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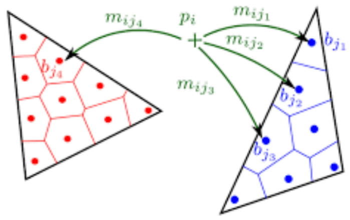

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Up-To Techniques for Behavioural Metrics via Fibrations

Abstract

Up-to techniques are a well-known method for enhancing coinductive proofs of behavioural equivalences. We introduce up-to techniques for behavioural metrics between systems modelled as coalgebras and we provide abstract results to prove their soundness in a compositional way.

In order to obtain a general framework, we need a systematic way to lift functors: we show that the Wasserstein lifting of a functor, introduced in a previous work, corresponds to a change of base in a fibrational sense. This observation enables us to reuse existing results about soundness of up-to techniques in a fibrational setting. We focus on the fibrations of predicates and relations valued in a quantale, for which pseudo-metric spaces are an example. To illustrate our approach we provide an example on distances between regular languages.

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1 Introduction

Checking whether two systems have an equivalent (or similar) behaviour is a crucial problem in computer science. In concurrency theory, one standard methodology for establishing behavioural equivalence of two systems is constructing a bisimulation relation between them. When the systems display a quantitative behaviour, the notion of behavioural equivalence is replaced with the more robust notion of behavioural metric [41, 14, 15].

Due to the sheer complexity of state-based systems, computing their behavioural equivalences and metrics can be very costly, therefore optimization techniques—the so called *up-to techniques*—have been developed to render these computations more efficient. These techniques found applications in various domains such as checking algorithms [9, 7], abstract interpretation [6] and proof assistants [13]. In the qualitative setting and in particular in concurrency, the theory of up-to techniques for bisimulations and various other coinductive predicates has been thoroughly studied [29, 33, 20]. On the other hand, in the quantitative setting, so far, only [12] has studied up-to techniques for behavioural metrics. However, the notion of up-to techniques therein and the accompanying theory of soundness are specific for probabilistic automata and are not instances of the standard lattice theoretic framework, which we will briefly recall next.

Suppose we want to verify whether two states in a system behave in the same way, (e.g. whether two states of an NFA accept the same language). The starting observation is that the relation of interest (e.g. behavioural equivalence or language equivalence) can be expressed as the greatest fixed point νb of a monotone function $b\colon \mathsf{Rel}_Q \to \mathsf{Rel}_Q$ on the complete lattice Rel_Q of relations on the state space Q of the system. Hence, in order to prove that two states x and y are behaviourally equivalent, i.e. $(x, y) \in \nu b$, it suffices to find a witness relation r which on one hand is a post-fixpoint of b, that is, $r \subseteq b(r)$ and on the other hand contains the pair (x, y). This is simply the coinduction proof principle. However, exhibiting such a witness relation r can be sometimes computationally expensive. In many situations this computation can be significantly optimized, if instead of computing a post-fixpoint of b one exhibits a relaxed invariant, that is a relation r such that $r \subseteq b(f(r))$ for a suitable function f. The function f is called a *sound* up-to technique when the proof principle

$$\frac{(x,y) \in r \quad r \subseteq b(f(r))}{(x,y) \in \nu b}$$

is valid. Establishing the soundness of up-to techniques on a case-by-case basis can be a tedious and sometimes delicate problem, see e.g. [28]. For this reason, several works [35, 31, 33, 20, 30, 32] have established a lattice-theoretic framework for proving soundness results in a modular fashion. The key notion is compatibility: for arbitrary monotone maps b and f on a complete lattice $(C, \leq)$, the up-to technique f is *b-compatible* iff $f \circ b \leq b \circ f$. Compatible techniques are sound and, most importantly, can be combined in several useful ways.

In this paper we develop a generic theory of up-to techniques for behavioural metrics applicable to different kinds of systems and metrics, which reuses established methodology. To achieve this we exploit the theory developed in [8] by modelling systems as *coalgebras* [34, 22] and behavioural metrics as coinductive predicates in a *fibration* [18]. In order to provide general soundness results, we need a principled way to lift functors from Set to metric spaces, a problem that has been studied in [19] and [3]. Our key observation is that these liftings arise from a change-of-base situation between $\mathcal{V}$-Rel and $\mathcal{V}$-Pred, namely the fibrations of relations, respectively predicates, valued over a quantale $\mathcal{V}$ (see Section 4 and 5).

In Section 6 we provide sufficient conditions ensuring the compatibility of basic quantitative up-to techniques, as well as proper ways to compose them. Interestingly enough, the conditions ensuring compatibility of the quantitative analogue of up-to reflexivity and up-to transitivity are subsumed by those used in [19] to extend monads to a bicategory of many-valued relations and generalize those in [3] (see the discussion after Theorem 21).

When the state space of a system is equipped with an algebraic structure, e.g. in process algebras, one can usually exploit this structure by reasoning up-to context. Assuming that the system forms a *bialgebra* [39, 26], intuitively the algebraic structure distributes over the coalgebraic behaviour as in GSOS specifications, we give sufficient conditions ensuring the compatibility of the quantitative version of contextual closure (Theorem 27).

In the qualitative setting, the sufficient conditions for compatibility are automatically met when taking as lifting the *canonical* relational one (see [8]). We show that the situation is similar in the quantitative setting for a certain notion of *quantitative canonical* lifting. In particular, up-to context is compatible for the canonical lifting under very mild assumptions (Theorem 30). As an immediate corollary we have that, in a bialgebra, syntactic contexts are *non-expansive* with respect to the behavioural metric induced by the canonical lifting. This property and weaker variants of it (such as non-extensiveness or uniform continuity), considered to be the quantitative analogue of behavioural equivalence being a congruence, have recently received considerable attention (see e.g. [15, 1, 38]).

To fix intuitions, Section 2 provides a motivating example, formally treated in Section 7. We conclude with a comparison to related work and a discussion of open problems in Section 8.

All proofs and additional material are provided in the full version of this paper [10].

2 Motivating example: distances between regular languages

Computing various distances (such as the edit-distance or Cantor metric) between strings, and more generally between regular languages or string distributions, has found various practical

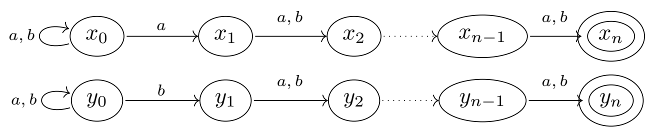


Figure 1 Example automaton

applications in various areas such as speech and handwriting recognition or computational biology. In this section we focus on a simple distance between regular languages, which we will call *shortest-distinguishing-word-distance* and is defined as $d_{\texttt{sdw}}(L, K) = c^{|w|}$ – where w is the shortest word which belongs to exactly one of the languages L, K and c is a constant such that $0 < c < 1$.

As a running example, which will be formally explained in Section 7, we consider the non-deterministic finite automaton in Figure 1 and the languages accepted by the states x_0, respectively y_0. We can similarly define a distance on the states of an automaton as the aforementioned distance between the languages accepted by the two states. The inequality

$$d_{\texttt{sdw}}(x_0, y_0) \le c^n \tag{1}$$

holds in this example since no word of length smaller than n is accepted by either state. Note that computing this distance is PSPACE-hard since the language equivalence problem for non-deterministic automata can be reduced to it.

One way to show this is to determinize the automaton in Figure 1 and to use the fact that for deterministic automata the shortest-distinguishing-word-distance can be expressed as the greatest fixpoint for a monotone function. Indeed, for a finite deterministic automaton $(Q, (\delta_a : Q \to Q)_{a \in A}, F \subseteq Q)$ over a finite alphabet A, we have that $d_{\texttt{sdw}} : Q \times Q \to [0, 1]$ is the greatest fixpoint of a function b defined on the complete lattice $[0, 1]^{Q \times Q}$ of functions ordered with the *reversed* pointwise order $\prec$ and given by

$$b(d)(q_1, q_2) = \begin{cases} 1, & \text{if only one of } q_1, q_2 \text{ is in } F \\ \max_{a \in A} c \cdot \{d(\delta_a(q_1), \delta_a(q_2))\}, & \text{otherwise} \end{cases} \tag{2}$$

Notice that we use the reversed order on $[0, 1]$, for technical reasons (see Section 4).

In order to prove (1) we can define a witness distance $\bar{d}$ on the states of the determinized automaton such that $\bar{d}(\{x_0\}, \{y_0\}) \le c^n$ and which is a post-fixpoint for b, i.e. $\bar{d} \prec b(\bar{d})$. Notice that this would entail $\bar{d} \prec d_{\texttt{sdw}}$ and hence $d_{\texttt{sdw}}(\{x_0\}, \{y_0\}) \le \bar{d}(\{x_0\}, \{y_0\}) \le c^n$.

This approach is problematic since the determinization of the automaton is of exponential size, so we have to define $\bar{d}$ for exponentially many pairs of sets of states. In order to mitigate the state space explosion we will use an up-to technique, which, just as up-to congruence in [9], exploits the join-semilattice structure of the state set $\mathcal{P}(Q)$ of the determinization of an NFA with state set Q. The crucial observation is the fact that given the states $Q_1, Q_2, Q_1', Q_2' \in \mathcal{P}(Q)$ in the determinization of an NFA, the following inference rule holds

$$\frac{d_{\texttt{sdw}}(Q_1, Q_2) \le r \qquad d_{\texttt{sdw}}(Q_1', Q_2') \le r}{d_{\texttt{sdw}}(Q_1 \cup Q_1', Q_2 \cup Q_2') \le r}$$

Based on this, we can define a monotone function f on $[0, 1]^{\mathcal{P}(Q) \times \mathcal{P}(Q)}$ that closes a function d according to such proof rules, producing $f(d)$, which is in general smaller (in the numerical sense) than d (the formal definition of f is given in Section 7). The general theory developed in this paper allows us to show in Section 7 that f is a sound up-to technique, i.e., it is sufficient to prove $\bar{d} \prec b(f(\bar{d}))$ in order to establish $\bar{d} \prec d_{\texttt{sdw}}$.

Using this technique it suffices to consider a quadratic number of pairs of sets of states. In particular we define a function $\bar{d}\colon \mathcal{P}(Q) \times \mathcal{P}(Q) \to [0,1]$ as follows:

$$\bar{d}(\{x_i\},\{y_j\}) = c^{n-\max\{i,j\}}$$

and $\bar{d}(X_1, X_2) = 1$ for all other values. Note that this function is not a metric but rather, what we will call in Section 4, a relation valued in $[0,1]$.

It holds that $\bar{d}(\{x_0\},\{y_0\}) = c^n$. It remains to show that $\bar{d} \prec b(f(\bar{d}))$. For this, it suffices to prove that

$$b(f(\bar{d}))(\{x_i\},\{y_j\}) \leq \bar{d}(\{x_i\},\{y_j\})\,.$$

For instance, when $i = j = 0$ we compute the sets of a-successors, which are $\{x_0, x_1\}$, $\{y_0\}$. We have that $\bar{d}(\{x_0\},\{y_0\}) = c^n \leq c^{n-1}$, $\bar{d}(\{x_0\},\{y_1\}) = c^{n-1}$ and using the up-to proof rule introduced above we obtain that $f(\bar{d})(\{x_0,x_1\},\{y_0\}) \leq c^{n-1}$. The same holds for the sets of b-successors and since x_0 and y_0 are both non-final we infer $b(f(\bar{d}))(\{x_0\},\{y_0\}) \leq c \cdot c^{n-1} = c^n = \bar{d}(\{x_0\},\{y_0\})$. The remaining cases (when $i \neq 0 \neq j$) are analogous.

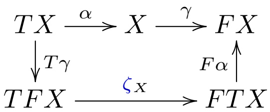
Our aim is to introduce such proof techniques for behavioural metrics, to make this kind of reasoning precise, not only for this specific example, but for coalgebras in general. Furthermore, we will not limit ourselves to metrics and distances, but we will consider more general relations valued in arbitrary quantales, of which the interval $[0,1]$ is an example.

3 Preliminaries

We recall here formal definitions for notions such as coalgebras, bialgebras or fibrations.

▶ **Definition 1.** *A coalgebra for a functor $F\colon \mathcal{C} \to \mathcal{C}$, or an F-coalgebra is a morphism $\gamma\colon X \to FX$ for some object X of $\mathcal{C}$, referred to as the carrier of the coalgebra γ. A morphism between two coalgebras $\gamma\colon X \to FX$ and $\xi\colon Y \to FY$ is a morphism $f\colon X \to Y$ such that $\xi \circ f = Ff \circ \gamma$. Algebras for the functor F, or F-algebras, are defined dually as morphisms of the form $\alpha\colon FX \to X$.*

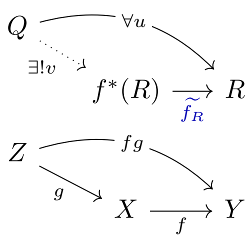
▶ **Definition 2.** *Consider two functors F, T and a natural transformation $\zeta\colon TF \Rightarrow FT$. A bialgebra for ζ is a tuple (X, α, γ) such that $\alpha\colon TX \to X$ is a T-algebra, $\gamma\colon X \to FX$ is an F-coalgebra so that the diagram on the left commutes. We call ζ the distributive law of the bialgebra (X, α, γ), even when T is not a monad.*



▶ **Example 3.** *The determinization of an NFA can be seen as a bialgebra with $X = \mathcal{P}Q$, the algebra $\mu_Q\colon \mathcal{P}\mathcal{P}Q \to \mathcal{P}Q$ given by the multiplication of the powerset monad, a coalgebra for the functor $F(X) = 2 \times X^A$, and a distributive law $\zeta\colon \mathcal{P}F \to F\mathcal{P}$ defined for $M \subseteq 2 \times X^A$ by $\zeta_X(M) = (\bigvee_{(b,f)\in M} b, [a \mapsto \{f(a) \mid (b,f) \in M\}])$. See [37, 23] for more details.*

We now introduce the notions of fibration and bifibration.

▶ **Definition 4.** *A functor $p\colon \mathcal{E} \to \mathcal{B}$ is called a fibration when for every morphism $f\colon X \to Y$ in $\mathcal{B}$ and every R in $\mathcal{E}$ with $p(R) = Y$ there exists a map $\widetilde{f_R}\colon f^*(R) \to R$ such that $p(\widetilde{f_R}) = f$, satisfying the following universal property: For all maps $g : Z \to X$ in $\mathcal{B}$ and $u\colon Q \to R$ in $\mathcal{E}$ sitting above fg (i.e., $p(u) = fg$) there is a unique map $v\colon Q \to f^*(R)$ such that $u = \widetilde{f_R}v$ and $p(v) = g$. For X in $\mathcal{B}$ we denote by $\mathcal{E}_X$ the fibre above X, i.e., the subcategory of $\mathcal{E}$ with objects mapped by p to X and arrows sitting above the identity on X.*



A map $\widetilde{f}$ as above is called a *Cartesian lifting* of f and is unique up to isomorphism. If we make a choice of Cartesian liftings, the association $R \mapsto f^*(R)$ gives rise to the so-called *reindexing functor* $f^* \colon \mathcal{E}_Y \to \mathcal{E}_X$. In what follows we will only consider *split fibrations*, that is, the Cartesian liftings are chosen such that we have $(fg)^* = g^* f^*$.

A functor $p \colon \mathcal{E} \to \mathcal{B}$ is called a *bifibration* if both $p \colon \mathcal{E} \to \mathcal{B}$ and $p^{op} \colon \mathcal{E}^{op} \to \mathcal{B}^{op}$ are fibrations. Interestingly, a fibration is a bifibration if and only if each reindexing functor $f^* \colon \mathcal{E}_Y \to \mathcal{E}_X$ has a left adjoint $\Sigma_f \dashv f^*$, see [21, Lemma 9.1.2]. We will call the functors Σ_f *direct images along* f.

Two important examples of bifibrations are those of relations over sets, $p \colon \mathsf{Rel} \to \mathsf{Set}$, and of predicates over sets, $p \colon \mathsf{Pred} \to \mathsf{Set}$, which played a crucial role in [8]. We do not recall their exact definitions here, as they arise as instances of the more general bifibrations of quantale-valued relations and predicates described in detail in the next section.

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\widehat{F}} & \mathcal{E}' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ \mathcal{B} & \xrightarrow{F} & \mathcal{B}' \end{array}$$

Given fibrations $p \colon \mathcal{E} \to \mathcal{B}$ and $p' \colon \mathcal{E}' \to \mathcal{B}'$ and a functor on the base categories $F \colon \mathcal{B} \to \mathcal{B}'$, we call $\widehat{F} \colon \mathcal{E} \to \mathcal{E}'$ a *lifting* of F when $p'\widehat{F} = Fp$. Notice that a lifting $\widehat{F}$ restricts to a functor between the fibres $\widehat{F}_X \colon \mathcal{E}_X \to \mathcal{E}'_{FX}$. We omit the subscript X when it is clear from the context.

Consider an arbitrary lifting $\widehat{F}$ of F and a morphism $f \colon X \to Y$ in $\mathcal{B}$. For any $R \in \mathcal{E}_Y$ the maps $\widetilde{Ff}_{\widehat{F}R} \colon (Ff)^*(\widehat{F}R) \to \widehat{F}R$ and $\widehat{F}(\widetilde{f_R}) \colon \widehat{F}(f^*R) \to \widehat{F}R$ sit above Ff. Using the universal property in Definition 4, we obtain a canonical morphism

$$\widehat{F} \circ f^*(R) \to (Ff)^* \circ \widehat{F}(R) \,. \tag{3}$$

A lifting $\widehat{F}$ is called a *fibred lifting* when the natural transformation in (3) is an isomorphism.

4 Moving towards a quantitative setting

We start by introducing two fibrations which are the foundations for our quantitative reasoning: predicates and relations valued in a quantale.

▶ **Definition 5.** *A quantale $\mathcal{V}$ is a complete lattice equipped with an associative operation $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ which is distributive on both sides over arbitrary joins $\bigvee$.*

This implies that for every $y \in \mathcal{V}$ the functor $- \otimes y$ has a right adjoint $[y, -]$. Similarly, for every $x \in \mathcal{V}$, the functor $x \otimes -$ has a right adjoint, denoted by $[\![x, -]\!]$. Thus, for every $x, y, z \in \mathcal{V}$, we have: $x \otimes y \leq z \iff x \leq [y, z] \iff y \leq [\![x, z]\!]$.

If $\otimes$ has an identity element or *unit* 1 for $\otimes$ the quantale is called *unital*. If $x \otimes y = y \otimes x$ for every $x, y \in \mathcal{V}$ the quantale is called *commutative* and we have $[x, -] = [\![x, -]\!]$. Hereafter, we only work with unital, commutative quantales.

▶ **Example 6.** *The Boolean algebra 2 with $\otimes = \wedge$ is a unital and commutative quantale: the unit is 1 and $[y, z] = y \to z$. The complete lattice $[0, \infty]$ ordered by the reversed order*[1] *of the reals, i.e. $\leq = \geq_{\mathbb{R}}$ and with $\otimes = +$ is a unital commutative quantale: the unit is 0 and for every $y, z \in [0, \infty]$ we have $[y, z] = z \mathbin{\dot{-}} y$ (truncated subtraction). Also $[0, 1]$ is a unital quantale where $r \otimes s = \min(r + s, 1)$ (truncated addition).*

▶ **Definition 7.** *Given a set X and a quantale $\mathcal{V}$, a $\mathcal{V}$-valued predicate on X is a map $p : X \to \mathcal{V}$. A $\mathcal{V}$-valued relation on X is a map $r : X \times X \to \mathcal{V}$.*

Given two $\mathcal{V}$-valued predicates $p, q : X \to \mathcal{V}$, we say that $p \leq q \iff \forall x \in X.\ p(x) \leq q(x)$.

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[1] To avoid confusion we use $\vee, \wedge$ in the quantale and $\inf, \sup$ in the reals.

▶ **Definition 8.** *A morphism between $\mathcal{V}$-valued predicates $p : X \to \mathcal{V}$ and $q : Y \to \mathcal{V}$ is a map $f : X \to Y$ such that $p \leq q \circ f$. We consider the category $\mathcal{V}$-Pred whose objects are $\mathcal{V}$-valued predicates and arrows are as above.*

▶ **Definition 9.** *A morphism between $\mathcal{V}$-valued relations $r : X \times X \to \mathcal{V}$ and $q : Y \times Y \to \mathcal{V}$ is a map $f : X \to Y$ such that $p \leq q \circ (f \times f)$. We consider the category $\mathcal{V}$-Rel whose objects are $\mathcal{V}$-valued relations and arrows are as above.*

The bifibration of $\mathcal{V}$-valued predicates. The forgetful functor $\mathcal{V}$-Pred $\to$ Set mapping a predicate $p : X \to \mathcal{V}$ to X is a bifibration. The fibre $\mathcal{V}\text{-Pred}_X$ is the lattice of $\mathcal{V}$-valued predicates on X. For $f : X \to Y$ in Set the reindexing and direct image functors on a predicate $p \in \mathcal{V}\text{-Pred}_Y$ are given by

$$f^*(p) = p \circ f \qquad \text{and} \qquad \Sigma_f(p)(y) = \bigvee \{p(x) \mid x \in f^{-1}(y)\}\,.$$

The bifibration of $\mathcal{V}$-valued relations. Notice that we have the following pullback in Cat, where $\Delta X = X \times X$. This is a change-of-base situation and thus the functor $\mathcal{V}$-Rel $\to$ Set mapping each $\mathcal{V}$-valued relation to its underlying set is also a bifibration. We denote by $\mathcal{V}\text{-Rel}_X$ the fibre above a set X. For each set X the functor ι restricts to an isomorphism $\iota_X : \mathcal{V}\text{-Rel}_X \to \mathcal{V}\text{-Pred}_{X \times X}$.

$$\begin{array}{ccc} \mathcal{V}\text{-Rel} & \xrightarrow{\ \iota\ } & \mathcal{V}\text{-Pred} \\ \downarrow & \lrcorner & \downarrow \\ \text{Set} & \xrightarrow[\ \Delta\ ]{} & \text{Set} \end{array}$$

For $f : X \to Y$ in Set the reindexing and direct image on $p \in \mathcal{V}\text{-Rel}_Y$ are given by

$$f^*(p) = p \circ (f \times f) \qquad \text{and} \qquad \Sigma_f(p)(y) = \bigvee \{p(x, x') \mid (x, x') \in (f \times f)^{-1}(y, y')\}\,.$$

For two relations $p, q \in \mathcal{V}\text{-Rel}_X$, we define their *composition* $p \cdot q \colon X \times X \to \mathcal{V}$ by $p \cdot q(x, y) = \bigvee \{p(x, z) \otimes q(z, y) \mid z \in X\}$. We define the *diagonal relation* $\mathit{diag}_X \in \mathcal{V}\text{-Rel}_X$ by $\mathit{diag}_X(x, y) = 1$ if $x = y$ and $\bot$ otherwise.

▶ **Definition 10.** *We say that a $\mathcal{V}$-valued relation $r : X \times X \to \mathcal{V}$ is*

- *reflexive if for all $x \in X$ we have $r(x, x) \geq 1$, (i.e. $r \geq \mathit{diag}_X$);*
- *transitive if $r \cdot r \leq r$;*
- *symmetric if $r = r \circ \mathsf{sym}_X$, where $\mathsf{sym}_X : X \times X \to X \times X$ is the symmetry isomorphism.*

We denote by $\mathcal{V}$-Cat the full subcategory of $\mathcal{V}$-Rel consisting of reflexive, transitive relations and by $\mathcal{V}\text{-Cat}_{\mathsf{sym}}$ the full subcategory of $\mathcal{V}$-Rel that are additionally symmetric.

Note that $\mathcal{V}$-Cat is the category of small categories enriched over the $\mathcal{V}$ in the sense of [25].

▶ **Example 11.** *For $\mathcal{V} = 2$, $\mathcal{V}$-valued relations are just relations. Reflexivity, transitivity and symmetry coincide with the standard notions, so $\mathcal{V}$-Cat is the category of preorders, while $\mathcal{V}\text{-Cat}_{\mathsf{sym}}$ is the category of equivalence relations.*

For $\mathcal{V} = [0, \infty]$, $\mathcal{V}$-Cat is the category of generalized metric spaces à la Lawvere [27] (i.e. directed pseudo-metrics and non-expansive maps), while $\mathcal{V}\text{-Cat}_{\mathsf{sym}}$ is the one of pseudo-metrics.

5 Lifting functors to $\mathcal{V}$-Pred and $\mathcal{V}$-Rel

In the previous section, we have introduced the fibrations of interest for quantitative reasoning. In order to deal with coinductive predicates in this setting, it is convenient to have a structured way to lift Set-functors to $\mathcal{V}$-valued predicates and relations, and eventually to $\mathcal{V}$-enriched categories. Our strategy is to first lift functors to $\mathcal{V}$-Pred and then, by exploiting the change of base, move these liftings to $\mathcal{V}$-Rel. A comparison with the extensions of Set-monads to the bicategory of $\mathcal{V}$-matrices [19] is provided in Section 8.

5.1 $\mathcal{V}$-predicate liftings

Liftings of Set-functors to the category Pred (for $\mathcal{V} = 2$) of predicates have been widely studied in the context of coalgebraic modal logic, as they correspond to modal operators (see e.g. [36]). For $\mathcal{V}$-Pred, we proceed in a similar way. Let us analyse what it means to have a fibred lifting $\widehat{F}$ to $\mathcal{V}$-Pred of an endofunctor F on Set. First, recall that the fibre $\mathcal{V}\text{-Pred}_X$ is just the preorder $\mathcal{V}^X$. So the restriction $\widehat{F}_X$ to such a fibre corresponds to a *monotone* map $\mathcal{V}^X \to \mathcal{V}^{FX}$. The fact that $\widehat{F}$ is a fibred lifting essentially means that the maps $(\mathcal{V}^X \to \mathcal{V}^{FX})_X$ form a natural transformation between the contravariant functors $\mathcal{V}^-$ and $\mathcal{V}^{F-}$. Furthermore, by Yoneda lemma we know that natural transformations $\mathcal{V}^- \Rightarrow \mathcal{V}^{F-}$ are in one-to-one correspondence with maps $ev : F\mathcal{V} \to \mathcal{V}$, which we will call hereafter *evaluation maps*. One can characterise the evaluation maps which correspond to the *monotone* natural transformations. These are the *monotone evaluation maps* $ev : (F\mathcal{V}, \ll) \to (\mathcal{V}, \leq)$ with respect to the usual order $\leq$ on $\mathcal{V}$ and an order $\ll$ on $F\mathcal{V}$ defined by applying the standard canonical relation lifting of F to $\leq$.

▶ **Proposition 12.** *There is a one-to-one correspondence between*

- *fibred liftings $\widehat{F}$ of F to $\mathcal{V}$-Pred,*
- *monotone natural transformations $\mathcal{V}^- \Rightarrow \mathcal{V}^{F-}$,*
- *monotone evaluation maps $ev : F\mathcal{V} \to \mathcal{V}$.*

Notice that the correspondence between fibred liftings and monotone evaluation maps is given in one direction by $ev = \widehat{F}(id_{\mathcal{V}})$, and conversely, by $\widehat{F}(p\colon X \to \mathcal{V}) = ev \circ F(p)$.

Evaluation maps as Eilenberg-Moore algebras. Evaluation maps have also been extensively considered in the coalgebraic approach to modal logics [36]. A special kind of evaluation map arises when the truth values $\mathcal{V}$ have an algebraic structure for a given monad (T, μ, η), that is, we have $\mathcal{V} = T\Omega$ for some object Ω and the evaluation map $T\mathcal{V} \to \mathcal{V}$ is an Eilenberg-Moore algebra for T. This notion of monadic modality has been studied in [17] where the category of free algebras for T was assumed to be order enriched. Under reasonable assumptions the evaluation map obtained as the free Eilenberg-Moore algebra on Ω (i.e., $ev\colon T\mathcal{V} \to \mathcal{V}$ is just $\mu_\Omega\colon T^2\Omega \to T\Omega$) is a monotone evaluation map, and hence gives rise to a fibred lifting of T (see [10] for more details.)

We provide next several examples of monotone evaluation maps which arise in this fashion.

▶ **Example 13.** *When T is the powerset monad $\mathcal{P}$ and $\Omega = 1$ we obtain $\mathcal{V} = 2$ and $\mu_1\colon \mathcal{P}2 \to 2$ corresponds to the $\Diamond$ modality, i.e. to an existential predicate transformer, see [17].*

▶ **Example 14.** *When T is the probability distribution functor $\mathcal{D}$ on Set and $\Omega = 2 = \{0, 1\}$ equipped with the order $1 \sqsubseteq 0$ we obtain $\mathcal{V} = \mathcal{D}\{0, 1\} \cong [0, 1]$ with the reversed order of the reals, i.e., $\leq\, = \,\geq_{\mathbb{R}}$. In this case $ev_{\mathcal{D}}(f) = \sum_{r\in[0,1]} r \cdot f(r)$ for $f\colon [0, 1] \to [0, 1]$ a probability distribution (expectation of the identity random variable).*

The canonical evaluation map. In the case $\mathcal{V} = 2$, there exists a simple way of lifting a functor $F\colon$ Set $\to$ Set: given a predicate $p\colon U \rightarrowtail X$, one defines the canonical predicate lifting $\widehat{F}_{\mathsf{can}}(U)$ of F as the epi-mono factorization of $Fp\colon FU \to FX$. This lifting corresponds to a canonical evaluation map true: $1 \to 2$ which maps the unique element of 1 into the element 1 of the quantale 2. For $\mathcal{V}$-relations, a generalized notion of canonical evaluation map was introduced in [19]. For $r \in \mathcal{V}$ consider the subset $\uparrow r = \{v \in \mathcal{V} \mid v \geq r\}$ and write $\mathsf{true}_r\colon \uparrow r \to \mathcal{V}$ for the inclusion. Given $u \in F\mathcal{V}$ we write $u \in F(\uparrow r)$ when u is in the image of the injective function $F(\mathsf{true}_r)$. Following [19], we define $ev_{\mathsf{can}} : F\mathcal{V} \to \mathcal{V}$ as follows:

$$ev_{\mathsf{can}}(u) = \bigvee\{r \mid u \in F(\uparrow r)\}.$$

▶ **Example 15.** *Assume F is the powerset functor $\mathcal{P}$ and let $u \in \mathcal{P}(\mathcal{V})$. We obtain that*

$$ev_{\mathsf{can}}(u) = \bigvee\{r \mid u \subseteq \uparrow r\}, \text{ or equivalently, } ev_{\mathsf{can}}(u) = \bigwedge u\,.$$

When $\mathcal{V} = 2$ we obtain $ev_{\mathsf{can}} \colon \mathcal{P}2 \to 2$ given by $ev_{\mathsf{can}}(u) = 1$ iff $u = \emptyset$ or $u = \{1\}$. This corresponds to the $\Box$ operator from modal logic. If $\mathcal{V} = [0,\infty]$ we have $ev_{\mathsf{can}}(u) = \sup u$.

▶ **Example 16.** *The canonical evaluation map for the distribution monad $\mathcal{D}$ and $\mathcal{V} = [0,1]$ is $ev_{\mathsf{can}}(f) = \sup_{r\in[0,1]} f(r)$, which is not the monad multiplication.*

The canonical evaluation map ev_{can} is monotone whenever the functor F preserves weak pullbacks (see [10]). For such functors, by Proposition 12, the map ev_{can} induces a fibred lifting $\widehat{F}_{\mathsf{can}}$ of F, called the canonical $\mathcal{V}$-Pred-lifting of F and defined by

$$\widehat{F}_{\mathsf{can}}(p)(u) = \bigvee\{r \mid F(p)(u) \in F(\uparrow r)\} \quad \text{for } p \in \mathcal{V}\text{-}\mathsf{Pred}_X \text{ and } u \in FX\,.$$

5.2 From predicates to relations via Wasserstein

We describe next how functor liftings to $\mathcal{V}$-Rel can be systematically obtained using the change-of-base situation described above. In particular, we see how the Wasserstein metric between probability distributions (defined in terms of couplings of distributions) can be naturally modelled in the fibrational setting.

Consider a $\mathcal{V}$-predicate lifting $\widehat{F}$ of a Set-functor F. A natural way to lift F to $\mathcal{V}$-relations using $\widehat{F}$ is to regard a $\mathcal{V}$-relation $r\colon X \times X \to \mathcal{V}$ as a $\mathcal{V}$-predicate on the product $X \times X$. Formally, we will use the isomorphism ι_X described in Section 4. We can apply the functor $\widehat{F}$ to the predicate $\iota_X(r)$ in order to obtain the predicate $\widehat{F} \circ \iota_X(r)$ on the set $F(X \times X)$. Ideally, we would want to transform this predicate into a relation on FX. So first, we have to transform it into a predicate on $FX \times FX$. To this end, we use the natural transformation

$$\lambda^F \colon F \circ \Delta \Rightarrow \Delta \circ F \text{ defined by } \lambda^F_X = \langle F\pi_1, F\pi_2\rangle \colon F(X \times X) \to FX \times FX\,. \tag{4}$$

We drop the superscript and simply write λ when the functor F is clear from the context. Additionally, the bifibrational structure of $\mathcal{V}$-Rel plays a crucial role, as we can use the direct image functor Σ_{λ_X} to transform $\widehat{F} \circ \iota_X(r)$ into a predicate on $FX \times FX$. Putting all the pieces together, we define a lifting of F on the fibre $\mathcal{V}$-Rel_X as the composite $\overline{F}_X$ given by:

$$\overline{F}_X \colon \mathcal{V}\text{-}\mathsf{Rel}_X \xrightarrow{\iota_X} \mathcal{V}\text{-}\mathsf{Pred}_{\Delta X} \xrightarrow{\widehat{F}_{\Delta X}} \mathcal{V}\text{-}\mathsf{Pred}_{F\Delta X} \xrightarrow{\Sigma_{\lambda_X}} \mathcal{V}\text{-}\mathsf{Pred}_{\Delta FX} \xrightarrow{\iota_{FX}{}^{-1}} \mathcal{V}\text{-}\mathsf{Rel}_{FX} \tag{5}$$

The aim is to define a lifting $\overline{F}$ of F to $\mathcal{V}$-Rel. The above construction provides the definition of $\overline{F}$ on the fibres and, in particular, on the objects of $\mathcal{V}$-Rel. For a morphism between $\mathcal{V}$-relations $p \in \mathcal{V}$-Rel_X and $q \in \mathcal{V}$-Rel_Y, i.e. a map $f\colon X \to Y$ such that $p \le f^*(q)$, we define $\overline{F}(f)$ as the map $Ff\colon FX \to FY$. To see that this is well defined it remains to show that $\overline{F}p \le (Ff)^*(\overline{F}q)$. This is the first part of the next proposition.

▶ **Proposition 17.** *The functor $\overline{F}$ defined above is a well defined lifting of F to $\mathcal{V}$-Rel. Furthermore, when F preserves weak pullbacks and $\widehat{F}$ is a fibred lifting of F to $\mathcal{V}$-Pred, then $\overline{F}$ is a fibred lifting of F to $\mathcal{V}$-Rel.*

Spelling out the concrete description of the direct image functor and of λ_X, we obtain for a relation $p \in \mathcal{V}$-Rel_X and $t_1, t_2 \in FX$, that

$$\overline{F}(p)(t_1,t_2) = \bigvee\{\widehat{F}(p)(t) \mid t \in F(X \times X), F\pi_i(t) = t_i\} \tag{6}$$

Unraveling the definition of $\widehat{F}(p)(t) = ev \circ F(p)$, we obtain for $\overline{F}(p)$ the same formula as for the extension of F on $\mathcal{V}$-matrices, as given in [19, Definition 3.4]. This definition in [19] is obtained by a direct generalisation of the Barr extensions of Set-functors to the bicategory of relations. In contrast, we obtained (6) by exploiting the fibrational change-of-base situation and by first considering a $\mathcal{V}$-Pred-lifting.

We call a lifting of the form $\overline{F}$ the *Wasserstein lifting* of F corresponding to $\widehat{F}$. This terminology is motivated by the next example.

▶ **Example 18.** *When $F = \mathcal{D}$ (the distribution functor), $\mathcal{V} = [0,1]$ and ev_F is as in Example 14 then $\overline{F}$ is the original Wasserstein metric from transportation theory [42], which – by the Kantorovich-Rubinstein duality – is the same as the Kantorovich metric. Here we compare two probability distributions $t_1, t_2 \in \mathcal{D}X$ and obtain as a result the coupling $t \in \mathcal{D}(X \times X)$ with marginal distributions t_1, t_2, giving us the optimal plan to transport the "supply" t_1 to the "demand" t_2. More concretely, given a metric $d\colon X \times X \to \mathcal{V}$, the (discrete) Wasserstein metric is defined as*

$$d^W(t_1, t_2) = \inf\{\sum_{x,y\in X} d(x,y)\cdot t(x,y) \mid \sum_y t(x,y) = t_1(x), \sum_x t(x,y) = t_2(y)\}.$$

On the other hand, when ev_F is the canonical evaluation map of Example 16 the corresponding $\mathcal{V}$-Rel-lifting $\overline{F}$ minimizes the longest distance (and hence the required time) rather than the total cost of transport.

▶ **Example 19.** *Let us spell out the definition when $F = \mathcal{P}$ (powerset functor), $\mathcal{V} = [0,1]$ and $ev_F\colon \mathcal{P}[0,1] \to [0,1]$ corresponds to* sup*, which is clearly monotone and is the canonical evaluation map as in Example 15.*

Then, given a metric $d\colon X \times X \to [0,1]$ and $X_1, X_2 \subseteq X$, the lifted metric is defined as follows (remember that the order is reversed on $[0,1]$):

$$\overline{F}(d)(X_1, X_2) = \inf\{\sup d[Y] \mid Y \subseteq X \times X, \pi_i[Y] = X_i\}$$

As explained in [5], this is the same as the Hausdorff metric *d^H defined by:*

$$d^H(X_1, X_2) = \sup\{\sup_{x_1\in X_1}\inf_{x_2\in X_2} d(x_1,x_2), \sup_{x_2\in X_2}\inf_{x_1\in X_1} d(x_1,x_2)\}$$

The next lemma establishes that this construction is functorial: liftings of natural transformations to $\mathcal{V}$-Pred can be converted into liftings of natural transformations between the corresponding Wasserstein liftings on $\mathcal{V}$-Rel.

▶ **Lemma 20.** *If there exists a lifting $\widehat{\zeta}\colon \widehat{F} \Rightarrow \widehat{G}$ of a natural transformation $\zeta\colon F \Rightarrow G$, then there exists a lifting $\overline{\zeta}\colon \overline{F} \Rightarrow \overline{G}$ between the corresponding Wasserstein liftings. Furthermore, when $\widehat{F}$ and $\widehat{G}$ correspond to monotone evaluation maps ev_F and ev_G, then the lifting $\widehat{\zeta}$ exists and is unique if and only if $ev_F \leq ev_G \circ \zeta_{\mathcal{V}}$.*

For $\mathcal{V} = [0,\infty]$, one is also interested in lifting functors to the category of (generalized) pseudo-metric spaces, not just of $[0,\infty]$-valued relations. This motivates the next question: when does the lifting $\overline{F}$ restrict to a functor on $\mathcal{V}$-Cat and $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym}}$? We have the following characterization theorem, where $\kappa_X\colon X \to \mathcal{V}$ is the constant function $x \mapsto 1$ and $u \otimes v\colon X \to \mathcal{V}$ denotes the pointwise tensor of two predicates $u, v\colon X \to \mathcal{V}$, i.e. $(u \otimes v)(x) = u(x) \otimes v(x)$.

▶ **Theorem 21.** *Assume $\widehat{F}$ is a lifting of F to $\mathcal{V}$-Pred and $\overline{F}$ is the corresponding $\mathcal{V}$-Rel Wasserstein lifting. Then*

- If $\widehat{F}(\kappa_X) \geq \kappa_{FX}$ then $\overline{F}(diag_X) \geq diag_{FX}$, hence $\overline{F}$ preserves reflexive relations;
- If $\widehat{F}$ is a fibred lifting, F preserves weak pullbacks and $\widehat{F}(p \otimes q) \geq \widehat{F}(p) \otimes \widehat{F}(q)$ then $\overline{F}(p \cdot q) \geq \overline{F}(p) \cdot \overline{F}(q)$, hence $\overline{F}$ preserves transitive relations;
- $\overline{F}$ preserves symmetric relations.

Consequently, when all the above hypotheses are satisfied, then the corresponding $\mathcal{V}$-Rel *Wasserstein lifting* $\overline{F}$ *restricts to a lifting of* F *to both* $\mathcal{V}$-Cat *and* $\mathcal{V}$-Cat$_{\mathsf{sym}}$*.*

For $\mathcal{V} = [0, \infty]$, the first condition of Theorem 21 is a relaxed version of a condition in [5, Definition 5.14] used to guarantee reflexivity. The second condition (for transitivity) is equivalent to a non-symmetric variant of a condition in [5] (see [10]).

We can establish generic sufficient conditions on a monotone evaluation map ev so that the corresponding $\mathcal{V}$-Pred-lifting $\widehat{F}$ satisfies the conditions of Theorem 21. In [10] we show that $\widehat{F}(p \otimes q) \geq \widehat{F}(p) \otimes \widehat{F}(q)$ holds whenever the map $\otimes\colon \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ is the carrier of a lax morphism in the category of F-algebras between $(\mathcal{V}, ev)^2 \to (\mathcal{V}, ev)$, i.e., $\otimes \circ (ev \times ev) \circ \lambda_{\mathcal{V}} \leq ev \circ F(\otimes)$. Furthermore, $\widehat{F}(\kappa_X) \geq \kappa_X$ holds whenever the map $\kappa_{\mathbb{1}}\colon \mathbb{1} \to \mathcal{V}$ is the carrier of a lax morphism from the one-element F-algebra $!\colon F\mathbb{1} \to \mathbb{1}$ to $(\mathcal{V}, ev)$, i.e., $\kappa_{\mathbb{1}} \circ ! \leq ev \circ F\kappa_{\mathbb{1}}$. These two requirements correspond to the conditions $(Q_\otimes)$, respectively (Q_k) satisfied by a topological theory in the sense of [19, Definition 3.1]. Since these two are satisfied by the canonical evaluation map ev_{can},[2] we immediately obtain

▶ **Proposition 22.** *Whenever* F *preserves weak pullbacks the canonical lifting* $\widehat{F}_{\mathsf{can}}$ *satisfies the conditions in Theorem 21:*

1. $\widehat{F}_{\mathsf{can}}(p \otimes q) \geq \widehat{F}_{\mathsf{can}}(p) \otimes \widehat{F}_{\mathsf{can}}(q)$*, for all* $p, q \in \mathcal{V}$-Pred$_X$*,*
2. $\widehat{F}_{\mathsf{can}}(\kappa_X) \geq \kappa_X$*.*

An immediate consequence of Proposition 22 and of Theorem 21 is that the Wasserstein lifting $\overline{F}_{\mathsf{can}}$ that corresponds to $\widehat{F}_{\mathsf{can}}$ restricts to a lifting of F to both $\mathcal{V}$-Cat and $\mathcal{V}$-Cat$_{\mathsf{sym}}$.

6 Quantitative up-to techniques

The fibrational constructions of the previous section provides a convenient setting to develop an abstract theory of quantitative up-to techniques. The coinductive object of interest is the greatest fixpoint of a monotone map b on $\mathcal{V}$-Rel, hereafter denoted by νb. Recall that an up-to technique, namely a monotone map f on $\mathcal{V}$-Rel, is sound whenever $d \leq b(f(d))$ implies $d \leq \nu b$, for all $d \in \mathcal{V}$-Rel$_X$; it is compatible if $f \circ b \leq b \circ f$ in the pointwise order. It is well-known that compatibility entails soundness. Another useful property is:

$$\text{if } f \text{ is compatible, then } f(\nu b) \leq \nu b\,. \tag{7}$$

Following [8], we assume hereafter that b can be seen as the composite

$$b\colon \mathcal{V}\text{-Rel}_X \xrightarrow{\overline{F}} \mathcal{V}\text{-Rel}_{FX} \xrightarrow{\xi^*} \mathcal{V}\text{-Rel}_X\,. \tag{8}$$

where $\xi\colon X \to FX$ is some coalgebra for $F\colon$ Set $\to$ Set. When F admits a final coalgebra $\omega\colon \Omega \to F\Omega$, the unique morphism $!\colon X \to \Omega$ induces the *behavioural closure* up-to technique

$$bhv\colon \mathcal{V}\text{-Rel}_X \xrightarrow{\Sigma_!} \mathcal{V}\text{-Rel}_\Omega \xrightarrow{!^*} \mathcal{V}\text{-Rel}_X \;\; \left(bhv(p)(x,y) = \bigvee\{p(x',y') \mid !(x) = !(x') \text{ and } !(y) = !(y')\}\right) \tag{9}$$

[2] The same observation is present in [19, Theorem 3.3(b)] but in a slightly different setting.

that, for $\mathcal{V} = 2$, corresponds to the usual up-to behavioural equivalence (bisimilarity). Other immediate generalisations are the *up-to reflexivity* (*ref*), *up-to transitivity* (*trn*) and *up-to symmetry* (*sym*) techniques. Whenever $\overline{F}$ is obtained through the Wasserstein construction of some $\widehat{F}$ satisfying the conditions of Theorem 21, these techniques are compatible (see [10] for more details).

As usual, compatible techniques can be combined together either by function composition ($\circ$) or by arbitrary joins ($\bigvee$). For instance compatibility of *up-to metric closure*, defined as the composite $mtr = trn \circ sym \circ ref$ follows immediately from compatibility of *trn*, *sym* and *ref*. In $\mathcal{V}$-Rel there is yet another useful way to combine up-to techniques – called *chaining* in [12] – and defined as the pointwise composition ($\bullet$) of relations.

▶ **Proposition 23.** *Let $f_1, f_2 \colon \mathcal{V}\text{-Rel}_X \to \mathcal{V}\text{-Rel}_X$ be compatible with respect to $b \colon \mathcal{V}\text{-Rel}_X \to \mathcal{V}\text{-Rel}_X$. If $\overline{F}(p \bullet q) \geq \overline{F}(p) \bullet \overline{F}(q)$ for all $p, q \in \mathcal{V}\text{-Rel}_X$, then $f_1 \bullet f_2$ is b-compatible.*

In the reminder of this section, we focus next on quantitative generalizations of the *up-to contextual closure* technique, which given an algebra $\alpha \colon TX \to X$, is seen as the composite:

$$f \colon \mathcal{V}\text{-Rel}_X \xrightarrow{\overline{T}} \mathcal{V}\text{-Rel}_{TX} \xrightarrow{\Sigma_\alpha} \mathcal{V}\text{-Rel}_X \,. \tag{10}$$

▶ **Example 24.** *Consider a signature Σ and the algebra of Σ-terms with variables in X $\mu_X \colon T_\Sigma T_\Sigma X \to T_\Sigma X$. The contextual closure $ctx \colon \mathcal{V}\text{-Rel}_{T_\Sigma X} \to \mathcal{V}\text{-Rel}_{T_\Sigma X}$ is defined as in* (10) *by taking the canonical lifting of the functor T_Σ. For all $t_1, t_2 \in T_\Sigma X$ and $d \in \mathcal{V}\text{-Rel}_{T_\Sigma X}$*

$$ctx(d)(t_1, t_2) = \bigvee_C \{ \bigwedge_j d(s^1_j, s^2_j) \mid t_i = C(s^i_0, \dots, s^i_n) \}$$

where C ranges over arbitrary contexts and s^j_i over terms. Notice that for $\mathcal{V} = 2$, this boils down to the usual notion of contextual closure of a relation. Details can be found in [10].

▶ **Example 25.** *In [12], the convex closure of $d \in \mathcal{V}\text{-Rel}_{\mathcal{D}(X)}$ is defined for $\Delta, \Theta \in \mathcal{D}(X)$ as*

$$cvx(d)(\Delta, \Theta) = \inf \{ \sum_i p_i \cdot d(\Delta_i, \Theta_i) \mid \Delta = \sum_i p_i \cdot \Delta_i, \Theta = \sum_i p_i \cdot \Theta_i \} \,.$$

This can be obtained as in (10) *by taking the lifting of $\mathcal{D}$ from Example 18 and the algebra given by the multiplication $\mu_X \colon \mathcal{D}\mathcal{D}X \to \mathcal{D}X$. Details can be found in [10].*

We consider next systems modelled as bialgebras $(X, \alpha \colon TX \to X, \xi \colon X \to FX)$ for a natural transformation $\zeta \colon T \circ F \Rightarrow F \circ T$. When b and f are as in (8), respectively (10), we use [8, Theorem 2] to obtain

▶ **Proposition 26.** *If there exists a lifting $\overline{\zeta} \colon \overline{T} \circ \overline{F} \Rightarrow \overline{F} \circ \overline{T}$ of ζ, then f is b-compatible.*

The next theorem establishes sufficient conditions for the existence of a lifting of ζ.

▶ **Theorem 27.** *Assume the natural transformation $\zeta \colon T \circ F \Rightarrow F \circ T$ lifts to a natural transformation $\widehat{\zeta} \colon \widehat{T} \circ \widehat{F} \Rightarrow \widehat{F} \circ \widehat{T}$ and that we have $\widehat{T} \circ \Sigma_{\lambda^F_X} \leq \Sigma_{T\lambda^F_X} \circ \widehat{T}$. Then ζ lifts to a distributive law $\overline{\zeta} \colon \overline{T} \circ \overline{F} \Rightarrow \overline{F} \circ \overline{T}$.*

Proof Sketch. Notice that $\widehat{T \circ F} := \widehat{T} \circ \widehat{F}$ and $\widehat{F \circ T} := \widehat{F} \circ \widehat{T}$ are liftings of the composite functors $T \circ F$, respectively $F \circ T$. We will denote by $\overline{T \circ F}$ and $\overline{F \circ T}$ the corresponding Wasserstein liftings obtained from $\widehat{T \circ F}$, respectively $\widehat{F \circ T}$ as in Section 5. We split the proof obligation into three parts:

$$\overline{T} \circ \overline{F} \underset{(1)}{\Rightarrow} \overline{T \circ F} \overset{\tilde{\zeta}}{\underset{(2)}{\Rightarrow}} \overline{F \circ T} \underset{(3)}{\Rightarrow} \overline{F} \circ \overline{T} \,.$$

(1) lifts the identity natural transformation on $T \circ F$. Its existence is proved using the hypothesis $\widehat{T} \circ \Sigma_{\lambda^F_X} \leq \Sigma_{T\lambda^F_X} \circ \widehat{T}$.
(2) is obtained by applying Lemma 20 to $\widehat{\zeta}$. Such liftings have already been studied in [4].
(3) lifts the identity natural transformation on $F \circ T$. ◀

The first requirement of the previous theorem holds for the canonical $\mathcal{V}$-Pred-liftings under mild assumptions on F and T.

▶ **Proposition 28.** *Assume that $\zeta\colon T \circ F \Rightarrow F \circ T$ is a natural transformation and that, furthermore, T preserves weak pullbacks and F preserves intersections. Then ζ lifts to a natural transformation $\widehat{\zeta}\colon \widehat{T}_{\mathsf{can}} \circ \widehat{F}_{\mathsf{can}} \Rightarrow \widehat{F}_{\mathsf{can}} \circ \widehat{T}_{\mathsf{can}}$.*

The next proposition establishes sufficient conditions for the second hypothesis of Theorem 27. We need a property on $\mathcal{V}$ that holds for the quantales in Example 6 and was also assumed in [19]. Given $u, v \in \mathcal{V}$ we write $u \lll v$ if for every $W \subseteq \mathcal{V}$, $v \leq \bigvee W$ implies that there exists $w \in W$ with $u \leq w$. The quantale $\mathcal{V}$ is *constructively completely distributive* iff for all $v \in \mathcal{V}$ it holds that $v = \bigvee\{u \in \mathcal{V} \mid u \lll v\}$. In [10] we prove a more general statement (not reported here for lack of space) in which the lifting of T is not assumed to be the canonical one, but that it is still useful to guarantee the result for interesting liftings, such as the one in Example 18.

▶ **Proposition 29.** *Assume that T preserves weak pullbacks and that $\mathcal{V}$ is constructively completely distributive. Then $\widehat{T}_{\mathsf{can}} \circ \Sigma_f \leq \Sigma_{Tf} \circ \widehat{T}_{\mathsf{can}}$.*

Combining Theorem 27 and Propositions 26, 28 and 29 we conclude:

▶ **Theorem 30.** *Let $(X, \alpha\colon TX \to X, \xi\colon X \to FX)$ be a bialgebra for a natural transformation $\zeta\colon T \circ F \Rightarrow F \circ T$. If $\mathcal{V}$ is constructively completely distributive, T preserves weak pullbacks and F preserves intersections, then $f = \overline{T}_{\mathsf{can}} \circ \Sigma_\alpha$ is compatible with respect to $b = \overline{F}_{\mathsf{can}} \circ \xi^*$.*

When α is the free algebra for a signature $\mu_X\colon T_\Sigma T_\Sigma X \to T_\Sigma X$ (as in Example 24), the above theorem guarantees that up-to contextual closure is compatible with respect to b. By (7), the following holds.

▶ **Corollary 31.** *For all terms t_1, t_2 and unary contexts C, $\nu b(t_1, t_2) \leq \nu b(C(t_1), C(t_2))$.*

For $\mathcal{V} = 2$, since the canonical quantitative lifting coincides with the canonical relational one, then νb is exactly the standard coalgebraic notion of behavioural equivalence [18]. Therefore the above corollary just means that behavioural equivalence is a congruence.

For $\mathcal{V} = [0, \infty]$ instead, this property boils down to *non-expansiveness* of contexts with respect to the behavioural metric. It is worth to mention that this property often fails in probabilistic process algebras when taking the standard Wasserstein lifting which, as shown in Example 18, is *not* the canonical one. We leave as future work to explore the implications of this insight.

7 Example: distance between regular languages

We will now work out the quantitative version of the up-to congruence technique for non-deterministic automata. We consider the shortest-distinguishing-word-distance $d_{\mathtt{sdw}}$, proposed in Section 2. As explained, we will assume an on-the-fly determinization of the non-deterministic automaton, i.e. formally we will work with a coalgebra that corresponds to a deterministic automaton on which we have a join-semilattice structure.

We explain next the various ingredients of the example:

Coalgebra and algebra. As outlined in Section 2 and Example 3 the determinization of an NFA with state space Q is a bialgebra (X, α, ξ) for the distributive law $\zeta_X : \mathcal{P}(2 \times X^A) \to 2 \times (\mathcal{P}(X))^A$, where $X = \mathcal{P}(Q)$, $\alpha : \mathcal{P}(X) \to X$ is given by union and $\xi : X \to 2 \times X^A$ specifies the DFA structure of the determinization. Hence, we instantiate the generic results in the previous section with $TX = \mathcal{P}(X)$, $FX = 2 \times X^A$ and ζ as defined in Example 3.

Lifting the functors. We take the quantale $\mathcal{V} = [0, 1]$ (Example 6) and consider the Wasserstein liftings of the endofunctors F and T to $\mathcal{V}$-Rel corresponding to the following evaluation maps:

- $ev_F(b, f) := c \cdot \max_{a \in A} f(a)$, where $b \in \{0, 1\}$, $f : A \to [0, 1]$ and c is the constant used in $d_{\mathtt{sdw}}$, and,
- $ev_T := ev^{\mathcal{P}}_{\mathsf{can}} = \sup$, the canonical evaluation map as in Example 15.

These are monotone evaluation maps that satisfy the hypothesis of Theorem 21. Hence the corresponding Wasserstein liftings restrict to $\mathcal{V}$-Cat. We computed the Wasserstein lifting of $T = \mathcal{P}$ in Example 19: applying the lifted functor $\overline{T}$ to a map $d : X \times X \to [0, 1]$, gives us the Hausdorff distance, i.e., $\overline{T}(d)(X_1, X_2) = d^H(X_1, X_2)$, where $X_1, X_2 \subseteq X$ and d^H denotes the Hausdorff metric based on d. On the other hand, the Wasserstein lifting of F corresponding to ev_F associates to a metric $d : X \times X \to [0, 1]$ the metric $\overline{F}(d) : FX \times FX \to [0, 1]$ given by

$$((b_1, f_1), (b_2, f_2)) \mapsto \begin{cases} 1 & \text{if } b_1 \neq b_2 \\ \max\limits_{a \in A} c \cdot \{d(f_1(a), f_2(a))\} & \text{otherwise} \end{cases}$$

Fixpoint equation. The map b for the fixpoint equation was defined in Section 6 as the composite $\xi^* \circ \overline{F}$. Using the above lifting $\overline{F}$, this computation yields exactly the map b defined in (2), whose largest fixpoint (smallest with respect to the natural order on the reals) is the shortest-distinguishing-word-distance introduced in Section 2.

Up-to technique. The next step is to determine the map f introduced in Section 6 for the up-to technique and defined as the composite $\Sigma_\alpha \circ \overline{T}$ on $\mathcal{V}$-Rel. Combining the definition of the direct image functors on $\mathcal{V}$-Rel with the lifting $\overline{T}$, we obtain for a given a map $d : X \times X \to [0, 1]$ that

$$f(d)(x_1, x_2) = \inf\{d^H(X_1, X_2) \mid X_1, X_2 \subseteq X, \alpha(X_i) = x_i\}$$

To show that $f(d)(Q_1, Q_2) \leq_{\mathbb{R}} r$ for two sets $Q_1, Q_2 \subseteq Q$ (i.e. $Q_1, Q_2 \in X$) and a constant r we use the following rules:

$$f(d)(\emptyset, \emptyset) \leq_{\mathbb{R}} r \qquad \frac{d(Q_1, Q_2) \leq_{\mathbb{R}} r}{f(d)(Q_1, Q_2) \leq_{\mathbb{R}} r} \qquad \frac{f(d)(Q_1, Q_2) \leq_{\mathbb{R}} r \quad f(d)(Q_1', Q_2') \leq_{\mathbb{R}} r}{f(d)(Q_1 \cup Q_1', Q_2 \cup Q_2') \leq_{\mathbb{R}} r}$$

Lifting of distributive law. In order to prove that the distributive law lifts to $\mathcal{V}$-Rel and hence that the up-to technique is sound by virtue of Proposition 26, we can prove that the two conditions of Theorem 27 are met by the $\mathcal{V}$-Pred liftings of F and T corresponding to the evaluation maps ev_F and ev_T, see [10].

Everything combined, we obtain a sound up-to technique, which implies that the reasoning in Section 2 is valid. Furthermore, as the example shows, the up-to technique can significantly simplify behavioural distance arguments and speed up computations.

8 Related and future work

Up-to techniques for behavioural metrics in a probabilistic setting have been considered in [12] using a generalization of the Kantorovich lifting [11]. In Section 6, we have shown that the basic techniques introduced in [12] (e.g., metric closure, convex closure and contextual closure) as well as the ways to combine them (composition, join and chaining) naturally fit within our framework. The main difference with our approach—beyond the fact that we consider arbitrary coalgebras while in [12] just coalgebras for a fixed functor—is that the definition of up-to techniques and the criteria to prove their soundness do not fit within the standard framework of [33]. Nevertheless, as illustrated by a detailed comparison in [10], the techniques of [12] can be reformulated within the standard theory and thus proved sound by means of our framework. An important observation brought to light by compositional methodology inherent to the fibrational approach, is that for probabilistic automata a bisimulation metric up-to convexity in the sense of [12] is just a bisimulation metric, see [10]. Nevertheless, the up-to convex closure technique can find meaningful applications in linear, trace-based behavioural metrics (see [4]).

The Wasserstein (respectively Kantorovich) lifting of the distribution functor involving couplings was first used for defining behavioural pseudometrics using final coalgebras in [40]. Our work is based instead on liftings for arbitrary functors, a problem that has been considered in several works (see e.g. [19, 2, 5, 24]), despite with different shades. The closest to our approach are [19] and [2] that we discuss next.

In [19] Hofmann introduces a generalization of the Barr extension (of Set-functors to Rel), namely he defines extensions of Set-monads to the bicategory of $\mathcal{V}$-matrices, in which 0-cells are sets and the $\mathcal{V}$-relations are 1-cells. Some of the definitions and techniques do overlap between the developments in [19] and the results we presented in Section 5. However, there are also some (subtle) differences which would not allow us to use off the shelf his results.

First, in order to reuse the results in [8], we need to recast the theory in a fibrational setting, rather than the bicategorical setting of [19]. The definition of *topological theory* [19, Definition 3.1] comprises what we call an evaluation map, but which additionally has to satisfy various conditions. An important difference with what we do is that the condition $(Q_{\bigvee})$ in the aforementioned definition entails that the predicate lifting one would obtain from such an evaluation map would be an *opfibred lifting*, rather than a fibred lifting as in our setting. Indeed, the condition $(Q_{\bigvee})$ can be equivalently expressed in terms of a natural transformation involving the *covariant* functor $P_{\mathcal{V}}$, as opposed to the *contravariant* one $\mathcal{V}^{-}$ that we used in Section 5.1. Lastly, in our framework we need to work with arbitrary functors, not necessarily carrying a monad structure.

In [5] we provided a generic construction for the Wasserstein lifting of a functor to the category of pseudo-metric spaces, rather than on arbitrary quantale-valued relations. The realisation that this construction is an instance as a change-of-base situation between $\mathcal{V}$-Rel and $\mathcal{V}$-Pred allows us to exploit the theory in [8] for up-to techniques and, as a side result, provides simpler (and cleaner) conditions for the restriction $\mathcal{V}$-Cat (Theorem 21).

We leave for future work several open problems. What is a universal property for the canonical Wasserstein lifting? Secondly, can the Wasserstein liftings presented here be captured in the framework of [2] or [24]? Preliminary discussions with the first author of the latter paper suggested that the codensity monad construction cannot accommodate, at least in a straightforward way, the Wasserstein lifting. We also leave for future work the development of up-to techniques for other quantales than 2 and $[0, 1]$. We are particularly interested in weighted automata [16] over quantales and in conditional transition systems, a variant of featured transition systems.

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