Cluster algebraic interpretation of infinite friezes

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Finite frieze patterns

**Definition**

A **frieze pattern** is an array such that:

1. the top row is a row of 1s
2. every diamond

\[
\begin{array}{c c c}
& b & \\
a & d & \\
& c & \\
\end{array}
\]

satisfies the rule \( ad - bc = 1 \).

**Example (a finite integer frieze)**

<table>
<thead>
<tr>
<th>1</th>
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<th>1</th>
<th>1</th>
<th>1</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row 2</td>
<td>...</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>1</td>
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<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

Note: every frieze pattern is completely determined by the 2nd row.
Conway and Coxeter (1970s)

Theorem

Finite integer frieze patterns $\leftrightarrow$ triangulations of polygons

\[
\begin{array}{cccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\
\ldots & 3 & 1 & 2 & 2 & 1 & 3 & 1 \\
2 & 2 & 1 & 3 & 1 & 2 & 2 & \ldots \\
\ldots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

\[
\begin{tikzpicture}
  \node at (0,0) (v1) [circle, draw, inner sep=1.5pt] {$v_1$};
  \node at (1,1) (v2) [circle, draw, inner sep=1.5pt] {$v_2$};
  \node at (2,2) (v3) [circle, draw, inner sep=1.5pt] {$v_3$};
  \node at (1,3) (v4) [circle, draw, inner sep=1.5pt] {$v_4$};
  \node at (0,2) (v5) [circle, draw, inner sep=1.5pt] {$v_5$};
  \draw (v1) -- (v2) -- (v3) -- (v1);
  \draw (v2) -- (v4) -- (v2);
  \draw (v3) -- (v5) -- (v3);
\end{tikzpicture}
\quad \quad
\begin{tikzpicture}
  \node at (0,0) (v1) [circle, draw, inner sep=1.5pt] {$v_1$};
  \node at (1,1) (v2) [circle, draw, inner sep=1.5pt] {$v_2$};
  \node at (2,2) (v3) [circle, draw, inner sep=1.5pt] {$v_3$};
  \node at (3,1) (v4) [circle, draw, inner sep=1.5pt] {$v_4$};
  \node at (2,3) (v5) [circle, draw, inner sep=1.5pt] {$v_5$};
  \draw (v1) -- (v2) -- (v3) -- (v1);
  \draw (v2) -- (v4) -- (v2);
  \draw (v3) -- (v5) -- (v3);
\end{tikzpicture}
\]
Broline, Crowe, and Isaacs (1970s)

Theorem

Entries of a finite integer frieze pattern $\leftrightarrow$ edges between two vertices.

\[
\begin{array}{cccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\
\text{Row 2} & \ldots & 3 & 1 & 2 & 2 & 1 & 3 \\
2 & 2 & 1 & 3 & 1 & 2 & \ldots \\
\ldots & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]
Cluster algebras (Fomin and Zelevinsky, 2000)

A cluster algebra is a commutative ring with a distinguished set of generators, called cluster variables.

Cluster algebras from surfaces
(Fomin, Shapiro, and Thurston, 2006, etc.)

- Fix a marked surface: a Riemann surface $S$ + marked points.
- Points are either on the boundary of $S$ or in the interior (called punctures).
- The cluster variables $\leftrightarrow$ arcs with no self-intersection.
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- The cluster variables $\leftrightarrow$ arcs with no self-intersection.

Remark

- A *cluster algebra of type A arises from a polygon*.
- A *cluster algebra of type D arises from a punctured polygon*.
Cluster variables and mutations

▶ Specify an initial set of cluster variables $\{x_1, \ldots, x_n\}$ (called cluster) and an initial triangulation $T_{\text{initial}}$.

▶ To produce all cluster variables, repeatedly perform a mutation $\mu_k$ in each of the $n$ positions:

- Replace diagonal $k$ with $k'$:

  $\begin{array}{ccc}
  a & b & c \\
  b & d & k \\
  c & k & a
  \end{array}$

  $\begin{array}{ccc}
  a & b & c \\
  b & d & k' \\
  c & k' & a
  \end{array}$

  The new cluster variable is $U'_k$, where $U_kU'_k = U_{ak} + U_{bk}$.

- Set weight of a boundary edge to 1.

  Remark: $\mu_k$ is an involution.
Cluster variables and mutations

▶ Specify an initial set of cluster variables \( \{ x_1, \ldots, x_n \} \) (called \textbf{cluster}) and an initial triangulation \( T_{\text{initial}} \).

▶ To produce all cluster variables, repeatedly perform a \textbf{mutation} \( \mu_k \) in each of the \( n \) positions:

\[
\begin{align*}
\text{Definition (mutations)} \\
\text{▶ Replace diagonal } k \text{ with } k' \\
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 b \\
 k \qquad \mu_k \qquad k' \\
 c \\
 a
\end{array}
\end{array}
\end{array}
\end{align*}
\]

▶ The new cluster variable is \( U'_k \), where

\[
U_k U'_k = U_a U_c + U_b U_d.
\]

Set weight of a boundary edge to 1.

▶ Remark: \( \mu_k \) is an involution.
Example: a sequence of flips for a polygon
Example: a sequence of mutations

\[ T_{\text{init}} \quad \mu_1(T_{\text{init}}) \quad \mu_2(\mu_1(T_{\text{init}})) \quad \mu_3(\mu_2(\mu_1(T_{\text{init}}))) \]

\{x_1, \ldots, x_4\} \quad \{x'_1, x_2, x_3, x_4\} \quad \{x'_1, x'_2, x_3, x_4\} \quad \{x'_1, x'_2, x'_3, x_4\}

Initial cluster

\[ x'_1 = \frac{x_2 + x_4}{x_1} \quad x'_2 = \frac{x'_1 x_4 + x_3 x_4}{x_2} \quad x'_3 = \frac{x'_1 + x'_2}{x_3} \]

\[ x'_2 = \frac{x_0 x_2 + x_1 x_3 + x_4}{x_1 x_2} \quad x'_3 = \frac{x_2 x_4 + x_1 x_3 x_4 + x_4^2 + x_2^2 + x_2 x_4}{x_1 x_2 x_3} \]

Remark

Arc 4 in the two right-most triangulations looks like it cannot be flipped, but there is a way to mutate at 4.
Cluster algebras

**Definition (Fomin-Zelevinsky 2001)**

The **cluster algebra** (corresponding to a triangulation $T$) is the subring of $\mathbb{Q}(x_1, \ldots, x_n)$ generated by all cluster variables.

**Theorem (Fomin-Zelevinsky 2001)**

**Laurent Phenomenon:** each cluster variable can be expressed as a Laurent polynomial in $\{x_1, \ldots, x_n\}$, that is, as

$$f(x_1, \ldots, x_n) \cdot \frac{x_1^{d_1} \cdots x_n^{d_n}}{x_1^{d_1} \cdots x_n^{d_n}},$$

where $f$ is a polynomial.

**Theorem (Lee - Schiffler, Gross - Hacking - Keel - Kontsevich, 2014, and special cases by others)**

**Positivity:** this polynomial $f$ has positive coefficients.
Finite type classification (Fomin-Zelevinsky 2002)

A cluster algebra is of **finite type** if there are finitely many cluster variables.

- The finite type cluster algebras are classified by the Dynkin diagrams.
- Type $A$ and $D$ are modeled by marked surfaces.

**Example (Type $D_4$)**

- Type $B$ and $C$ are modeled by orbifolds.
Caldero-Chapoton (2006)

Theorem

*The cluster variables of a cluster algebra from a triangulated polygon (type A) form a finite frieze pattern.*

\[
\begin{array}{cccccccc}
\ldots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\frac{1+a+b}{ab} & a & \frac{1+b}{a} & \frac{1+a}{b} & b & \frac{1+a+b}{ab} \\
\frac{1+a}{b} & b & \frac{1+a+b}{ab} & a & \frac{1+b}{a} & \frac{1+a}{b} & \frac{1+a+b}{ab} \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

(Example: type $A_2$)

- Remark: When the variables $a$ and $b$ are specialized to 1, we recover the integer frieze pattern.
An infinite frieze patterns

Level 1

Level 2

Level 3

Level 4
Infinite frieze patterns

Theorem (Baur, Fellner, Parsons, and Tschabold, 2015-2016)

Any infinite frieze can be constructed from a triangulation of a punctured disk or an annulus/infinite strip.
Theorem (G., Musiker, Vogel)

We construct an infinite frieze pattern of Laurent polynomials corresponding to arcs (allowing self-intersections) between the boundary vertices of a punctured disk or annulus.

▶ Convention: the boundary is to the right of the curve.
Remark: When the variables are specialized to 1, we recover the integer frieze pattern.
Theorem (G., Musiker, Vogel)

We construct an infinite frieze pattern of Laurent polynomials corresponding to arcs (allowing self-intersections) between the boundary vertices of a punctured disk or annulus.

Proof: The self-intersecting arcs correspond to elements of the algebra via skein relation

\[
\begin{align*}
\begin{tikzpicture}
\node (X) at (0,0) [circle, draw, very thick, dashed] {X};
\end{tikzpicture}
& = \begin{tikzpicture}
\node (X) at (0,0) [circle, draw, very thick, dashed] {X};
\end{tikzpicture} + \begin{tikzpicture}
\node (X) at (0,0) [circle, draw, very thick, dashed] {X};
\end{tikzpicture}
\end{align*}
\]

due to Musiker, Schiffler, and Williams (2011), etc.

Example (Example of resolving a self-crossing)
Complementary arcs

Definition (complementary arc)

Let $\gamma_k := \gamma_k(i,j)$ be the arc from $i$ to $j$ with $k - 1$ self-crossings. The complementary arc $\gamma_k^C$ of $\gamma_k$ is the arc from $j$ to $i$ with $k - 1$ self-crossings.

Example ($\gamma_1$ and $\gamma_1^C$)

Their concatenation is a loop with twice as many self-crossings.
Glide symmetry for finite friezes

In a polygon vs a punctured disk/annulus
Complementary arcs in infinite friezes
Progression formulas

Theorem (G., Musiker, and Vogel)

Let $\gamma_1$ be an arc starting and finishing at vertices $i$ and $j$. For $k = 1, 2, \ldots$ and $1 \leq m \leq k - 1$, we have

$$x(\gamma_k) = x(\gamma_m)x(\text{Brac}_{k-m}) + x(\gamma_{k-2m+1}),$$

where:

- for $r \geq 0$, $\gamma_r^C$ is the curve $\gamma_{r+1}$ with a kink, so that $x(\gamma_r^C) = -x(\gamma_{r+1})$, and

- a bracelet $\text{Brac}_k$ is obtained by following a (non-contractible, non-self-crossing, kink-free) loop $k$ times, creating $(k - 1)$ self-crossings.

$$x(\gamma_4) = x(\gamma_1)x(\text{Brac}_3) + x(\gamma_3^C)$$ for $k = 4$, $m = 1$
Arithmetic progressions in frieze patterns from punctured disks (Tschabold)
Geometric interpretation of the arithmetic progression

Lemma

The arc from vertex blue to vertex green with \( k \) self-intersections

\[=\]

the arc from the vertex blue to vertex green with \( k - 1 \) self-intersections

Proof: Progression formulas and induction.
Constant growth factor across rows (Baur, Fellner, Parsons, Tschabold)

\[
\begin{align*}
\begin{array}{cccccccc}
-1 & -1 & -1 & -1 & -1 & -1 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\end{align*}
\]

\[s_0 = 2\]

\[
\begin{align*}
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 2 & 6 & 1 & 2 & 6 & \\
5 & 9 & 4 & 5 & 9 & 4 & \\
4 & 7 & 19 & 4 & 7 & 19 & \\
3 & 33 & 15 & 3 & 33 & 15 & \\
\end{array}
\end{align*}
\]

\[s_1 = 3\]

\[s_2 = 7\]

\[s_3 = 18\]
In punctured disk case, this growth factor is always 2.
**Geometric interpretation of the growth factor**

The “jump” between frieze level $k$ and $k + 1$ correspond to the bracelet which crosses itself $k - 1$ times.

**Bracelets with 0, 1, and 2 self-crossings**

---

**Definition**

Define the **normalized Chebyshev polynomial** by

\[
T_0(x) = 2, \quad T_1(x) = x, \quad \text{and}
\]

the recurrence relation

\[
T_k(x) = x \ T_{k-1}(x) - T_{k-2}(x).
\]

For punctured disk, every bracelet corresponds to the integer 2.
Definition (Broline, Crowe, and Isaacs, 1970s)

Let $R_1, R_2, \ldots, R_r$ be the boundary vertices to the right of $\gamma$. A **BCI tuple** for $\gamma$ is an $r$-tuple $(t_1, \ldots, t_r)$ such that:

1. **(B1)** the $i$-th entry $t_i$ is a triangle of $T$ having $R_i$ as a vertex. (We say that the vertex $R_i$ is matched to the triangle in the $i$-th entry of the tuple).

2. **(B2)** the entries are pairwise distinct.
Definition (Studied by Carroll-Price, 2003 and others)

A **BCI trail** $w$ for $(t_1, \ldots, t_r)$ is a walk from the beginning to the ending point of $\gamma$ along $T$ such that:

**(TR 1)** the triangles $t_1, \ldots, t_r$ are to the right of $w$,

**(TR 2)** the other triangles are to the left of $w$.

Proposition (G., Musiker, Vogel)

*There is a lattice-preserving bijection between the BCI tuples and $T$-paths (of Schiffler-Thomas, 2006-2007).*
Corollary (G., Musiker, Vogel)

1. BCI-trail formula: the Laurent polynomial expansion corresponding to $\gamma$ written in the variables of $T$ is

$$x_\gamma = \frac{a}{b} + \frac{1}{b}$$

2. Starting from the minimal BCI-tuple for $\gamma$, we get to all the BCI-tuples by toggling a triangle to get closer to the starting point of $\gamma$. 
From ideal triangulation $T$ to its polygon cover
\[ L_5 = P \]
Figure: Based on the setup of Figure ?? (redrawn at the top). Left: the lattice $L_{BCI}(\gamma)$ of the BCI tuples for $\gamma$. Right: the poset $Q_\gamma$. 

\[(\Delta_3, A, B, C, D, \Delta_5, E, F)\] 

\[(\Delta_3, A, B, C, D, \Delta_4, E, F)\] 

\[(\Delta_2, A, B, C, D, \Delta_5, E, F)\] 

\[(\Delta_2, A, B, C, D, \Delta_4, E, F)\] 

\[(\Delta_1, A, B, C, D, \Delta_5, E, F)\] 

\[(\Delta_1, A, B, C, D, \Delta_4, E, F)\] 

\[(\Delta_0, A, B, C, D, \Delta_5, E, F)\] 

\[(\Delta_0, A, B, C, D, \Delta_4, E, F)\] 

\[(\Delta_0, A, B, C, D, \Delta_3, E, F)\] 

\[(\Delta_0, A, B, C, D, \Delta_3, E, F)\]
The 11 BCI tuples correspond to the 11 terms of the expansion of $x_\gamma$:

$$x_\gamma = \frac{x_0 x_1 x_4 + 2x_1 x_3 x_4 + 2x_0^2 + 4x_0 x_3 + 2x_3^2}{x_0 x_1 x_4}$$

For example, from the minimal BCI tuple $b = (\Delta_0, A, B, C, D, \Delta_3, E, F)$, we get a BCI trail $(b_{40}, \tau_5, \tau_1, \tau_1, \tau_3)$.

With weight $b_{40} x_0^{-1} x_1 x_1^{-1} x_3 = \frac{1 x_1 x_3}{x_0 x_1}$
If \( w \) is an ordinary arc, let an associated \( A_n \) such that 
\[
A_n \text{ is homotopic to } \cdots, n
\]
Definition: ordinary arcs = (disk is a maximum collection of distinct arcs that
\[
is not homotopic to the segment from \( 0 \) to \( x \)).\]
In addition, we have:
\[
\text{Definition: ordinary arcs that are pairwise...}
\]
\[
\text{Cluster algebras from orientable surfaces...}
\]
\[
\text{Fig. 2: Four of the nine...}
\]
\[
\text{The bracelets collection forms a basis for unpunctured surfaces...}
\]
\[
\text{Theorem (atomic basis)...}
\]