y : !_s\mathsf{unit} \multimap \sigma.\mathbf{case}\ y\ \mathbf{of}\ \{!x \to C[y*]\})(!\lambda.e)$$

It is immediate to see that $|(\lambda y.\mathbf{case}\ y\ \mathbf{of}\ \{!x \to C[y*]\})(!\lambda.e)|$ captures $|C[e]|$ (although the expression has not been defined). Moreover, an easy calculation shows that for any compatible λ-term V-relation α, and for all terms e, e' of type σ we have:

$$\begin{aligned} & s \circ \alpha_\sigma(e, e') \\ & \leq \alpha_\tau((\lambda y.\mathbf{case}\ y\ \mathbf{of}\ \{!x \to C[y*]\})(!\lambda.e), (\lambda y.\mathbf{case}\ y\ \mathbf{of}\ \{!x \to C[y*]\})(!\lambda.e')). \end{aligned}$$

[19]. A different approach is investigated in [17] where a denotational semantics combining ordinary metric spaces and domains is given to *pure* (i.e. without effects) Fuzz. The main theorem of [17] is a denotational version of the so-called *metric preservation* [41] (whose original proof requires the introduction of a suitable *step-indexed metric logical relation*). Our Corollary 1 is the operational counterpart of such result generalised to arbitrary algebraic effects.

A different, although deeply related, line of research has been recently proposed in [12, 13] where coinductive, operationally-based distances have been studied for probabilistic λ-calculi. In particular, in [12] a notion of applicative distance based on the Wasserstein lifting is proposed for a probabilistic *affine* λ-calculus. Restricting to affine programs only makes the calculus strongly normalising and remove copying capabilities of programs by construction. In this way programs cannot amplify distances between their inputs and therefore are forced to behave as non-expansive functions. This limitation is overcame in [13], where a coinductive notion of distance is proposed for a full linear λ-calculus, and distance trivialisation phenomena are studied in depth. The price to pay for such generality is that the distance proposed is not applicative, but a trace distance somehow resembling environmental bisimilarity [45].

10 Conclusion

In this work we have introduced an abstract framework for studying quantale-valued behavioural relations for higher-order effectful languages. Such framework has been instantiated to define the quantitative refinements of Abramsky's applicative similarity and bisimilarity for V-Fuzz, a universal λ-calculus with a linear type system tracking program sensitivity enriched with algebraic effects. Our main theorems state that under suitable conditions the quantitative notions of applicative similarity and bisimilarity obtained are a compatible generealised metric and pseudometric, respectively. These results can be instantiated to obtain compatible pseudometrics for several concrete calculi.

A future research direction is to study how the abstract framework developed can be used to investigate quantitative refinements of behavioural relations different from applicative (bi)similarity. In particular, investigating contextual distances (see [20] for some preliminary observations), denotationally-based distances (along the lines of [17]), and distances based on suitable logical relations (such as the one in [41]) are interesting topics for further research.

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A Appendix: Technical Development

This appendix provides proofs of propositions and lemmas stated in the main body of this paper.

A.1 Proofs of Section 3

Lemma 3. *For any $e \in \Lambda_\sigma$, we have $|e|^\sigma_n \sqsubseteq_{\mathcal{V}_\sigma} |e|^\sigma_{n+1}$, for any $n \geq 0$.*

Proof. By induction on n. We show the case for sequential composition. We have to prove $|\mathbf{let}\ x = e\ \mathbf{in}\ f|_{n+1} \sqsubseteq |\mathbf{let}\ x = e\ \mathbf{in}\ f|_{n+2}$ (for readability we omit subscripts). By definition of $|-|_n$ we have:

$$|\mathbf{let}\ x = e\ \mathbf{in}\ f|_{n+1} = |f[x := -]|^*_n(|e|_n),$$
$$|\mathbf{let}\ x = e\ \mathbf{in}\ f|_{n+2} = |f[x := -]|^*_{n+1}(|e|_{n+1}).$$

By induction hypothesis, for any closed value v of the appropriate type we have the inequality $|f[x := v]|_n \sqsubseteq |f[x := v]|_{n+1}$, from which follows $|f[x := -]|_n \sqsubseteq |f[x := -]|_{n+1}$. By ω-cppo enrichment the latter implies $|f[x := -]|^*_n \sqsubseteq |f[x := -]|^*_{n+1}$. Finally, by induction hypothesis we have $|e|_n \sqsubseteq |e|_{n+1}$, so that we can conclude the thesis as follows[12]:

$$|f[x := -]|^*_n(|e|_n) \sqsubseteq |f[x := -]|^*_{n+1}(|e|_n) \sqsubseteq |f[x := -]|^*_{n+1}(|e|_{n+1}).$$

□

A.2 Proofs of Section 4

Proposition 2. *Let T, U be set endofunctors. Then:*

1. *If Γ and Δ are V-relators for T and U, respectively, then $\Delta \cdot \Gamma$ defined by $(\Delta \cdot \Gamma)\alpha \triangleq \Delta\Gamma\alpha$ is a V-relator for UT.*
2. *If $\{\Gamma\}_{i\in I}$ is a family of V-relators for T, then $\bigwedge_{i\in I}\Gamma_i$ defined by $(\bigwedge_{i\in I}\Gamma_i)\alpha \triangleq \bigwedge_{i\in I}\Gamma_i\alpha$ is a V-relator for T.*
3. *If Γ is a V-relator for T, then Γ° defined by $\Gamma^\circ\alpha \triangleq (\Gamma\alpha^\circ)^\circ$ is a V-relator for T.*
4. *For any V-relator Γ, $\Gamma \wedge \Gamma^\circ$ is the greatest conversive V-relator smaller than Γ.*

Proof. The proof consists of a number of straightforward calculations. As an example, we show that $\bigwedge_{i\in I}\Gamma_i$ in point 2 satisfies condition (V-rel 2). Concretely, we have to prove

$$\bigwedge_{i\in I}\Gamma_i\beta \cdot \bigwedge_{i\in I}\Gamma_i\alpha \leq \bigwedge_{i\in I}\Gamma_i(\beta\cdot\alpha).$$

For that it is sufficient to prove that for any $j \in I$ we have:

$$\bigwedge_{i\in I}\Gamma_i\beta \cdot \bigwedge_{i\in I}\Gamma_i\alpha \leq \Gamma_j(\beta\cdot\alpha).$$

Observe that we have $\bigwedge_{i\in I}\Gamma_i\beta \leq \Gamma_j\beta$ and $\bigwedge_{i\in I}\Gamma_i\alpha \leq \Gamma_j\alpha$, so that by monotonicity of composition (recall that V-**Rel** is a quantaloid) we infer $\bigwedge_{i\in I}\Gamma_i\beta \cdot \bigwedge_{i\in I}\Gamma_i\alpha \leq \Gamma_j\beta \cdot \Gamma_j\alpha$. The thesis now follows from (V-rel 2). □

[12] Note that by ω-cppo-enrichment f^* is monotone, for any $f : X \to TY$. Let $t, u : Z \to TX$ with $t \sqsubseteq u$, i.e. $u = \bigsqcup\{t, u\}$. Then:

$$f^* \cdot u = f^* \cdot \bigsqcup\{t, u\} = \bigsqcup\{f^* \cdot t, f^* \cdot u\}$$

holds, i.e. $f^* \cdot t \sqsubseteq f^* \cdot u$. This specialises to usual pointwise monotonicity, by taking $t, u : 1 \to TX$.

Proposition 3. *Let $\mu \in \mathcal{D}(X), \nu \in \mathcal{D}(Y)$ be countable distributions and $\alpha : X \nrightarrow Y$ be a $[0,1]$-relation. Then:*

$$\begin{aligned} W\alpha(\mu,\nu) &= \min\{\sum_{x,y} \alpha(x,y)\cdot\omega(x,y) \mid \omega \in \Omega(\mu,\nu)\} \\ &= \max\{\sum_x a_x\cdot\mu(x) + \sum_y b_y\cdot\nu(y) \\ &\qquad \mid a_x + b_y \le \alpha(x,y), a_x, b_y \text{ bounded}\}, \end{aligned}$$

where a_x, b_y bounded means that there exist $\bar{a}, \bar{b} \in \mathbb{R}$ such that $\forall x.\ a_x \le \bar{a}$, and $\forall y.\ b_y \le \bar{b}$.

Proof. The proof is a direct consequence of the following duality theorem for countable transportation problems [29].

Fact 1. *Let $i, j, \ldots$ range over natural numbers. Let m_i, n_j, c_{ij} be non-negative real number, for all i, j. Define*

$$M \triangleq \inf\{\sum_{i,j} c_{ij}x_{ij} \mid x_{ij} \ge 0, \sum_j x_{ij} = m_i, \sum_i x_{ij} = n_j,\}$$
$$M^* \triangleq \sup\{\sum_i m_i a_i + \sum_j n_j b_j \mid a_i + b_j \le c_{ij}, a_i, b_j \text{ bounded}\}.$$

where a_i, b_j bounded means that there exist $\bar{a}, \bar{b} \in \mathbb{R}$ such that $a_i \le \bar{a}$, and $b_j \le \bar{b}$, for all i, j. Then the following hold:

1. *$M = M^*$.*
2. *The linear problem P induced by M has optimal solution.*
3. *The linear problem P^* induced by M^* has optimal solution.*

Now, we first of all notice that $\Gamma\alpha(\mu,\nu)$ is nothing but

$$\begin{aligned} \inf\{&\sum_{x,y} \alpha(x,y)\cdot\omega(x,y) \\ &\mid \omega(x,y) \ge 0, \sum_y \omega(x,y) = \mu(x), \sum_x \omega(x,y) = \nu(y)\}. \end{aligned}$$

In fact, $\omega(x,y) \ge 0$, $\sum_y \omega(x,y) = \mu(x)$ and $\sum_x \omega(x,y) = \nu(y)$ imply $\omega \in \mathcal{D}(X \times Y)$. Moreover, since α is a $[0,1]$-relation, $\alpha(x,y) \in [0,1]$ (recall that Fact 1 requires c_{ij} to be a non-negative real number). We conclude the thesis by Fact 1. In particular, it follows that there exists $\omega \in \Omega(\mu,\nu)$ such that:

$$W\alpha(\mu,\nu) = \sum_{x,y} \alpha(x,y)\cdot\omega(x,y).$$

Since $\alpha(x,y), \omega(x,y) \in [0,1]$ we have $\alpha(x,y)\cdot\omega(x,y) \le \omega(x,y)$, for all x, y. It follows

$$0 \le \sum_{x,y} \alpha(x,y)\cdot\omega(x,y) \le \sum_{x,y} \omega(x,y) = 1$$

so that $W\alpha$ is indeed a $[0,1]$-relation. □

Proposition 9. *Wasserstein lifting W satisfies conditions in Definition 8.*

Proof. We start by showing that W satisfies condition (Lax unit). Let $|x\rangle$ denotes the Dirac distribution on x. We have to show that for any $z \in X, w \in Y$, $\alpha(z,w) \ge W\alpha(|z\rangle, |w\rangle)$ holds. By duality (Proposition 3) we have:

$$W\alpha(|z\rangle,|w\rangle) = \max\{\sum_x a_x\cdot|z\rangle(x) + \sum_y b_y\cdot|w\rangle(y) \mid a_x + b_y \le \alpha(x,y)\},$$

where a_x, b_y are bounded. Clearly $W\alpha(|z\rangle,|w\rangle) = a_x + b_y$, for suitable $x \in X$ and $y \in Y$. Since $a_x + b_y \le \alpha(x,y)$ we are done.

We now observe that condition (L-Strong lax bind) can actually be split in two different conditions:

$$\Gamma(s\circ\alpha) = s\circ\Gamma\alpha, \tag{L-dist}$$
$$\gamma\otimes(s\circ\alpha) \le g^\circ\cdot\Gamma\beta\cdot f \implies \gamma\otimes(s\circ\Gamma\alpha) \le (g^*)^\circ\cdot\Gamma\beta\cdot f^*, \tag{Strong lax bind}$$

where $s \le 1$. In particular, we can write condition (Strong lax bind) as follows:

$$\begin{array}{ccc} Z\times X & \xrightarrow{f} & TY \\ {\scriptstyle\gamma\otimes\alpha}\downarrow & \le & \downarrow{\scriptstyle\Gamma\beta} \\ Z'\times X' & \xrightarrow[g]{} & TY' \end{array} \implies \begin{array}{ccc} Z\times TX & \xrightarrow{f^*} & TY \\ {\scriptstyle\gamma\otimes\Gamma\alpha}\downarrow & \le & \downarrow{\scriptstyle\Gamma\beta}. \\ Z'\times TX' & \xrightarrow[g^*]{} & TY' \end{array}$$

(notice that the latter, together with condition (Lax unit), is equivalent to stating non-expansiveness of unit, multiplication, and strength of $\mathbb{T}$).

Proving that W satisfies condition (L-dist) is straightforward. We prove it satisfies condition (Strong lax bind). Concretely, we have to prove the following implication:

$$\begin{array}{ccc} U\times X & \xrightarrow{f} & \mathcal{D}Z \\ {\scriptstyle\gamma+\alpha}\downarrow & \ge & \downarrow{\scriptstyle W\beta} \\ V\times Y & \xrightarrow[g]{} & \mathcal{D}W \end{array} \implies \begin{array}{ccc} U\times\mathcal{D}X & \xrightarrow{f^*} & \mathcal{D}Z \\ {\scriptstyle\gamma+W\alpha}\downarrow & \ge & \downarrow{\scriptstyle W\beta}. \\ V\times\mathcal{D}Y & \xrightarrow[g^*]{} & \mathcal{D}W \end{array}$$

We show that for any $u \in U, v \in V, \mu \in \mathcal{D}X, \nu \in \mathcal{D}Y$ we have:

$$W\beta(f^*(u,\mu), g^*(v,\nu)) \le \gamma(u,v) + W\alpha(\mu,\nu).$$

(note that in the right hand side of the above equations we can assume without loss of generality to have ordinary addition in place of a truncated sum). By very definition of strong Kleisli extension we have:

$$f^*(u,\mu)(z) = \sum_x \mu(x)\cdot f(u,x)(z),$$
$$g^*(v,\nu)(w) = \sum_y \nu(y)\cdot g(v,y)(w).$$

Let $M \triangleq W\beta(f^*(u,\mu), g^*(v,\nu))$. By duality we have:

$$\begin{aligned} M = \max\{&\sum_z a_z\cdot\sum_x \mu(x)\cdot f(u,x)(z) \\ &+ \sum_w b_w\cdot\sum_y \nu(y)\cdot g(v,y)(w) \\ &\mid a_z + b_w \le \beta(z,w)\}, \end{aligned}$$

where a_z and b_w are bounded. By Proposition 3 there exists an $\omega \in \Omega(\mu,\nu)$ such that $W\alpha(\mu,\nu) = \sum_{x,y} \omega(x,y)\cdot\alpha(x,y)$. We have to prove:

$$M \le \gamma(u,v) + \sum_{x,y} \omega(x,y)\cdot\alpha(x,y).$$

From $\omega \in \Omega(\mu,\nu)$ we obtain $\mu(x) = \sum_y \omega(x,y)$, $\nu(y) = \sum_x \omega(x,y)$. We apply the above equalities to M, obtaining (for readability we

omit the constraint $a_z + b_w \leq \beta(z,w)$):

$$\begin{aligned}
M &= \max\{\sum_z a_z \cdot \sum_x \mu(x) \cdot f(u,x)(z) \\
&\quad + \sum_w b_w \cdot \sum_y \nu(y) \cdot g(v,y)(w)\} \\
&= \max\{\sum_z a_z \cdot \sum_{x,y} \omega(x,y) \cdot f(u,x)(z) \\
&\quad + \sum_w b_w \cdot \sum_{x,y} \omega(x,y) \cdot g(v,y)(w)\} \\
&= \max\{\sum_{x,y} \omega(x,y)(\sum_z a_z \cdot f(u,x)(z) \\
&\quad + \sum_w b_w \cdot g(v,y)(w))\} \\
&= \sum_{x,y} \omega(x,y) \cdot \max\{\sum_z a_z \cdot f(u,x)(z) \\
&\quad + \sum_w b_w \cdot g(v,y)(w)\} \\
&= \sum_{x,y} \omega(x,y) \cdot W\beta(f(u,x), g(v,y)).
\end{aligned}$$

We are now in position to use our hypothesis, namely the inequality:

$$W\beta(f(u,x), g(v,y)) \leq \gamma(u,v) + \alpha(f(u,x), g(v,y))$$

(note that the hypothesis we have is actually stronger, since it gives an inequality for truncated addition). We conclude:

$$\begin{aligned}
M &\leq \sum_{x,y} \omega(x,y) \cdot (\gamma(u,v) + \alpha(x,y)) \\
&= \sum_{x,y} \omega(x,y) \cdot \gamma(u,v) + \sum_{x,y} \omega(x,y) \cdot \alpha(x,y) \\
&= \gamma(u,v) + \sum_{x,y} \omega(x,y) \cdot \alpha(x,y)
\end{aligned}$$

(where in the last equality we used the fact that $\omega(x,y) \in \Omega(\mu,\nu)$ implies $\sum_{x,y} \omega(x,y) = 1$). We are done. □

Proposition 10. *Wasserstein lifting $W_\perp$ satisfy conditions in Definition 8.*

Proof. Showing that $W_\perp$ satisfies conditions (Lax unit) and (L-dist) is straightforward (but notice that for the latter we need the hypothesis $s \leq 1$). We prove it satisfies condition (Strong lax bind) as well. First of all define for $f : U \times X \to \mathcal{D}(Y_\perp)$ the map $f^\perp : U \times X_\perp \to \mathcal{D}(Y_\perp)$ by:

$$\begin{aligned}
f^\perp(u, \perp_X) &\triangleq |\perp_Y\rangle, \\
f^\perp(u,x) &\triangleq f(u,x).
\end{aligned}$$

We see that the Kleisli extension f^* with respect to the subdistribution monad $\mathcal{D}_{\leq 1}$ of $f : U \times X \to \mathcal{D}(Y_\perp)$ is equal to $f^{\perp\sharp}$, where $-^\sharp$ denotes the (strong) Kleisli extension with respect to the (full) distribution monad. Moreover, we have the following implication:

$$\begin{array}{ccc} U \times X & \xrightarrow{f} & \mathcal{D}Z_\perp \\ {\scriptstyle \gamma+\alpha}\downarrow & \geq & \downarrow{\scriptstyle W_\perp\beta} \\ V \times Y & \xrightarrow[g]{} & \mathcal{D}W_\perp \end{array} \implies \begin{array}{ccc} U \times X_\perp & \xrightarrow{f^\perp} & \mathcal{D}Z_\perp \\ {\scriptstyle \gamma+\alpha_\perp}\downarrow & \geq & \downarrow{\scriptstyle W_\perp\beta}. \\ V \times Y_\perp & \xrightarrow[g^\perp]{} & \mathcal{D}W_\perp \end{array}$$

Proving $W_\perp(f^\perp(u,\chi), g^\perp(v,y)) \leq \gamma(u,v) + \alpha_\perp(\chi,y)$ is trivial except if $\chi = \perp$, meaning that $f^\perp(u,\chi) = |\perp_Z\rangle$. In that case we observe that for any distribution $\nu \in \mathcal{D}(Y_\perp)$ and $[0,1]$-relation $\alpha : X \nrightarrow Y$ we have $W_\perp(|\perp_X\rangle, \nu) = 0$. Consider an expression of the form

$$\sum_{(\chi,y) \in X_\perp \times Y_\perp} \omega(\chi,y) \cdot \alpha_\perp(\chi,y),$$

where $\omega \in \Omega(|\perp_X\rangle, \nu)$. We can expand such expression as:

$$\begin{aligned}
&\sum_{(x,y) \in X \times Y} \omega(x,y) \cdot \alpha_\perp(x,y) + \sum_{x \in X} \omega(x, \perp_Y) \cdot \alpha_\perp(x, \perp_Y) + \\
&\quad \sum_{y \in Y} \omega(\perp_X, y) \cdot \alpha_\perp(\perp_X, y) + \omega(\perp_X, \perp_Y) \cdot \alpha_\perp(\perp_X, \perp_Y).
\end{aligned}$$

By very definition of $\alpha_\perp$ the latter reduces to:

$$\sum_{(x,y) \in X \times Y} \omega(x,y) \cdot \alpha(x,y) + \sum_{x \in X} \omega(x, \perp_Y).$$

Since $\omega \in \Omega(|\perp_X\rangle, \nu)$ we have $\sum_{y \in Y_\perp} \omega(\chi,y) = |\perp_X\rangle(\chi)$, meaning that for any $x \in X$ and $y \in Y_\perp$, $\omega(x,y) = 0$. We can conclude $W_\perp(|\perp_X\rangle, \nu) = 0$.

Finally, since $W_\perp$ satisfies condition (Strong lax bind) we can infer the desired thesis as follows:

$$\begin{aligned}
\gamma + \alpha \geq g^\circ \cdot W_\perp\beta \cdot f &\implies \gamma + \alpha_\perp \geq (g^\perp)^\circ \cdot W_\perp\beta \cdot f^\perp \\
&\iff \gamma + \alpha_\perp \geq (g^\perp)^\circ \cdot W\beta_\perp \cdot f^\perp \\
&\implies \gamma + W\alpha_\perp \geq (g^{\perp\sharp})^\circ \cdot W\beta_\perp \cdot f^{\perp\sharp} \\
&\iff \gamma + W_\perp\alpha \geq (g^*)^\circ \cdot W_\perp\beta \cdot f^*.
\end{aligned}$$

□

A.3 Proofs of Section 5

Lemma 13. *For all subdistributions $\mu \in \mathcal{D}(X_\perp)$ and $\nu \in \mathcal{D}(Y_\perp)$, and $[0,1]$-relation (with respect to the unit interval quantale) $\alpha : X \nrightarrow Y$, we have:*

$$\sum \mu - \sum \nu \leq W_\perp\alpha(\mu,\nu),$$

where $\sum \mu$ denotes the 'probability of convergence' of μ defined by $\sum \mu \triangleq \sum_{x \in X} \mu(x)$ (and similarity for ν), and $-$ denotes truncated subtraction.

Proof. We have:

$$\begin{aligned}
W_\perp\alpha(\mu,\nu) = \max\{&\sum_x a_x \cdot \mu(x) + a_{\perp_X} \cdot \mu(\perp_X) \\
&+ \sum_y b_y \cdot \nu(y) + b_{\perp_Y} \cdot \nu(\perp_Y)\},
\end{aligned}$$

where $a_x, a_{\perp_X}, b_y, b_{\perp_Y}$ are bounded and satisfy the following constraints (already simplified according to the definition of $\alpha_\perp$):

$$\begin{aligned}
a_x + b_y &\leq \alpha(x,y), & a_{\perp_X} + b_y &\leq 0, \\
a_x + b_{\perp_Y} &\leq 1, & a_{\perp_X} + b_{\perp_Y} &\leq 0.
\end{aligned}$$

Choosing $a_x \triangleq 1$, $b_y \triangleq -1$, $a_{\perp_X} \triangleq b_{\perp_Y} \triangleq 0$ we obtain the desired inequality. □

Proposition 5. *Applicative Γ-similarity δ is a reflexive and transitive λ-term V-relation.*

Proof. The proof is by coinduction. Let us show that δ is transitive, i.e. that $\delta \cdot \delta \leq \delta$. We prove that the λ-term V-relation $(\delta^\Lambda \cdot \delta^\Lambda, \delta^\mathsf{V} \cdot \delta^\mathsf{V})$ is an applicative Γ-simulation. We split the proof into five cases:

1. We show that for all terms $e, f \in \Lambda_\sigma$ we have:

$$\bigvee_{g \in \Lambda_\sigma} \delta^\Lambda_\sigma(e,g) \otimes \delta^\Lambda_\sigma(g,f) \leq \Gamma(\delta^\mathcal{V}_\sigma \cdot \delta^\mathcal{V}_\sigma)(|e|,|f|).$$

   By (V-rel 2) it is sufficient to prove:

$$\bigvee_{g \in \Lambda_\sigma} \delta^\Lambda_\sigma(e,g) \otimes \delta^\Lambda_\sigma(g,f) \leq \bigvee_{\mathcal{V} \in T\mathcal{V}_\sigma} \Gamma\delta^\mathcal{V}_\sigma(|e|,\mathcal{V}) \otimes \Gamma\delta^\mathcal{V}_\sigma(\mathcal{V},|f|).$$

   For any $g \in \Lambda_\sigma$ instantiate $\mathcal{V}$ as $|g|$. Since $\delta^\Lambda_\sigma(e,g) \leq \Gamma\delta^\mathcal{V}_\sigma(|e|,|g|)$ and $\delta^\Lambda_\sigma(g,f) \leq \Gamma\delta^\mathcal{V}_\sigma(|g|,|f|)$, we are done by very definition of δ.
2. We prove that

$$(\delta^\mathcal{V}_{\sigma\multimap\tau} \cdot \delta^\mathcal{V}_{\sigma\multimap\tau})(v,w) \leq \bigwedge_{u \in \mathcal{V}_\sigma} (\delta^\Lambda_\tau \cdot \delta^\Lambda_\tau)(vu, wu)$$

   holds for all values $v, w \in \mathcal{V}_{\sigma\multimap\tau}$. For that it is sufficient to prove that for any $u \in \mathcal{V}_\sigma$ and for any $z \in \mathcal{V}_{\sigma\multimap\tau}$ there exists a term $e \in \Lambda_\tau$ such that:

$$\delta^\mathcal{V}_{\sigma\multimap\tau}(v,z) \otimes \delta^\mathcal{V}_{\sigma\multimap\tau}(z,w) \leq \delta^\Lambda_\tau(vu,e) \otimes \delta^\Lambda_\tau(e,wu).$$

   By very definition of $\delta^\mathcal{V}_{\sigma\multimap\tau}$ we have:

$$\begin{aligned}\delta^\mathcal{V}_{\sigma\multimap\tau}(v,z) &\otimes \delta^\mathcal{V}_{\sigma\multimap\tau}(z,w)\\ &\leq \bigwedge_{u' \in \mathcal{V}_\sigma} \delta^\Lambda_\tau(vu', zu') \otimes \bigwedge_{u' \in \mathcal{V}_\sigma} \delta^\Lambda_\tau(zu', wu')\\ &\leq \delta^\Lambda_\tau(vu, zu) \otimes \delta^\Lambda_\tau(zu, wu),\end{aligned}$$

   so that it is sufficient to instantiate e as zu.
3. We prove that

$$\begin{aligned}(\delta^\mathcal{V}_{\sum_{i\in I}\sigma_i} \cdot \delta^\mathcal{V}_{\sum_{i\in I}\sigma_i})(\langle \hat{\imath}, v\rangle, \langle \hat{\jmath}, u\rangle) &\leq \bot\!\!\!\bot,\\ (\delta^\mathcal{V}_{\sum_{i\in I}\sigma_i} \cdot \delta^\mathcal{V}_{\sum_{i\in I}\sigma_i})(\langle \hat{\imath}, v\rangle, \langle \hat{\imath}, w\rangle) &\leq (\delta^\mathcal{V}_{\sigma_{\hat{\imath}}} \cdot \delta^\mathcal{V}_{\sigma_{\hat{\imath}}})(v,w),\end{aligned}$$

   hold for all $v, w \in \mathcal{V}_{\sigma_{\hat{\imath}}}$ and $u \in \mathcal{V}_{\sigma_{\hat{\jmath}}}$, with $\hat{\imath} \neq \hat{\jmath}$. We have

$$\begin{aligned}&(\delta^\mathcal{V}_{\sum_{i\in I}\sigma_i} \cdot \delta^\mathcal{V}_{\sum_{i\in I}\sigma_i})(\langle \hat{\imath}, v\rangle, \langle \hat{\jmath}, u\rangle)\\ &= \bigvee_{\langle \hat{\ell}, z\rangle \in \mathcal{V}_{\sum_{i\in I}\sigma_i}} \delta^\mathcal{V}_{\sum_{i\in I}\sigma_i}(\langle \hat{\imath}, v\rangle, \langle \hat{\ell}, z\rangle) \otimes \delta^\mathcal{V}_{\sum_{i\in I}\sigma_i}(\langle \hat{\ell}, z\rangle, \langle \hat{\jmath}, w\rangle).\end{aligned}$$

   Since $\hat{\imath} \neq \hat{\jmath}$ at least one among $\hat{\imath} \neq \hat{\ell}$ and $\hat{\ell} \neq \hat{\jmath}$ holds, for any $\langle \hat{\ell}, z\rangle \in \mathcal{V}_{\sum_{i\in I}\sigma_i}$. As a consequence, by very definition of δ, the right hand side of the above inequality is equal to something of the form $\bot\!\!\!\bot \otimes a$, which is itself equal to $\bot\!\!\!\bot$. To prove the second inequality, we have to show that for any $\langle \hat{\imath}, u\rangle \in \mathcal{V}_{\sum_{i\in I}\sigma_i}$ there exists $z \in \mathcal{V}_{\sigma_{\hat{\imath}}}$ such that

$$\begin{aligned}\delta^\mathcal{V}_{\sum_{i\in I}\sigma_i}(\langle \hat{\imath}, v\rangle, \langle \hat{\imath}, u\rangle) &\otimes \delta^\mathcal{V}_{\sum_{i\in I}\sigma_i}(\langle \hat{\imath}, u\rangle, \langle \hat{\imath}, w\rangle)\\ &\leq \delta^\mathcal{V}_{\sigma_{\hat{\imath}}}(v,z) \otimes \delta^\mathcal{V}_{\sigma_{\hat{\imath}}}(z,w).\end{aligned}$$

   Notice that for a value $\langle \hat{\jmath}, u\rangle \in \mathcal{V}_{\sum_{i\in I}\sigma_i}$ with $\hat{\jmath} \neq \hat{\imath}$ we would have, by very definition of δ, $\delta^\mathcal{V}_{\sum_{i\in I}\sigma_i}(\langle \hat{\imath}, v\rangle, \langle \hat{\jmath}, u\rangle) = \bot\!\!\!\bot$, and thus we would be trivially done. Proving the above inequality is straightforward: simply instantiate z as u and observe that by definition of δ we have

$$\begin{aligned}\delta^\mathcal{V}_{\sum_{i\in I}\sigma_i}(\langle \hat{\imath}, v\rangle, \langle \hat{\imath}, u\rangle) &\leq \delta^\mathcal{V}_{\sigma_{\hat{\imath}}}(v,u),\\ \delta^\mathcal{V}_{\sum_{i\in I}\sigma_i}(\langle \hat{\imath}, u\rangle, \langle \hat{\imath}, w\rangle) &\leq \delta^\mathcal{V}_{\sigma_{\hat{\imath}}}(u,w).\end{aligned}$$

4. The case for $\mu t.\sigma$ follows the same pattern of the above one.
5. We prove:

$$(\delta^\mathcal{V}_{!_s\sigma} \cdot \delta^\mathcal{V}_{!_s\sigma})(!v, !w) \leq s \circ (\delta^\mathcal{V}_\sigma \cdot \delta^\mathcal{V}_\sigma)(v,w).$$

   For that we notice that for every $!u \in !_s\sigma$ we have:

$$\begin{aligned}\delta^\mathcal{V}_{!_s\sigma}(!v,!u) \otimes \delta^\mathcal{V}_{!_s\sigma}(!u,!w) &\leq (s \circ \delta^\mathcal{V}_\sigma)(v,u) \otimes (s \circ \delta^\mathcal{V}_\sigma)(u,w)\\ &\leq ((s \circ \delta^\mathcal{V}_\sigma) \cdot (s \circ \delta^\mathcal{V}_\sigma))(v,w)\\ &\leq s \circ (\delta^\mathcal{V}_\sigma \cdot \delta^\mathcal{V}_\sigma)(v,w).\end{aligned}$$

□

Proposition 6. *Define applicative Δ_Γ-similarity $\preceq$ by instantiating Definition 13 with the 2-relator Δ_Γ and replacing the clause for types of the form $!_s\sigma$ as follows: $!v\ \mathcal{R}_{!_s\sigma}\ !w$ implies $(\varphi \cdot s \cdot \psi) \circ \mathcal{R}_\sigma(v,w)$. Then the kernel $\varphi \circ \delta$ of δ coincide with $\preceq$.*

Proof. The proof is by coinduction. We start proving that $\varphi \circ \delta$ is an applicative Δ_Γ-simulation. Since $\delta^\Lambda_\sigma(e,f) \leq \Gamma\delta^\mathcal{V}_\sigma(|e|,|f|)$ holds for all terms $e, f \in \Lambda_\sigma$, we can apply Lemma 4 and infer the inequality $\varphi \circ \delta^\Lambda_\sigma(e,f) \leq \Delta_\Gamma(\varphi \circ \delta^\mathcal{V}_\sigma)(|e|,|f|)$. Let us now move to the value clauses.

1. We prove that for all values $v, w \in \mathcal{V}_{\sigma\multimap\tau}$ we have:

$$\varphi \circ \delta^\mathcal{V}_{\sigma\multimap\tau}(v,w) \leq \bigwedge_{u \in \mathcal{V}_\sigma} \varphi \circ \delta^\Lambda_\tau(vu, wu).$$

   Suppose $\varphi \circ \delta^\mathcal{V}_{\sigma\multimap\tau}(v,w) = \text{true}$, so that $\delta^\mathcal{V}_{\sigma\multimap\tau}(v,w) = k$. We show that $\varphi \circ \delta^\Lambda_\tau(vu, wu) = \text{true}$ holds for any $u \in \mathcal{V}_\sigma$. By very definition of applicative Γ-similarity, $\delta^\mathcal{V}_{\sigma\multimap\tau}(v,w) = k$ implies $\bigwedge_{u\in\mathcal{V}_\sigma} \delta^\Lambda_\tau(vu, wu) = k$. Since V is integral (i.e. $k = \top\!\!\!\top$), we must have $\delta^\Lambda_\tau(vu, wu) = k$ (and thus $\varphi \circ \delta^\Lambda_\tau(vu, wu) = \text{true}$) for any $u \in \mathcal{V}_\sigma$.
2. Clauses for sum and recursive types are straightforward.
3. We show that for all values $!v, !w \in \mathcal{V}_{!_s\sigma}$, $\varphi \circ \delta^\mathcal{V}_{!_s\sigma}(!v,!w) = \text{true}$ implies $(\varphi \cdot s \cdot \psi) \circ (\varphi \circ \delta^\mathcal{V}_\sigma)(v,w) = \text{true}$. By algebra of CBFs we have:

$$\begin{aligned}(\varphi \cdot s \cdot \psi) \circ (\varphi \circ \delta^\mathcal{V}_\sigma) &= (\varphi \cdot s \cdot \psi \cdot \varphi) \circ \delta^\mathcal{V}_\sigma\\ &= (\varphi \cdot s) \circ \delta^\mathcal{V}_\sigma\\ &= \varphi \circ (s \circ \delta^\mathcal{V}_\sigma).\end{aligned}$$

   Since $\varphi \circ \delta^\mathcal{V}_{!_s\sigma}(!v,!w) = \text{true}$, and thus $\delta^\mathcal{V}_{!_s\sigma}(!v,!w) = k$, by very definition of δ we infer $s \circ \delta^\mathcal{V}_\sigma(v,w) = k$. We conclude $(\varphi \circ (s \circ \delta^\mathcal{V}_\sigma))(v,w) = \text{true}$.

We now prove by coinduction $(\psi \circ \preceq) \leq \delta$, from which follows $((\varphi \cdot \psi) \circ \preceq) \subseteq (\varphi \circ \delta)$ and thus $\preceq \subseteq (\varphi \circ \delta)$. The clause for terms directly follows from Lemma 4. The clauses for values follow the same structure of the previous part of the proof. We show the case for values of type $!_s\sigma$. Suppose $\psi \circ \preceq^\mathcal{V}_{!_s\sigma}(!v,!w) = k$ to hold (otherwise we are trivially done), meaning that $!v \preceq^\mathcal{V}_{!_s\sigma} !w$ holds as well. As a consequence, we have $((\varphi \cdot s \cdot \psi) \circ \preceq^\mathcal{V}_\sigma)(v,w) = \text{true}$, and thus $s \circ (\psi \circ \preceq^\mathcal{V}_\sigma)(v,w) = k$. □

A.4 Howe's Method

Lemma 5. *The following hold:*

1. *Given well-typed values $\Gamma \vdash^\mathsf{v} v, w : \sigma$, let*

$$A \triangleq \{a \mid \Gamma \models^\mathsf{v} a \leq \alpha^H(v,w) : \sigma\}$$

   be non-empty. Then $\Gamma \models^\mathsf{v} \bigvee A \leq \alpha^H(v,w)$ is derivable.

2. *Given well-typed terms* $\Gamma \vdash e, f : \sigma$, *let*

$$A \triangleq \{a \mid \Gamma \models^{\mathsf{c}} a \leq \alpha^H(e,f) : \sigma\}$$

be non-empty. Then $\Gamma \models^{\mathsf{c}} \bigvee A \leq \alpha^H(e,f)$ *is derivable.*

Proof sketch. We simultaneously prove statements 1 and 2 by induction on (v, e). We show a couple of cases as illustrative examples:

1. Suppose
$$A \triangleq \{a \mid \Gamma \models^{\mathsf{v}} a \leq \alpha^H(x,w) : \sigma\}$$
to be non-empty. If the judgment $\Gamma \models^{\mathsf{v}} a \leq \alpha^H(x,w) : \sigma$ is provable, then it must be the conclusion of an instance of rule (H-var) from the premise:
$$a \leq (\Delta, x :_s \sigma \vdash^{\mathsf{v}} \alpha(x,w) : \sigma),$$
so that $\Gamma = \Delta, x :_s \sigma$. As a consequence, we see that the set A is just $\{a \mid a \leq (\Delta, x :_s \sigma \vdash^{\mathsf{v}} \alpha(x,w) : \sigma)\}$. In particular, we have $\Delta, x :_s \sigma \vdash^{\mathsf{v}} \alpha(x,w) : \sigma = \bigvee A \in A$.
2. Suppose
$$A \triangleq \{a \mid \Gamma \models a \leq \alpha^H(\mathbf{let}\ x = e\ \mathbf{in}\ f, g) : \tau\}$$
to be non-empty. That means there exists $a \in \mathsf{V}$ such that $\Gamma \models a \leq \alpha^H(\mathbf{let}\ x = e\ \mathbf{in}\ f, g) : \tau$ is derivable. The latter judgment must be the conclusion of an instance of rule (H-let) from premisses:
$$\Sigma \models b \leq \alpha^H(e, e') : \sigma,$$
$$\Delta, x :_s \sigma \models c \leq \alpha^H(f, f') : \tau,$$
$$d \leq (s \wedge 1) \cdot \Sigma \otimes \Delta \vdash \alpha(\mathbf{let}\ x = e'\ \mathbf{in}\ f', g) : \tau,$$
so that $\Gamma = (s \wedge 1) \cdot \Sigma \otimes \Delta$ and $a = (s \wedge 1)(b) \otimes c \otimes d$. In particular, the sets
$$B = \{b \mid \Sigma \models b \leq \alpha^H(e, e') : \sigma\},$$
$$C = \{c \mid \Delta, x :_s \sigma \models c \leq \alpha^H(f, f') : \tau\},$$
are non-empty. By induction hypothesis we have $\bigvee B \in B$ and $\bigvee C \in C$. Let $\underline{d} = (s \wedge 1) \cdot \Sigma \otimes \Delta \vdash \alpha(\mathbf{let}\ x = e'\ \mathbf{in}\ f', g) : \tau$. We can now apply rule (H-let) obtaining $(s \wedge 1)\Big(\bigvee B\Big) \otimes \Big(\bigvee C\Big) \otimes \underline{d} \in A$. To see that the latter is actually $\bigvee A$ it is sufficient to show that for any $a \in A$ we have $a \leq (s \wedge 1)\Big(\bigvee B\Big) \otimes \Big(\bigvee C\Big) \otimes \underline{d}$. But any $a \in A$ (with $a \neq \mathbb{1}$) is of the form $(s \wedge 1)(b) \otimes c \otimes d$ for $b \in B$, $c \in C$, and $d \leq \underline{d}$. We are done since both $(s \wedge 1)$ and $\otimes$ are monotone.

□

It is now easy to show that the above definition of Howe's extension coincide with the one of Definition 16. In particular, for an open λ-term V-relation α, α^H is the least compatible open λ-term V-relation satisfying the inequality $\alpha \cdot \beta \leq \beta$.

The following are standard results on Howe's extension. Proofs are straightforward but tedious (they closely resemble their relational counterparts), and thus are omitted.

Lemma 7 (Substitutivity)**.** *Let* α *be a value substitutive* λ*-term* V*-preorder. For all values,* $\Gamma, x :_s \sigma \vdash^{\mathsf{v}} u, z : \tau$ *and* $\emptyset \vdash v, w : \sigma$*, and terms* $\Gamma, x :_s \sigma \vdash e, f : \tau$*, let* $\underline{a} \triangleq \emptyset \vdash^{\mathsf{v}} \alpha^H(v,w) : \sigma$*. Then:*

$$(\Gamma, x :_s \sigma \vdash^{\mathsf{v}} \alpha^H(u,z) : \tau) \otimes s(\underline{a}) \leq \Gamma \vdash^{\mathsf{v}} \alpha^H(u[v/x], z[w/x]) : \tau,$$

$$(\Gamma, x :_s \sigma \vdash \alpha^H(e,f) : \tau) \otimes s(\underline{a}) \leq \Gamma \vdash \alpha^H(e[x := v], f[x := w]) : \tau.$$

Proof. We simultaneously prove the following statements.

(i) For any $a \in \mathsf{V}$ if $\Gamma, x :_s \sigma \models a \leq \alpha^H(e,f) : \tau$ is derivable, then $a \otimes s(\underline{a}) \leq \Gamma \vdash \alpha^H(e[x := v], f[x := w]) : \tau$ holds.

(ii) For any $a \in \mathsf{V}$ if $\Gamma, x :_s \sigma \models^{\mathsf{v}} a \leq \alpha^H(u,z) : \tau$ is derivable, then $a \otimes s(\underline{a}) \leq \Gamma \vdash \alpha^H(u[v/x], z[w/x]) : \tau$ holds.

The proof is by induction on the derivation of the judgments:

$$\mathcal{J} \triangleq \Gamma, x :_s \sigma \models a \leq \alpha^H(e,f) : \tau,$$
$$\mathcal{J}' \triangleq \Gamma, x :_s \sigma \models^{\mathsf{v}} a \leq \alpha^H(u,z) : \tau.$$

1. Suppose $\mathcal{J}'$ has been inferred via an instance of rule (H-var). We have two subcases to consider.
   1.1 $\mathcal{J}'$ has been inferred via an instance of rule (H-var) from premisses:
   $$\frac{a \leq \Gamma, x :_s \sigma \vdash^{\mathsf{v}} \alpha(x,u) : \sigma}{\Gamma, x :_s \sigma \models^{\mathsf{v}} a \leq \alpha^H(x,u) : \sigma}\ \text{(H-var)},$$
   so that $s \leq 1$ and $\mathcal{J}'$ is $\Gamma, x :_s \sigma \models^{\mathsf{v}} a \leq \alpha^H(x,u) : \sigma$. We have to prove $a \otimes s \circ (\emptyset \vdash^{\mathsf{v}} \alpha^H(v,w)) \leq \Gamma \vdash^{\mathsf{v}} \alpha^H(v, u[w/x]) : \sigma$. Since α is value substitutive, from $\Gamma, x :_s \sigma \models^{\mathsf{v}} a \leq \alpha^H(x,u) : \sigma$ we infer $a \leq \Gamma \vdash^{\mathsf{v}} \alpha(w, u[w/x]) : \sigma$. Moreover, since α^H is an open λ-term V-relation (and thus closed under weakening), we have $\emptyset \vdash^{\mathsf{v}} \alpha^H(v,w) : \sigma \leq \Gamma \vdash^{\mathsf{v}} \alpha^H(v,w) : \sigma$. We can now conclude the thesis as follows:
   $$\begin{aligned} a \otimes s(\underline{a}) &\leq (\Gamma \vdash^{\mathsf{v}} \alpha^H(v,w) : \sigma) \otimes s \circ (\Gamma \vdash^{\mathsf{v}} \alpha(w, u[w/x]) : \sigma) \\ &\leq (\Gamma \vdash^{\mathsf{v}} \alpha^H(v,w) : \sigma) \otimes (\Gamma \vdash^{\mathsf{v}} \alpha(w, u[w/x]) : \sigma) \\ &\qquad [\text{ since } s \leq 1 \text{ }] \\ &\leq \Gamma \vdash^{\mathsf{v}} \alpha^H(v, u[w/x]) : \sigma \\ &\qquad [\text{ by pseudo-transitivity }]. \end{aligned}$$
   1.2 $\mathcal{J}'$ has been inferred via an instance of rule (H-var) from premisses:
   $$\frac{a \leq \Gamma, y :_r \tau, x :_s \sigma \vdash^{\mathsf{v}} \alpha(y,u) : \tau}{\Gamma, y :_r \tau, x :_s \sigma \models^{\mathsf{v}} a \leq \alpha^H(y,u) : \tau}\ \text{(H-var)}$$
   so that $\mathcal{J}'$ is $\Gamma, y :_r \tau, x :_s \sigma \models^{\mathsf{v}} a \leq \alpha^H(y,u) : \tau$. We have to prove $a \otimes s \circ (\emptyset \vdash^{\mathsf{v}} \alpha^H(v,w)) \leq \Gamma, y :_r \tau \vdash^{\mathsf{v}} \alpha^H(y, u[w/x]) : \tau$. As V is integral and α is value-substitutive, we have:
   $$a \otimes s \circ (\emptyset \vdash^{\mathsf{v}} \alpha^H(v,w)) \leq a \leq \Gamma, y :_r \tau \vdash^{\mathsf{v}} \alpha(y, u[w/x]).$$
   Since $\alpha \leq \alpha^H$ we are done.
2. Suppose $\mathcal{J}$ has been inferred via an instance of rule (H-let) from premisses:
$$\Gamma, x :_s \sigma \models a \leq \alpha^H(e, e') : \sigma', \tag{1}$$
$$\Delta, x :_r \sigma, y :_p \sigma' \models b \leq \alpha^H(f, f') : \tau, \tag{2}$$
$$c \leq (p \wedge 1) \cdot (\Gamma, x :_s \sigma) \otimes (\Delta, x :_r \sigma) \vdash \alpha^H(\mathbf{let}\ y = e'\ \mathbf{in}\ f', g) : \tau. \tag{3}$$
   so that J is:
   $$(p \wedge 1) \cdot \Gamma \otimes \Delta, x :_{(p \wedge 1) \cdot s \otimes r} \sigma \models (p \wedge 1)(a) \otimes b \otimes c \leq \alpha^H(\mathbf{let}\ y = e\ \mathbf{in}\ f, g) : \tau.$$
   We have to prove:
   $$(p \wedge 1)(s(\underline{a})) \otimes r(\underline{a}) \otimes (p \wedge 1)(a) \otimes b \otimes c \leq (p \wedge 1) \cdot \Gamma \otimes \Delta \vdash \alpha^H(\mathbf{let}\ y = e[x := v]\ \mathbf{in}\ f[x := v], g[x := w]) : \tau.$$

We apply the induction hypothesis on (1) and (2) obtaining:

$$s(\underline{a}) \otimes a \leq \Gamma \vdash \alpha^H(e[x := v], e'[x := w]) : \sigma', \tag{4}$$

$$r(\underline{a}) \otimes b \leq \Delta, y :_p \sigma' \vdash \alpha^H(f[x := v], f[x := w]) : \tau. \tag{5}$$

From (4) and (5) by compatibility of α^H (and lax equations of change of base functors) we infer:

$$\begin{aligned}(p \wedge 1)(s(\underline{a})) \otimes (p \wedge 1)(a) \otimes r(\underline{a}) \otimes b &\leq (p \wedge 1) \cdot \Gamma \otimes \Delta \\ &\vdash \alpha^H(\mathbf{let}\ y = e[x := v]\ \mathbf{in}\ f[x := a], \\ &\qquad \mathbf{let}\ y = e'[x := w]\ \mathbf{in}\ f'[x := w]) : \tau. \end{aligned} \tag{6}$$

Finally, since α is value-substitutive, from (3) we obtain:

$$c \leq (p \wedge 1) \cdot \Gamma \otimes \Delta \vdash \alpha^H(\mathbf{let}\ y = e'[x := w]\ \mathbf{in}\ f'[x := w], g) : \tau,$$

and thus conclude the thesis from the latter and (6) by pseudo-transitivity.

3. Suppose $\mathcal{J}$ has been inferred via an instance of rule (H-op) from premisses (as usual we write $\vec{x_i}$ for items $x_1, \ldots, x_n$):

$$\forall i.\ \Gamma_i, x :_{s_i} \sigma \models a_i \leq \alpha^H(e_i, e_i') : \tau, \tag{7}$$

$$b \leq op_{\mathsf{V}}(\vec{\Gamma_i}), x :_{op_{\mathsf{V}}(\vec{s_i})} \sigma \vdash \alpha(\mathbf{op}(\vec{e_i'}), f) : \tau, \tag{8}$$

so that J is

$$op_{\mathsf{V}}(\vec{\Gamma_i}), x :_{op_{\mathsf{V}}(\vec{s_i})} \sigma \models op_{\mathsf{V}}(\vec{a_i}) \otimes b \leq \alpha^H(\mathbf{op}(\vec{e_i}), f) : \tau.$$

We have to prove

$$\begin{aligned}op_{\mathsf{V}}(\overrightarrow{s_i(\underline{a})}) \otimes op_{\mathsf{V}}(\vec{a_i}) \otimes b \\ \leq op_{\mathsf{V}}(\vec{\Gamma_i}) \vdash \alpha^H(\overrightarrow{e_i[x := v]}, f[x := w]) : \tau.\end{aligned}$$

We apply the induction hypothesis on (7) obtaining:

$$\forall i.\ s(\underline{a}) \otimes a_i \leq \Gamma_i \vdash \alpha^H(e_i[x := v], e_i'[x := w]) : \tau. \tag{9}$$

Monotonicity of op_{V} on (9) followed by compatibility gives:

$$\begin{aligned}op_{\mathsf{V}}(\overrightarrow{s_i(\underline{a}) \otimes a_i}) \\ \leq op_{\mathsf{V}}(\vec{\Gamma_i}) \vdash \alpha^H(\mathbf{op}(\overrightarrow{e_i[x := v]}), \mathbf{op}(\overrightarrow{e_i'[x := w]})).\end{aligned} \tag{10}$$

Finally, as α is value-substitutive, from (8) we obtain:

$$b \leq op_{\mathsf{V}}(\vec{\Gamma_i}), \vdash \alpha(\mathbf{op}(\overrightarrow{e_i'[x := w]}) f[x := w]) : \tau.$$

The latter together with (5) implies

$$op_{\mathsf{V}}(\overrightarrow{s_i(\underline{a}) \otimes a_i}) \otimes b \leq op_{\mathsf{V}}(\vec{\Gamma_i}) \vdash \alpha^H(\mathbf{op}(\overrightarrow{e_i[x := v]}), f[x := w])$$

by pseudo-transitivity. We conclude the thesis as Definition 3 entails:

$$op_{\mathsf{V}}(\overrightarrow{s_i(\underline{a})}) \otimes op_{\mathsf{V}}(\vec{a_i}) \leq op_{\mathsf{V}}(\overrightarrow{s_i(\underline{a}) \otimes a_i}).$$

The remaining cases follow the same pattern. □

Lemma 8 (Key Lemma). *Let α be a reflexive and transitive applicative Γ-simulation. Then the Howe's extension of α restricted to closed terms/values in an applicative Γ-simulation.*

Proof. Let us write α^H for the Howe's extension of α restricted to closed terms/values. It is easy to see that α^H satisfies the simulation clauses for values. For instance, we prove the inequation $\alpha^H_{!_s\sigma}(!v, !w) \leq s \circ \alpha^H_\sigma(v, w)$, where for readability we omit values superscript in α and α^H. It is sufficient to show that for any $a \in \mathsf{V}$ such that $\mathcal{J} \triangleq \emptyset \models a \leq \alpha^H(!v, !w) : !_s\sigma$ is derivable, the inequation $a \leq s \circ \alpha^H_\sigma(v, w)$ holds. The judgment $\mathcal{J}$ must have been inferred via an instance of rule (H-bang), so that without loss of generality we can assume $a = s(b) \otimes \alpha_{!_s\sigma}(!u, !w)$, with $\emptyset \models b \leq \alpha^H(v, u) : \sigma$ derivable, for some value u. We conclude the thesis as follows:

$$\begin{aligned}a &\leq s \circ \alpha^H_\sigma(v, u) \otimes \alpha_{!_s\sigma}(!u, !w) \\ &\leq s \circ \alpha^H_\sigma(v, u) \otimes s \circ \alpha_\sigma(u, w) \\ &\qquad [\alpha \text{ is an applicative } \Gamma\text{-simulation}] \\ &\leq s \circ (\alpha^H_\sigma(v, u) \otimes \alpha_\sigma(u, w)) \\ &\leq s \circ (\alpha_\sigma \cdot \alpha^H_\sigma)(v, w) \\ &\leq s \circ \alpha^H_\sigma(v, w) \\ &\qquad [\text{pseudo-transitivity}]\end{aligned}$$

The crucial part of the proof is to show that α^H satisfies the clause for terms. We prove that for any $n \geq 0$,

$$(\alpha^H)^\wedge_\sigma(e, f) \leq \Gamma(\alpha^H)^\vee_\sigma(|e|_n, |f|)$$

holds for all terms $e, f \in \Lambda_\sigma$. Since Γ is inductive the above inequality gives the thesis as follows:

$$\begin{aligned}(\alpha^H_\sigma)^\wedge(e, f) &\leq \bigwedge_n \Gamma(\alpha^H_\sigma)^\vee(|e|_n, |f|) \\ &\leq \Gamma(\alpha^H_\sigma)^\vee(\bigsqcup_n |e|_n, |f|) \\ &= \Gamma(\alpha^H_\sigma)^\vee(|e|, |f|).\end{aligned}$$

The proof is by induction on n with a case analysis on the term structure in the inductive case. For readability we simply write α in place of $\alpha^\wedge$ and $\alpha^\vee$. Moreover, to avoid confusion it is useful to explicitly distinguishing between (ordinary) Kleisli extension and strong Kleisli extension. Given a monoidal category $\langle \mathbb{C}, I, \otimes \rangle$, we denote by $f^* : Z \otimes TX \to TY$ the *strong* Kleisli extension of $f : Z \otimes X \to TY$ and by $g^\dagger : TX \to TY$ the Kleisli extension of $g : X \to TY$. The latter can be defined in terms of the former as $g^\dagger \triangleq (g \cdot \lambda_X)^* \cdot \lambda_{TX}^{-1}$, where $\lambda_X : I \otimes X \xrightarrow{\cong} X$ is the natural isomorphism given by the monoidal structure of $\mathbb{C}$. Note that, in particular, $g^\dagger \cdot \lambda_{TX} = (g \cdot \lambda_X)^*$.

1. We have to prove:

$$\alpha^H_\sigma(e, f) \leq \Gamma\alpha^H_\sigma(|e|_0, |f|).$$

Since Γ is inductive and $|e|_0 = \bot_{\mathcal{V}_\sigma}$, it is sufficient to prove $\alpha^H_\sigma(e, f) \leq k$. Because the quantale is integral the latter trivially holds.

2. We have to prove:

$$\alpha^H_\sigma(\mathbf{val}\ v, w) \leq \Gamma\alpha^H_\sigma(|\mathbf{val}\ v|_{n+1}, |w|).$$

Since $|\mathbf{val}\ v|_{n+1} = \eta(v)$, it is sufficient to prove that for any a such that the judgment $\emptyset \models a \leq \alpha^H(\mathbf{val}\ v, w) : \sigma$ is derivable, $a \leq \Gamma\alpha^H_\sigma(\eta(v), |w|)$ holds. Suppose $\emptyset \models a \leq \alpha^H(\mathbf{val}\ v, w) : \sigma$ to be derivable. The latter must have been inferred via an instance of rule (H-val) from premisses:

$$\emptyset \models b \leq \alpha^H(v, v') : \sigma, \tag{11}$$

$$c \leq \alpha_\sigma(\mathbf{val}\ v', w). \tag{12}$$

In particular, we have $b \leq \alpha^H_\sigma(v, v')$ and thus, by condition (Lax unit), $b \leq \Gamma\alpha^H_\sigma(\eta(v), \eta(v'))$. From, (11) we infer, by very definition of applicative Γ-simulation, $c \leq \Gamma\alpha_\sigma(\eta(v'), |w|)$, and thus $b \otimes c \leq \Gamma\alpha^H_\sigma(\eta(v), \eta(v')) \otimes \Gamma\alpha_\sigma(\eta(v'), |w|)$. We conclude the thesis by Γ-pseudo-transitivity.

3. We have to prove:

$$\alpha^H_\tau((\lambda x.e)v, f) \leq \Gamma\alpha^H_\tau(|(\lambda x.e)v|_{n+1}, |f|).$$

As $|(\lambda x.e)v|_{n+1} = |e[x := v]|_n$, it is sufficient to show that for any a such that $\emptyset \models a \leq \alpha^H((\lambda x.e)v, f) : \tau$ holds, we have $a \leq \Gamma\alpha^H_\tau(|e[x := v]|_n, |f|)$. Assume $\emptyset \models a \leq \alpha^H((\lambda x.e)v, f) : \tau$. The latter must have been inferred via an instance of rule (H-app) from premisses:

$$\emptyset \models b \leq \alpha^H(v, w) : \sigma, \tag{13}$$
$$\emptyset \models c \leq \alpha^H(\lambda x.e, u) : \sigma \multimap \tau, \tag{14}$$
$$d \leq \alpha_\tau(uw, f). \tag{15}$$

Let us examine premise (14). First of all, since u is a closed value of type $\sigma \multimap \tau$ it must be of the form $\lambda x.g$. Moreover, (14) must have been inferred via an instance rule rule (H-abs) from premisses:

$$x :_1 \sigma \models c_1 \leq \alpha^H(e, h) : \tau, \tag{16}$$
$$c_2 \leq \alpha_{\sigma\multimap\tau}(\lambda x.h, \lambda x.g). \tag{17}$$

In particular, we have the equality $c_1 \otimes c_2 = c$. From (16) we deduce $c_1 \leq x :_1 \sigma \vdash \alpha^H(e, h) : \tau$, whereas from (13) we infer $b \leq \alpha^H_\sigma(v, w)$. We are now in position to apply the Substitution Lemma, obtaining $c_1 \otimes b \leq \alpha^H_\tau(e[x := v], h[x := w])$. By very definition of applicative Γ-simulation, (17) implies the inequality $c_2 \leq \alpha_\tau(h[x := w], g[x := w])$. Applying pseudo-transitivity followed by the induction hypothesis we obtain:

$$c_1 \otimes c_2 \otimes b \leq \alpha^H_\tau(e[x := v], g[x := w]) \leq \Gamma\alpha^H_\tau(|e[x := v]|_n, |g[x := w]|).$$

Finally, from (15), by definition of applicative Γ-simulation we infer $d \leq \Gamma\alpha_\tau(|g[x := w]|, |f|)$ (recall that $u = \lambda x.g$, so that $|uw| = |g[x := w]|$). We can now conclude the thesis by Γ-pseudo-transitivity.

4. Cases for pattern matching against folds and sums are standard (they follow the same pattern of point 5 but are simpler).
5. We have to prove:

$$\alpha^H_\tau(\mathbf{case}\ !v\ \mathbf{of}\ \{!x \to e\}, f) \leq \Gamma\alpha^H_\tau(|\mathbf{case}\ !v\ \mathbf{of}\ \{!x \to e\}|_{n+1}, |f|).$$

As $|\mathbf{case}\ !v\ \mathbf{of}\ \{!x \to e\}|_{n+1} = |e[x := v]|_n$, we show that for any a such that $\emptyset \models a \leq \alpha^H(\mathbf{case}\ !v\ \mathbf{of}\ \{!x \to e\}, f) : \tau$ is derivable, the inequality $a \leq \Gamma\alpha^H_\tau(|e[x := v]|_n, |f|)$ holds. Suppose $\emptyset \models a \leq \alpha^H(\mathbf{case}\ !v\ \mathbf{of}\ \{!x \to e\}, f) : \tau$. The latter must have been inferred via an instance of rule (H-bang-cases) from premisses:

$$\emptyset \models b \leq \alpha^H(!v, u) : !_s\sigma, \tag{18}$$
$$x :_{r\cdot s} \sigma \models c \leq \alpha^H(e, e') : \tau, \tag{19}$$
$$d \leq \alpha_\tau(\mathbf{case}\ u\ \mathbf{of}\ \{!x \to e'\}, f). \tag{20}$$

In particular, we have $a = r(b) \otimes c \otimes d$. Let us examine premise (18). First of all, since u is a closed value of type $!_s\sigma$ it must be of the form $!v'$. Moreover, (18) must have been inferred via an instance of rule (H-bang) from premisses:

$$\emptyset \models b_1 \leq \alpha^H(v, w) : \sigma, \tag{21}$$
$$b_2 \leq \alpha_{!_s\sigma}(!w, !v'). \tag{22}$$

In particular, $b = s(b_1) \otimes b_2$. From (22), by definition of applicative Γ-simulation we infer $b_2 \leq s \circ \alpha_\sigma(w, v')$. Since (21) implies $b_1 \leq \alpha^H_\sigma(v, w)$, we have:

$$\begin{aligned} b &= s(b_1) \otimes b_2 \\ &\leq s \circ \alpha^H_\sigma(v, w) \otimes s \circ \alpha_\sigma(w, v') \\ &\leq s \circ (\alpha^H_\sigma(v, w) \otimes \alpha_\sigma(w, v')) \\ &\leq s \circ \alpha^H_\sigma(v, v'), \end{aligned}$$

where the last inequality follows by pseudo-transitivity. From (19) we infer the inequality $c \leq x :_{r\cdot s} \sigma \vdash \alpha^H(e, e') : \tau$. We are now in position to apply the Substitution Lemma obtaining:

$$(r \cdot s) \circ \alpha^H_\sigma(v, v') \otimes c \leq \alpha^H_\tau(e[x := v], e'[x := v']).$$

The latter, together with the inequality $b \leq s \circ \alpha^H_\sigma(v, v')$, implies $r(b) \otimes c \leq \alpha^H_\tau(e[x := v], e'[x := v'])$. Applying the induction hypothesis we conclude:

$$r(b) \otimes c \leq \Gamma\alpha^H_\tau(|e[x := v]|_n, |e'[x := v']|).$$

Finally, from (20) by definition of applicative Γ-simulation we infer $d \leq \Gamma\alpha_\tau(|e'[x := v']|, |f|)$ (recall that $u = !v'$) and thus conclude the thesis by Γ-pseudo-transitivity.

6. We have to prove:

$$\alpha^H_\tau(\mathbf{let}\ x = e\ \mathbf{in}\ f, g) \leq \Gamma\alpha^H_\tau(|\mathbf{let}\ x = e\ \mathbf{in}\ f|_{n+1}, |g|).$$

As $|\mathbf{let}\ x = e\ \mathbf{in}\ f|_{n+1} = |f[x := \_]|^\dagger_n|e|_n$, it is sufficient to prove that for any a such that $\emptyset \models a \leq \alpha^H(\mathbf{let}\ x = e\ \mathbf{in}\ f, g) : \tau$ is derivable, we have $a \leq \Gamma\alpha^H_\tau(|f[x := \_]|^\dagger_n|e|_n, |g|)$. Suppose $\emptyset \models a \leq \alpha^H(\mathbf{let}\ x = e\ \mathbf{in}\ f, g) : \tau$. The latter must have been inferred via an instance of rule (H-let) from premisses:

$$\emptyset \models b \leq \alpha^H(e, e') : \sigma, \tag{23}$$
$$x :_s \sigma \models c \leq \alpha^H(f, f') : \tau, \tag{24}$$
$$d \leq \alpha_\tau(\mathbf{let}\ x = e'\ \mathbf{in}\ f', g). \tag{25}$$

In particular, we have $a = (s \wedge 1)(b) \otimes c \otimes d$. We now claim to have:

$$(x :_s \sigma \vdash \alpha^H(f, f') : \tau) \otimes (s \wedge 1) \circ \alpha^H_\sigma(e, e') \leq \Gamma\alpha^H_\tau(|\mathbf{let}\ x = e\ \mathbf{in}\ f|_{n+1}, |\mathbf{let}\ x = e'\ \mathbf{in}\ f'|). \tag{26}$$

By very definition of Howe's extension, the latter obviously entails $(s\wedge1)(b)\otimes c \leq \Gamma\alpha^H_\tau(|\mathbf{let}\ x = e\ \mathbf{in}\ f|_{n+1}, |\mathbf{let}\ x = e'\ \mathbf{in}\ f'|)$. Moreover, by definition of applicative Γ-simulation, (25) implies $d \leq \Gamma\alpha_\tau(|\mathbf{let}\ x = e'\ \mathbf{in}\ f'|, |g|)$, which allows to conclude the thesis by Γ-pseudo-transitivity. Let us now turn to the proof of (25). First of all we apply the induction hypothesis on $\alpha^H_\sigma(e, e')$. By monotonicity of $s \wedge 1$ we have thus reduced the proof of (25) to proving the inequality:

$$(x :_s \sigma \vdash \alpha^H(f, f') : \tau) \otimes (s \wedge 1) \circ \Gamma\alpha^H_\sigma(|e|_n, |e'|) \leq \Gamma\alpha^H_\tau(|f[x := \_]|^\dagger_n|e|_n, |f'[x := \_]|^\dagger|e'|). \tag{27}$$

Consider the diagram:

$$\begin{array}{ccc} I \times T\mathcal{V}_\sigma & \xrightarrow{|f[x:=\_]|^\dagger_n \cdot \lambda_{T\mathcal{V}_\sigma}} & T\mathcal{V}_\tau \\ {\scriptstyle \gamma\otimes(s\wedge1)\circ\Gamma\alpha^H_\sigma}\downarrow & \leq & \downarrow{\scriptstyle \Gamma\alpha^H_\tau} \\ I \times T\mathcal{V}_\sigma & \xrightarrow[|f'[x:=\_]|^\dagger \cdot \lambda_{T\mathcal{V}_\sigma}]{} & T\mathcal{V}_\tau \end{array} \tag{28}$$

where $I = \{*\}$ and $\gamma(*,*) = (x :_s \sigma \vdash \alpha^H(f,f') : \tau)$. It is easy to see that (27) follows from (28), since e.g.:

$$(|f[x := \_]|_n^\dagger \cdot \lambda_T \mathcal{V}_\sigma)(*, |e|_n) = |e[x := \_]|_n^\dagger |e|_n.$$

To prove (28) we first observe that by very definition of strong monad we have $|f[x := \_]|_n^\dagger \cdot \lambda_T \mathcal{V}_\sigma = (|f[x := \_]|_n \cdot \lambda_{\mathcal{V}_\sigma})^*$. We can now apply condition (*L*-Strong lax bind). As a consequence, to prove (28) it is sufficient to prove that for all closed values v, w of type σ, we have:

$$(x :_s \sigma \vdash \alpha^H(f,f') : \tau) \otimes (s \wedge 1) \circ \alpha_\sigma^H(v,w) \leq \Gamma \alpha_\tau^H(|f[x := v]|_n, |f'[x := w]|).$$

By Substitution Lemma and induction hypothesis we have:

$$(x :_s \sigma \vdash \alpha^H(f,f') : \tau) \otimes s \circ \alpha_\sigma^H(v,w) \leq \Gamma \alpha_\tau^H(|f[x := v]|_n, |f'[x := w]|).$$

We conclude the thesis since $s \wedge 1 \leq s$.

7. We have to prove:

$$\alpha_\sigma^H(\mathbf{op}(e_1,\dots,e_m), f) \leq \Gamma \alpha_\sigma^H(|\mathbf{op}(e_1,\dots,e_m)|_{n+1}, |f|),$$

where **op** is an m-ary operation symbol in Σ. As usual, we use the notation $\vec{x_i}$ for items $x_1,\dots,x_m$.
We show that for any a such that $\emptyset \models a \leq \alpha^H(\mathbf{op}(\vec{e_i}), f) : \sigma$ is derivable, $a \leq \Gamma \alpha_\tau^H(|\mathbf{op}(\vec{e_i})|_n, |f|)$ holds. Suppose to have $\emptyset \models a \leq \alpha^H(\mathbf{op}(\vec{e_i}), f) : \tau$. The latter must have been inferred via an instance of rule (H-op) from premisses:

$$\forall i \leq m.\, \emptyset \models a_i \leq \alpha^H(e_i, f_i) : \sigma, \tag{29}$$

$$b \leq \alpha_\tau(\mathbf{op}(f_1,\dots,f_m), f). \tag{30}$$

In particular, we have $a = op_{\mathcal{V}}(a_1,\dots,a_m) \otimes b$. We apply the induction hypothesis on (29) obtaining, for each $i \leq m$, the inequality $a_i \leq \Gamma \alpha^H(|e_i|_n, |f_i|)$. By monotonicity of $op_{\mathcal{V}}$ we thus infer:

$$\begin{aligned} op_{\mathcal{V}}(\vec{a_i}) &\leq op_{\mathcal{V}}(\Gamma \alpha^H(|e_1|_n, |f_1|), \dots, \Gamma \alpha^H(|e_m|_n, |f_m|)) \\ &\leq \Gamma \alpha_\sigma^H(op_{\mathcal{V}_\sigma}(|e_1|_n,\dots,|e_m|_n), op_{\mathcal{V}_\sigma}(|f_1|,\dots,|f_m|)) \\ &= \Gamma \alpha_\sigma^H(|\mathbf{op}(e_1,\dots,e_m)|_{n+1}, |\mathbf{op}(f_1,\dots,f_m)|), \end{aligned}$$

where the second inequality follows since Γ is Σ-compatible. We conclude the thesis from (30) by Γ-pseudo-transitivity and definition of applicative Γ-simulation.

□

A.5 Applicative Γ-bisimilarity

In this last section we expand on some technical details necessary to prove that applicative Γ-bisimilarity is compatible.

Proposition 8. *Let Γ be a $\vee$-relator. Define the λ-term $\vee$-relation γ' as follows:*

$$\gamma' \triangleq \bigvee \{\alpha \mid \alpha^\circ = \alpha,\ \alpha \leq [\alpha]\}.$$

Then:

1. *γ' is a symmetric applicative Γ-simulation, and therefore the largest such λ-term $\vee$-relation.*
2. *γ' coincide with applicative $(\Gamma \wedge \Gamma^\circ)$-similarity γ.*

Proof. Obviously γ is an applicative Γ-simulation. Moreover, γ is symmetric and thus we have $\gamma \leq \gamma'$. To see that $\gamma' \leq \gamma$ it is sufficient to prove that γ' is an applicative $(\Gamma \wedge \Gamma^\circ)$-simulation. Clauses on values are trivially satisfied. We now show that for any symmetric applicative Γ-simulation α, we have the inequality $\alpha_\sigma^\wedge(e,e') \leq \Gamma \alpha_\sigma^\vee(|e|,|e'|) \wedge \Gamma(\alpha_\sigma^\vee)^\circ(|e'|,|e|)$ for all terms $e, e' \in \Lambda_\sigma$. For that it is sufficient to prove $\alpha_\sigma^\wedge(e,e') \leq \Gamma(\alpha_\sigma^\vee)^\circ(|e'|,|e|)$, which obviously holds since α is symmetric. □

Lemma 10. *Assume CBEs in Π to be finitely continuous. Define the transitive closure α^T of a $\vee$-relation α as $\alpha^T \triangleq \bigvee_n \alpha^{(n)}$, where $\alpha^{(0)} \triangleq id$, and $\alpha^{(n+1)} \triangleq \alpha^{(n)} \cdot \alpha$.*

1. *Let α be a reflexive and transitive λ-term $\vee$-relation. Then $(\alpha^H)^T$ is compatible.*
2. *Let α be an reflexive, symmetric, and transitive open λ-term $\vee$-relation. Then $(\alpha^H)^T$ is symmetric.*

Proof. We start with point 1. First of all observe that by Lemma 6 α^H is compatible. To prove compatibility of $(\alpha^H)^T$ we have to check that it satisfies all clauses in Figure 5. We show the case for sequential composition as an illustrative example (the other cases are proved in a similar, but easier, way). We have to prove:

$$(s \wedge 1) \circ (\Gamma \vdash (\alpha^H)^T(e,e') : \sigma) \otimes (\Delta, x :_s \sigma \vdash (\alpha^H)^T(f,f') : \tau) \leq (s \wedge 1) \cdot \Gamma \vdash (\alpha^H)^T(\mathbf{let}\ x = e\ \mathbf{in}\ f, \mathbf{let}\ x = e'\ \mathbf{in}\ f') : \tau.$$

Let $c \triangleq ((s \wedge 1) \cdot \Gamma \vdash (\alpha^H)^T(\mathbf{let}\ x = e\ \mathbf{in}\ f, \mathbf{let}\ x = e'\ \mathbf{in}\ f') : \tau)$. By definition of transitive closure we have to prove:

$$(s \wedge 1) \circ \bigvee_n (\Gamma \vdash (\alpha^H)^{(n)}(e,e') : \sigma) \otimes \bigvee_m (\Delta, x :_s \sigma \vdash (\alpha^H)^{(m)}(f,f') : \tau) \leq c.$$

By finite continuity either $s \wedge 1 = \infty$ or it is continuous with respect to joints. In the former case we are trivially done. So suppose the latter case, so that thesis becomes:

$$\bigvee_n (s \wedge 1) \circ (\Gamma \vdash (\alpha^H)^{(n)}(e,e') : \sigma) \otimes \bigvee_m (\Delta, x :_s \sigma \vdash (\alpha^H)^{(m)}(f,f') : \tau) \leq c.$$

In particular, we also have $s \neq \infty$. We prove that for any $n, m \geq 0$ the following holds: for all e, e', f, f' (of appropriate type),

$$(s \wedge 1) \circ (\Gamma \vdash (\alpha^H)^{(n)}(e,e') : \sigma) \otimes (\Delta, x :_s \sigma \vdash (\alpha^H)^{(m)}(f,f') : \tau) \leq ((s \wedge 1) \cdot \Gamma \vdash (\alpha^H)^T(\mathbf{let}\ x = e\ \mathbf{in}\ f, \mathbf{let}\ x = e'\ \mathbf{in}\ f') : \tau)$$

holds. First of all we observe that since α^H is reflexive, we can assume $n = m$. In fact, if e.g. $n = m + l$, then we can 'complete' $(\alpha^H)^{(m)}$ as follows:

$$(\alpha^H)^{(m)} = (\alpha^H)^{(m)} \underbrace{\cdot id \cdot \dots \cdot id}_{l\text{-times}} \leq (\alpha^H)^{(m)} \underbrace{\cdot \alpha^H \cdot \dots \cdot \alpha^H}_{l\text{-times}} = (\alpha^H)^{(n)}.$$

We now do induction on n. The base case is trivial. Let us turn on the inductive step. We have to prove:

$$(s \wedge 1) \circ \Big(\bigvee_{e''} (\Gamma \vdash \alpha^H(e,e'') : \sigma) \otimes (\Gamma \vdash (\alpha^H)^{(n)}(e'',e') : \sigma) \Big) \otimes \bigvee_{f''} (\Delta, x :_s \sigma \vdash \alpha^H(f,f'') : \tau) \otimes (\Delta, x :_s \sigma \vdash (\alpha^H)^{(n)}(f'',f') : \tau) \leq c.$$

Since $s \wedge 1$ is continuous it is sufficient to prove that for all terms e'', f'' we have:

$$(s \wedge 1) \circ (\Gamma \vdash \alpha^H(e, e'') : \sigma) \otimes (s \wedge 1) \circ (\Gamma \vdash (\alpha^H)^{(n)}(e'', e') : \sigma)$$
$$\otimes(\Delta, x :_s \sigma \vdash \alpha^H(f, f'') : \tau) \otimes (\Delta, x :_s \sigma \vdash (\alpha^H)^{(n)}(f'', f') : \tau) \leq c,$$

i.e.

$$(s \wedge 1) \circ (\Gamma \vdash \alpha^H(e, e'') : \sigma) \otimes (\Delta, x :_s \sigma \vdash \alpha^H(f, f'') : \tau)$$
$$\otimes (s \wedge 1) \circ (\Gamma \vdash (\alpha^H)^{(n)}(e'', e') : \sigma)$$
$$\otimes (\Delta, x :_s \sigma \vdash (\alpha^H)^{(n)}(f'', f') : \tau) \leq c.$$

We can now apply compatibility of α^H plus the induction hypothesis, thus reducing the thesis to:

$$\Big((s \wedge 1) \cdot \Gamma \otimes \Delta \vdash \alpha^H(\mathbf{let}\ x = e\ \mathbf{in}\ f, \mathbf{let}\ x = e''\ \mathbf{in}\ f'') : \sigma)\Big)$$
$$\otimes\Big((s \wedge 1) \cdot \Gamma \otimes \Delta \vdash (\alpha^H)^T(\mathbf{let}\ x = e''\ \mathbf{in}\ f'', \mathbf{let}\ x = e'\ \mathbf{in}\ f) : \sigma)\Big) \leq c.$$

We can now conclude the thesis by very definition of $(\alpha^H)^T$.

To prove point 2 we have to show $(\alpha^H)^T \leq ((\alpha^H)^T)^\circ$. For that it is sufficient to show $\alpha^H \leq ((\alpha^H)^T)^\circ$. That amounts to prove that for all terms $\Gamma \vdash e, e' : \sigma$ and values $\Gamma \vdash^{\mathsf{v}} v, v'$, and for any $a \in \mathsf{V}$ such that $\Gamma \models a \leq \alpha^H(e, e') : \sigma$ is derivable we have $a \leq \Gamma \vdash (\alpha^H)^T(e, e') : \sigma$ (and similarity for $\Gamma \vdash^{\mathsf{v}} v, v' : \sigma$). The proof is by induction on the derivation of $\Gamma \models a \leq \alpha^H(e, e') : \sigma$ using point 1. □