Cluster algebraic interpretation of infinite friezes

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January, 2017

Finite frieze patterns

Definition

A **frieze pattern** is an array such that:

1. the top row is a row of 1s
2. every diamond

$$\begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array}$$

   satisfies the rule $ad - bc = 1$.

Example (a **finite** integer frieze)

$$\begin{array}{llllllllllllllllll}
 & & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \cdots \\
\text{Row 2} & \cdots & & & \color{blue}{\mathbf{3}} & & \color{red}{\mathbf{1}} & & \color{green}{\mathbf{2}} & & \color{purple}{\mathbf{2}} & & \color{magenta}{\mathbf{1}} & & 3 & & 1 & \\
 & & & 2 & & 2 & & 1 & & 3 & & 1 & & 2 & & 2 & & \cdots \\
 & \cdots & & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 &
\end{array}$$

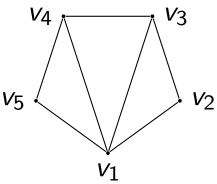
Note: every frieze pattern is completely determined by the 2nd row.

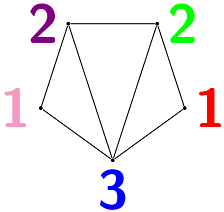
Conway and Coxeter (1970s)

Theorem

Finite integer frieze patterns $\longleftrightarrow$ triangulations of polygons

		1		1		1		1		1		1		1		$\cdots$
Row 2	$\cdots$		3		1		2		2		1		3		1	
		2		2		1		3		1		2		2		$\cdots$
	$\cdots$		1		1		1		1		1		1		1	



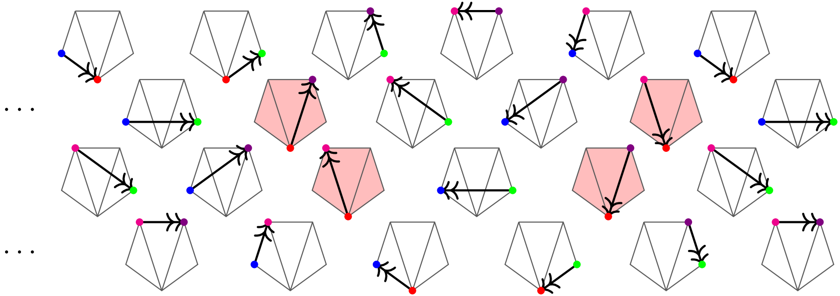


Broline, Crowe, and Isaacs (1970s)

Theorem

Entries of a finite integer frieze pattern $\longleftrightarrow$ *edges between two vertices.*

		1		1		1		1		1		1		…
Row 2	…		3		**1**		2		2		**1**		3	
		2		2		**1**		3		**1**		2		…
	…		1		1		1		1		1		1	



Cluster algebras (Fomin and Zelevinsky, 2000)

A **cluster algebra** is a commutative ring with a distinguished set of generators, called **cluster variables**.

Cluster algebras from surfaces (Fomin, Shapiro, and Thurston, 2006, etc.)

- Fix a marked surface: a Riemann surface S + marked points.
- Points are either on the boundary of S or in the interior (called punctures).
- The cluster variables $\longleftrightarrow$ arcs with no self-intersection.

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Remark

- *A cluster algebra of type A arises from a polygon.*
- *A cluster algebra of type D arises from a punctured polygon.*

Cluster variables and mutations

- Specify an initial set of cluster variables $\{x_1, \ldots, x_n\}$ (called **cluster**) and an initial triangulation T_{initial}.
- To produce all cluster variables, repeatedly perform a **mutation** μ_k in each of the n positions:

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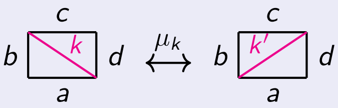
Definition (mutations)

- Replace diagonal k with k'



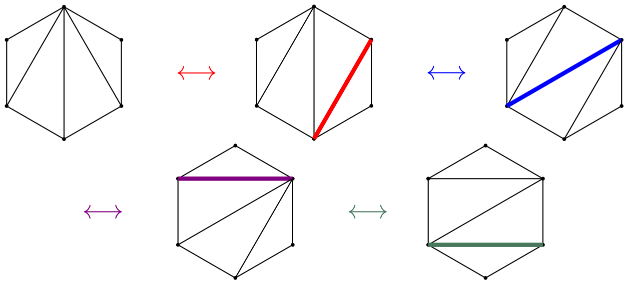
- The new cluster variable is U'_k, where

$$U_k\, U'_k = U_a\, U_c + U_b\, U_d\,.$$

Set weight of a boundary edge to 1.

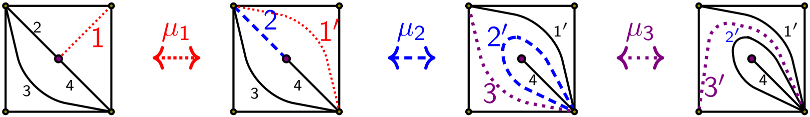
- Remark: μ_k is an involution.

Example: a sequence of flips for a polygon



Example: a sequence of mutations

T_{init} $\qquad$ $\mu_1(T_{\text{init}})$ $\qquad$ $\mu_2(\mu_1(T_{\text{init}}))$ $\qquad$ $\mu_3(\mu_2(\mu_1(T_{\text{init}})))$



$\{x_1, \ldots, x_4\}$ $\qquad$ $\{x_1', x_2, x_3, x_4\}$ $\qquad$ $\{x_1', x_2', x_3, x_4\}$ $\qquad$ $\{x_1', x_2', x_3', x_4\}$

Initial cluster $\qquad$ $x_1' = \frac{x_2+x_4}{x_1}$ $\qquad$ $x_2' = \frac{x_1'x_4+x_3x_4}{x_2}$ $\qquad$ $x_3' = \frac{x_1'+x_2'}{x_3}$

$x_2' = \frac{x_0x_2+x_1x_3+x_4}{x_1x_2}$ $\qquad$ $x_3' = \frac{x_2x_4+x_1x_3x_4+x_4^2+x_2^2+x_2x_4}{x_1x_2x_3}$

Remark

Arc 4 in the two right-most triangulations looks like it cannot be flipped, but there is a way to mutate at 4.

Cluster algebras

Definition (Fomin-Zelevinsky 2001)

The **cluster algebra** (corresponding to a triangulation T) is the subring of $\mathbb{Q}(x_1, \ldots, x_n)$ generated by all cluster variables.

Theorem (Fomin-Zelevinsky 2001)

Laurent Phenomenon: *each cluster variable can be expressed as a Laurent polynomial in* $\{x_1, \ldots, x_n\}$, *that is, as*

$$\frac{f(x_1, \ldots, x_n)}{x_1^{d_1} \ldots x_n^{d_n}},$$

where f *is a polynomial.*

Theorem (Lee - Schiffler, Gross - Hacking - Keel - Kontsevich, 2014, and special cases by others)

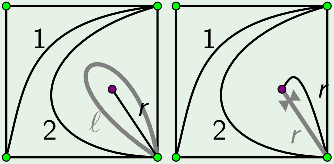
Positivity: *this polynomial* f *has positive coefficients.*

Finite type classification (Fomin-Zelevinsky 2002)

A cluster algebra is of **finite type** if there are finitely many cluster variables.

- The finite type cluster algebras are classified by the Dynkin diagrams.
- Type A and D are modeled by marked surfaces.

Example (Type D_4)



$$\begin{array}{c} & & r \\ & \nearrow & \\ 1 \leftarrow 2 & & \\ & \searrow & \\ & & r^{\bowtie} \end{array}$$

- Type B and C are modeled by orbifolds.

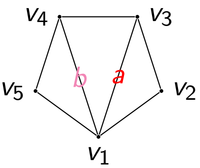
Caldero-Chapoton (2006)

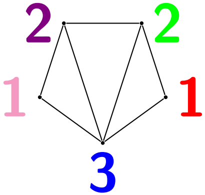
Theorem

The cluster variables of a cluster algebra from a triangulated polygon (type A) form a finite frieze pattern.

$$
\begin{array}{ccccccccccccc}
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 \\
& \frac{1+a+b}{a\,b} & & a & & \frac{1+b}{a} & & \frac{1+a}{b} & & b & & \frac{1+a+b}{a\,b} & \\
\cdots & & \frac{1+a}{b} & & b & & \frac{1+a+b}{a\,b} & & a & & \frac{1+b}{a} & & \frac{1+a}{b} \\
& 1 & & 1 & & 1 & & 1 & & 1 & & 1 &
\end{array}
$$

(Example: type A_2)





- Remark: When the variables a and b are specialized to 1, we recover the integer frieze pattern.

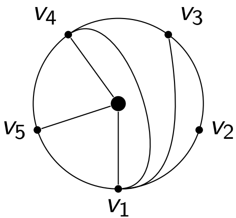
An infinite frieze patterns

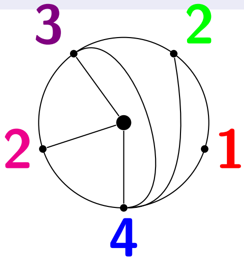
```
  1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1
    4   1   2   3   2   4   1   2   3   2   4   1   2   3   2
      3   1   5   5   7   3   1   5   5   7   3   1   5   5   7
        2   2   8  17   5   2   2   8  17   5   2   2   8  17
Level 1   3   3  27  12   3   3   3  27  12   3   3   3  27  12
          -----------------------------------------------------
            4  10  19   7   4   4  10  19   7   4   4  10  19
             13   7  11   9   5  13   7  11   9   5  13   7  11
                9   4  14  11  16   9   4  14  11  16   9   4
Level 2           5   5  17  35  11   5   5  17  35  11   5   5
                    6   6  54  24   6   6   6  54  24   6   6
                    -------------------------------------------
                      7  19  37  13   7   7  19  37  13   7   7
Level 3                22  13  20  15   8  22  13  20  15   8
                         15   7  23  17  25  15   7  23  17  25
                            8   8  26  53  17   8   8  26  53
                              9   9  81  36   9   9   9  81  36
                              ---------------------------------
                               10  28  55  19  10  10  28  55
Level 4                          31  19  29  21  11  31  19  29
                                   21  10  32  23  34  21  10
```

Infinite frieze patterns

Theorem (Baur, Fellner, Parsons, and Tschabold, 2015-2016)

*Any **infinite** frieze can be constructed from a triangulation of a punctured disk or an annulus/ infinite strip.*





1 1 1 1 1 1 1 1 1 1 1

4 1 2 3 2 4 1 2 3 2

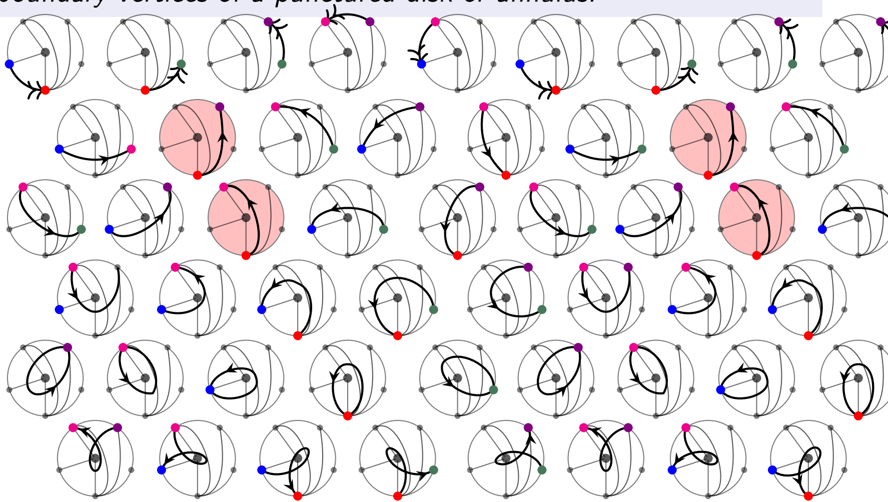
3 1 5 5 7 3 1 5 5 7

2 2 8 17 5 2 2 8 17 5

3 3 27 12 3 3 3 27 12

Theorem (G., Musiker, Vogel)

We construct an infinite frieze pattern of Laurent polynomials corresponding to arcs (allowing self-intersections) between the boundary vertices of a punctured disk or annulus.



- Convention: the boundary is to the right of the curve.

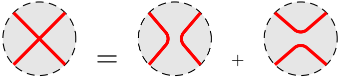
- Remark: When the variables are specialized to 1, we recover the integer frieze pattern.

```
1   1   1   1   1   1   1   1   1   1   1
  4   1   2   3   2   4   1   2   3   2   4
    3   1   5   5   7   3   1   5   5   7
      2   2   8  17   5   2   2   8  17   5
        3   3  27  12   3   3   3  27  12
        ------------------------------------
          4  10  19   7   4   4  10  19   7
           13   7  11   9   5  13   7  11
              9   4  14  11  16   9   4  14
                5   5  17  35  11   5   5
                  6   6  54  24   6   6   6
                  --------------------------
                    7  19  37  13   7   7
```

Theorem (G., Musiker, Vogel)

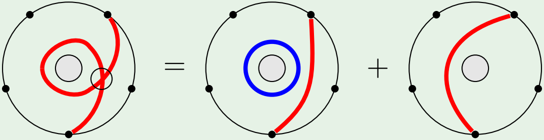
We construct an infinite frieze pattern of Laurent polynomials corresponding to arcs (allowing self-intersections) between the boundary vertices of a punctured disk or annulus.

Proof: The self-intersecting arcs correspond to elements of the algebra via skein relation



due to Musiker, Schiffler, and Williams (2011), etc.

Example (Example of resolving a self-crossing)

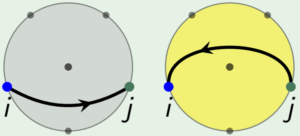


Complementary arcs

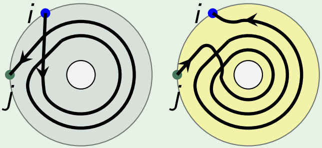
Definition (complementary arc)

Let $\gamma_k := \gamma_k(i,j)$ be the arc from i to j with $k-1$ self-crossings. The **complementary arc** γ_k^C of γ_k is the arc from j to i with $k-1$ self-crossings.

Example (γ_1 and γ_1^C)

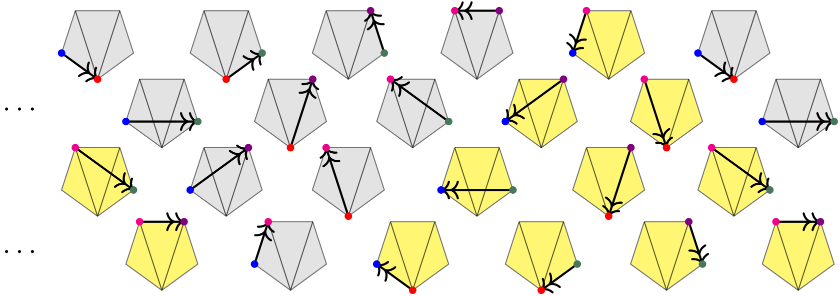


Example (γ_3 and γ_3^C)

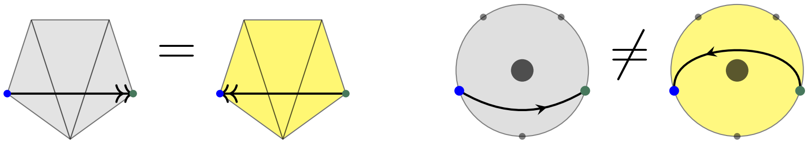


- Their concatenation is a loop with twice as many self-crossings.

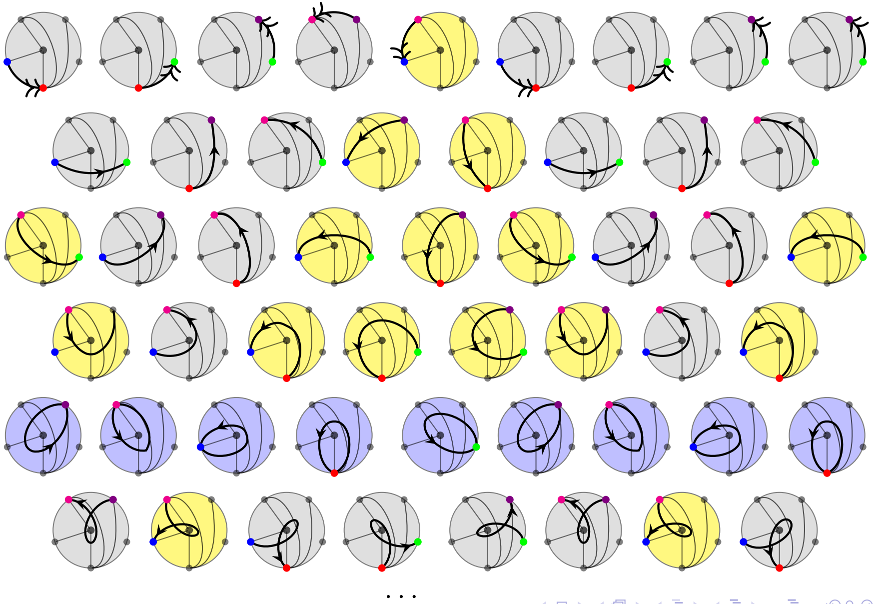
Glide symmetry for finite friezes



In a polygon vs a punctured disk/annulus



Complementary arcs in infinite friezes



Progression formulas

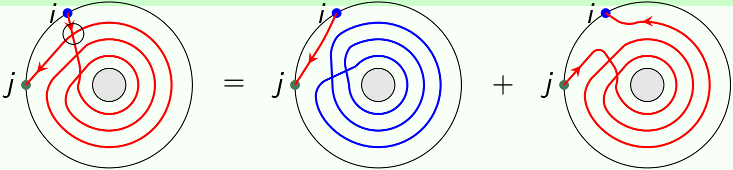
Theorem (G., Musiker, and Vogel)

Let γ_1 be an arc starting and finishing at vertices i and j. For $k = 1, 2, \ldots$ and $1 \leq m \leq k-1$, we have

$$x(\gamma_k) = x(\gamma_m)x(Brac_{k-m}) + x(\gamma^C_{k-2m+1}), \text{ where:}$$

- *for $r \geq 0$, γ^C_{-r} is the curve γ_{r+1} with a kink, so that $x(\gamma^C_{-r}) = -x(\gamma_{r+1})$, and*
- *a* **bracelet** *$Brac_k$ is obtained by following a (non-contractible, non-self-crossing, kink-free) loop k times, creating $(k-1)$ self-crossings.*

$$x(\gamma_4) = x(\gamma_1)x(Brac_3) + x(\gamma_3^C) \text{ for } k = 4,\ m = 1$$



Arithmetic progressions in frieze patterns from punctured disks (Tschabold)

1 1 1 1 1 **1** 1 1 1 1 **1** 1 1 1 1 **1**
4 1 2 3 2 4 1 2 3 2 4 1 2 3 2
3 **1** 5 5 7 3 **1** 5 5 7 3 **1** 5 5 7
2 2 8 17 5 2 2 8 17 5 2 2 8 17
3 3 27 **12** 3 3 3 27 **12** 3 3 3 27 **12**

---

4 10 19 7 4 **4** 10 19 7 4 **4** 10 19
13 7 11 9 5 13 7 11 9 5 13 7 11
9 **4** 14 11 16 9 **4** 14 11 16 9 **4**
5 5 17 35 11 5 5 17 35 11 5 5
6 6 54 **24** 6 6 6 54 **24** 6 6

---

7 19 37 13 7 **7** 19 37 13 7 **7**
22 13 20 15 8 22 13 20 15 8
15 **7** 23 17 25 15 **7** 23 17 25
8 8 26 53 17 8 8 26 53
9 9 81 **36** 9 9 9 81 **36**

---

10 28 55 19 10 **10** 28 55
31 19 29 21 11 31 19 29

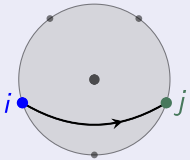
Geometric interpretation of the arithmetic progression

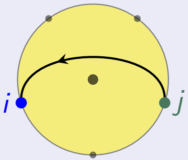
Lemma

The arc from vertex *blue* *to vertex* *green* *with k self-intersections*

$$=$$

the arc from the vertex *blue* *to vertex* *green* *with $k-1$ self-intersections*

$+$ 

$+$ 

Proof: Progression formulas and induction.

Constant **growth factor** across rows (Baur, Fellner, Parsons, Tschabold)

$$
\begin{array}{lccccccccccccccccccccccc}
& -1 && -1 && -1 && -1 && -1 && -1 \\
s_0 = 2 && \mathbf{0} && \mathbf{0} && \mathbf{0} && \mathbf{0} && \mathbf{0} && \mathbf{0} \\
&&& 1 && 1 && 1 && 1 && 1 && 1 \\
&&&& 1 && 2 && 6 && 1 && 2 && 6 \\
s_1 = 3 &&&&& \mathbf{1} && \mathbf{11} && \mathbf{5} && \mathbf{1} && \mathbf{11} && \mathbf{5} \\
&&&&&& 5 && 9 && 4 && 5 && 9 && 4 \\
&&&&&&& 4 && 7 && 19 && 4 && 7 && 19 \\
s_2 = 7 &&&&&&&& \mathbf{3} && \mathbf{33} && \mathbf{15} && \mathbf{3} && \mathbf{33} && \mathbf{15} \\
&&&&&&&&& 14 && 26 && 11 && 14 && 26 && 11 \\
&&&&&&&&&& 11 && 19 && 51 && 11 && 19 && 51 \\
s_3 = 18 &&&&&&&&&&& \mathbf{8} && \mathbf{88} && \mathbf{40} && \mathbf{8} && \mathbf{88} && \mathbf{40} \\
&&&&&&&&&&&& 37 && 69 && 29 && 37 && 69 && 29 \\
&&&&&&&&&&&&& 29 && 50 && 134 && 29 && 50 && 134 \\
&&&&&&&&&&&&&&&& \ddots &&&&&& \ddots
\end{array}
$$

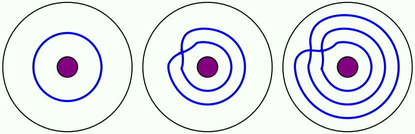
In punctured disk case, this growth factor is always 2.

```
   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1
     4   1   2   3   2   4   1   2   3   2   4   1   2   3   2   4   1   2   3
       3   1   5   5   7   3   1   5   5   7   3   1   5   5   7   3   1   5
         2   2   8  17   5   2   2   8  17   5   2   2   8  17   5   2   2   8
           3   3  27  12   3   3   3  27  12   3   3   3  27  12   3   3   3   2
        ----------------------------------------------------------------------
             4  10  19   7   4   4  10  19   7   4   4  10  19   7   4   4  10
              13   7  11   9   5  13   7  11   9   5  13   7  11   9   5  13
                 9   4  14  11  16   9   4  14  11  16   9   4  14  11  16   9
                   5   5  17  35  11   5   5  17  35  11   5   5  17  35  11
                     6   6  54  24   6   6   6  54  24   6   6   6  54  24   6
                  ------------------------------------------------------------
                       7  19  37  13   7   7  19  37  13   7   7  19  37  13
                        22  13  20  15   8  22  13  20  15   8  22  13  20  15
                          15   7  23  17  25  15   7  23  17  25  15   7  23   1
                             8   8  26  53  17   8   8  26  53  17   8   8  26
                               9   9  81  36   9   9   9  81  36   9   9   9   8
                            --------------------------------------------------
                                10  28  55  19  10  10  28  55  19  10  10  28
                                  31  19  29  21  11  31  19  29  21  11  31   1
                                    21  10  32  23  34  21  10  32  23  34  21
                                      11  11  35  71  23  11  11  35  71  23   1
                                        12  12 108  48  12  12  12 108  48  12
                                      ----------------------------------------
                                         13  37  73  25  13  13  37  73  25   1
                                           40  25  38  27  14  40  25  38  27
```

Geometric interpretation of the growth factor

The "jump" between frieze level k and $k+1$ correspond to the bracelet which crosses itself $k-1$ times.

Bracelets with 0, 1, and 2 self-crossings



Definition

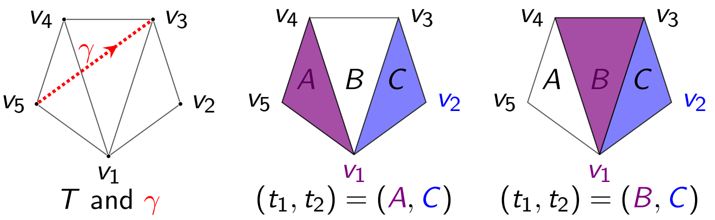
Define the **normalized Chebyshev polynomial** by

$$T_0(x) = 2,\, T_1(x) = x, \text{ and}$$

the recurrence relation

$$T_k(x) = x\, T_{k-1}(x) - T_{k-2}(x).$$

For punctured disk, every bracelet corresponds to the integer 2.

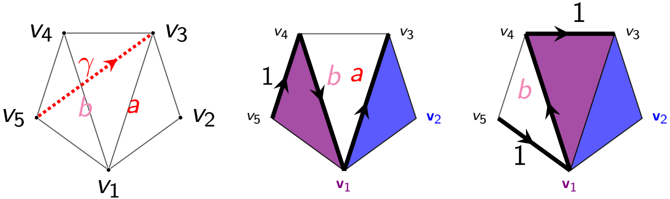


Definition (Broline, Crowe, and Isaacs, 1970s)

Let R_1, R_2, ..., R_r be the boundary vertices to the right of γ. A **BCI tuple** for γ is an r-tuple $(t_1, \ldots, t_r)$ such that:

(B1) the i-th entry t_i is a triangle of T having R_i as a vertex. (We say that the vertex R_i is matched to the triangle in the i-th entry of the tuple).

(B2) the entries are pairwise distinct.



T and γ | $1\, b^{-1}\, a$ | $1\, b^{-1}\, 1$

Definition (Studied by Carroll-Price, 2003 and others)

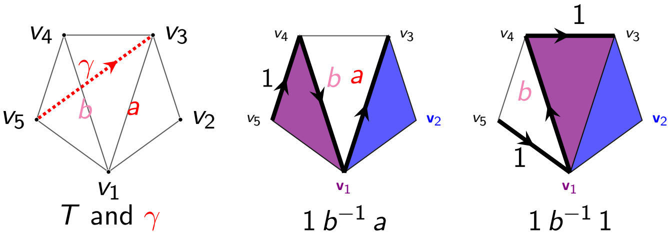
A **BCI trail** w for $(t_1, \ldots, t_r)$ is a walk from the beginning to the ending point of γ along T such that:

(TR 1) the triangles $t_1, \ldots, t_r$ are to the right of w,

(TR 2) the other triangles are to the left of w.

Proposition (G., Musiker, Vogel)

There is a lattice-preserving bijection between the BCI tuples and T-paths (of Schiffler-Thomas, 2006-2007).



$$x_{\gamma} = \frac{a}{b} + \frac{1}{b}$$

Corollary (G., Musiker, Vogel)

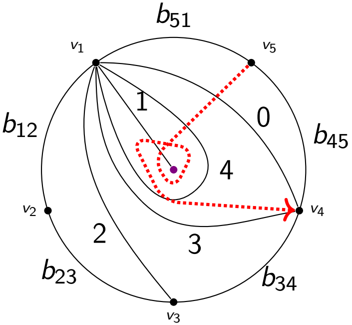
1. BCI-trail formula: the Laurent polynomial expansion corresponding to γ written in the variables of T is

$$x_{\gamma} = \sum_{w} \frac{\prod \textit{odd steps of } w}{\prod \textit{even steps of } w}$$

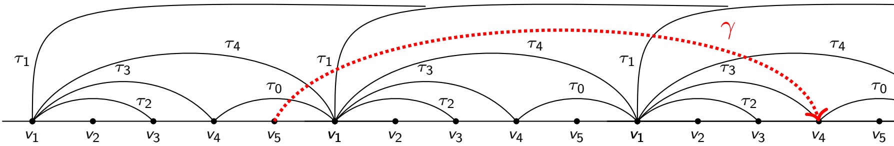
where the sum is over all BCI-trails w for γ.

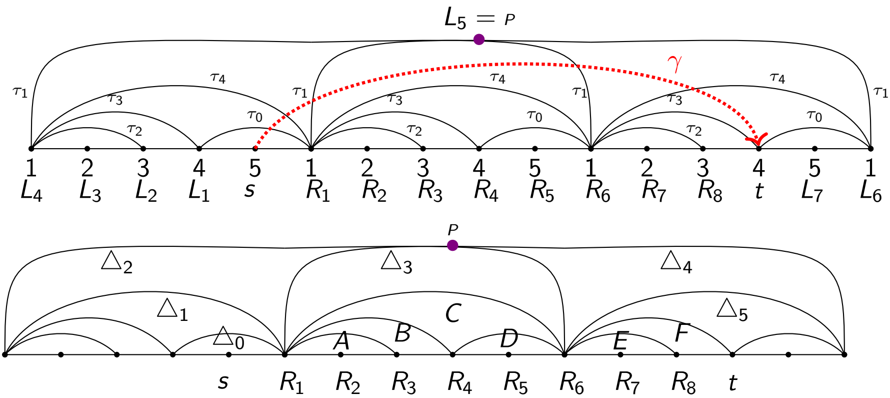
2. Starting from the minimal BCI-tuple for γ, we get to all the BCI-tuples by toggling a triangle to get closer to the starting point of γ.

From ideal triangulation T to its polygon cover

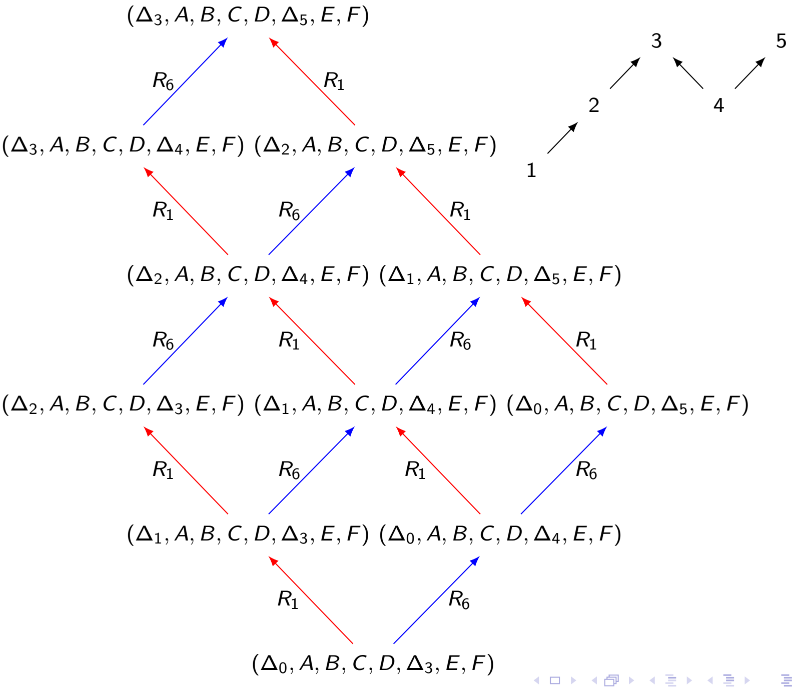


$\rightsquigarrow$





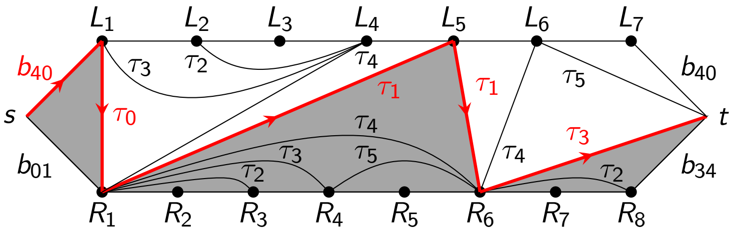
$L_5 = P$
γ
τ_1 τ_4 τ_3 τ_2 τ_0
1 2 3 4 5 1 2 3 4 5 1 2 3 4 5 1
L_4 L_3 L_2 L_1 s R_1 R_2 R_3 R_4 R_5 R_6 R_7 R_8 t L_7 L_6
P
$\triangle_2$ $\triangle_3$ $\triangle_4$
$\triangle_1$ C $\triangle_5$
$\triangle_0$ A B D E F
s R_1 R_2 R_3 R_4 R_5 R_6 R_7 R_8 t


(Δ3, A, B, C, D, Δ5, E, F)
R6
R1
(Δ3, A, B, C, D, Δ4, E, F)
(Δ2, A, B, C, D, Δ5, E, F)
R1
R6
R1
(Δ2, A, B, C, D, Δ4, E, F)
(Δ1, A, B, C, D, Δ5, E, F)
R6
R1
R6
R1
(Δ2, A, B, C, D, Δ3, E, F)
(Δ1, A, B, C, D, Δ4, E, F)
(Δ0, A, B, C, D, Δ5, E, F)
R1
R6
R1
R6
(Δ1, A, B, C, D, Δ3, E, F)
(Δ0, A, B, C, D, Δ4, E, F)
R1
R6
(Δ0, A, B, C, D, Δ3, E, F)
3
5
2
4
1

The 11 BCI tuples correspond to the 11 terms of the expansion of x_γ:

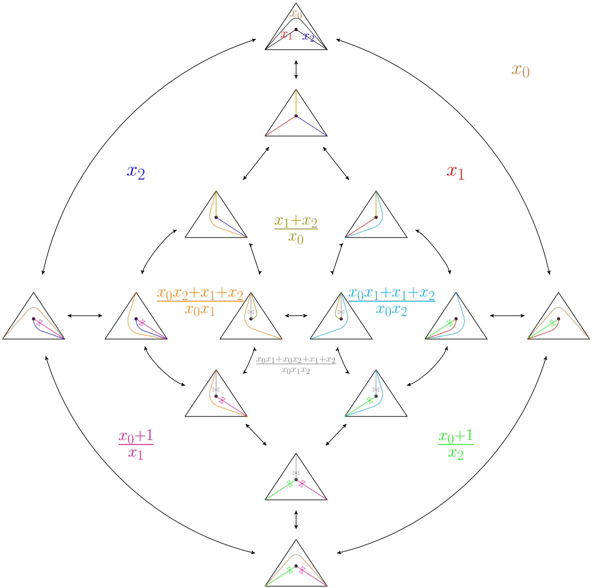
$$x_\gamma = \frac{\mathbf{x_0 x_1 x_4} + 2x_1x_3x_4 + 2x_0^2 + 4x_0x_3 + 2x_3^2}{x_0x_1x_4}$$

For example, from the minimal BCI tuple $b = (\Delta_0, A, B, C, D, \Delta_3, E, F)$, we get a BCI trail $(b_{40}, \tau_5, \tau_1, \tau_1, \tau_3)$.



$$\text{with weight } b_{40}\, x_0^{-1}\, x_1\, x_1^{-1}\, x_3 = \frac{1\, x_1\, x_3}{x_0\, x_1}$$

Thank you



x_0
x_1
x_2
x_0
x_2
x_1
$\frac{x_1+x_2}{x_0}$
$\frac{x_0x_2+x_1+x_2}{x_0x_1}$
$\frac{x_0x_1+x_1+x_2}{x_0x_2}$
$\frac{x_0x_1+x_0x_2+x_1+x_2}{x_0x_1x_2}$
$\frac{x_0+1}{x_1}$
$\frac{x_0+1}{x_2}$