Edge-colored graphs as higher-dimensional maps

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LIPN, Paris 13

May 18, 2016
GT Combi LIX
Discretization of manifolds

- 2D discrete surfaces: triangulations, $p$–angulations and combinatorial maps
- 3D triangulations: gluings of tetrahedra
Discretization of manifolds

- 2D discrete surfaces: triangulations, $p$–angulations and combinatorial maps
- 3D triangulations: gluings of tetrahedra
- How to represent them in a suitable fashion for combinatorics?
- Equivalent of $p$–angulations?
- Enumeration?
Colored triangulations and colored graphs

Bijection with (stuffed colored Walsh) maps
Colored triangulations and colored graphs

Bijection with (stuffed colored Walsh) maps
Combinatorial maps

Graph with cyclic ordering of edges incident to each vertex

\[ \neq \]
Combinatorial maps

Graph with cyclic ordering of edges incident to each vertex

Cyclic ordering defines **faces**: follow the corners
2p–angulation

- Faces of degree 2p
- Duality: vertices of degree 2p

- Euler’s relation with $E(M) = pV(M)$

$$F(M) - E(M) + V(M) = F(M) - (p-1)V(M) = 2 - 2g(M)$$

- $g(M) \geq 0 \Rightarrow$ bound on $F(M)$ linear in $V(M)$
- Maximizing $F(M)$ at fixed $V(M)$ equivalent to $g(M) = 0$
2p-angulation

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2$p$–angulation

- Faces of degree $2p$
- Duality: vertices of degree $2p$

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What do we know?

Maps: from Tutte to today

- Enumeration [Tutte’s equations, matrix models]
- Bijections [Cori-Vauquelin-Schaeffer, Bouttier-Di Francesco-Guitter]
- Topological recursion [Eynard]
- Continuum limit [Brownian sphere]
- More being developed nowadays [Hurwitz, integrable hierarchies, etc.]
What do we know?

Simplicial complexes in higher dim

▸ ??
What do we know?

Simplicial complexes in higher dim

- ??
- (Mostly) numerical works (Ambjorn, Jurkewic, Jonsson, Loll, etc.)
- Why? Combinatorics difficult to control!
- Most recent analytical attempts via colored graphs
  - Mostly by physicists
    [Gurau, Krajewski, Rivasseau, Tanasa, Vignes-Tourneret and students]
  - Try a more systematic combinatorial study
    [Gurau-Schaeffer, Bonzom-Lionni and wip w/ Monteil]
The physics

- Einstein’s 2nd revolution
  Gravitation = Geometry of space–time

- Quantum physics
  Quantum = probabilistic, random

Gravitation and quantum together
Space–time metric is a random variable

  Quantum gravity = random geometry
Discrete quantum gravity

Define quantum gravity at the discrete level

Two approaches
Discrete quantum gravity

Define quantum gravity at the discrete level

**Two approaches**

<table>
<thead>
<tr>
<th>Regge calculus, LQG, Spin foams</th>
<th>Dynamical triangulations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fix a discretization</td>
<td>Edges have fixed lengths</td>
</tr>
<tr>
<td>Geometry = edge lengths</td>
<td>Geometry = discretization</td>
</tr>
<tr>
<td>Quantization = $\int \prod_e d\ell_e$</td>
<td>Quantization</td>
</tr>
<tr>
<td>or $\sum_{\text{quantum numbers}}$</td>
<td>$\sum_{\text{Geometries}} = \sum_{\text{Triangulations}}$</td>
</tr>
</tbody>
</table>

In 2nd approach, $\sum_{\text{Triangulations}} \Rightarrow$ generating function!
How to represent triangulations?

Triangulations

- Gluing of simplices (tetrahedra, pentachora, etc.)
- Defined by attaching maps
- Ensemble of triangulations defined by constraints on attaching maps

Various ensembles

- Various ensembles in topology (simplicial, CW, Δ–complexes, etc.)
- Not suitable for combinatorics (too wild)
- Digging through old work, found colored triangulations
  - [Italian school: crystallization, graph–encoded manifold]
- Represented by edge–colored graphs
(d + 1)-colored graphs

- Bipartite graphs
  - black and white vertices
- Edges colored with d + 1 possible colors
- Vertices of degree d + 1
- All colors incident exactly once at each vertex
Colored graphs

(d + 1)-colored graphs

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Faces are closed cycles with only two colors.
Colored graphs

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Triangulations from colored graphs

\[
\text{duality} \quad \begin{cases} 
\text{vertex} & \rightarrow & d\text{-simplex} \\
\text{edge} & \rightarrow & (d - 1)\text{-simplex} \\
\text{face} & \rightarrow & (d - 2)\text{-simplex} \\
k\text{-bubble} & \rightarrow & (d - k)\text{-simplex}
\end{cases}
\]

- Boundary triangles labeled by a color \( c = 0, \ldots, d \)

Colors identify all sub-simplices
Triangulations from colored graphs

Duality

\[
\begin{align*}
\text{vertex} & \rightarrow d\text{–simplex} \\
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- Boundary triangles labeled by a color \( c = 0, \ldots, d \)
- Induced colorings
- Edges labeled by pair of colors

Colors identify all sub-simplices
Triangulations from colored graphs

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\end{cases}
\end{align*}
\]

- Boundary triangles labeled by a color \( c = 0, \ldots, d \)
- Induced colorings
- Edges labeled by pair of colors
- Nodes labeled by three colors

Colors identify all sub-simplices
Colored attaching maps

Gluing respecting all induced colorings

Theory of crystallization and GEMs (graph–encoded manifolds): 
\((d + 1)\)-colored graphs are dual to triangulations of pseudo-manifolds of 
dimension \(d\)  \[\text{Pezzana, Ferri, Cagliardi, Lins}\].
The 2D case

- Gluing of 2 \( p \) triangles with boundary of color 0
- Dually: Components with all colors but 0
The 2D case

Gluing of 2

Dually: Components with all colors but 0
The 2D case

Gluing of 2\(p\) triangles with boundary of color 0

Dually: Components with all colors but 0
The 2D case

2\(p\)–angle

- Gluing of \(2p\) triangles with boundary of color 0
- Dually: Components with all colors but 0
The 3D case
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The 3D case
Bubbles

- Colored graph with colors 0, 1, \ldots, d
  (triangulation in dim d)
- **Bubble**: connected piece with colors 1, \ldots, d
  Obtained by removing the color 0
- All graphs obtained by gluing bubbles along edges of color 0
- \( \mathcal{G}(B) \) set of \((d + 1)\)-colored graphs where all bubbles are \( B \)
Bubbles II

- 2D: only bubbles with $2p$ vertices
  Cycles of colors (1, 2)
Bubbles II

- 2D: only bubbles with \(2p\) vertices
  Cycles of colors \((1, 2)\)

- Many more in higher dimensions
- Vast world to explore
Vertices: two types

- cycle with colors (0, 1)
- cycle with colors (0, 2)
The problem

- Set $B$ a bubble, $G \in \mathcal{G}(B)$
- Enumerate w.r.t.
  - # bubbles $b(G)$
  - # subsimplices of codimension 2 which belong to bubble boundary
- Face of colors $(0, c)$: cycle with colors $(0, c)$

Number of faces $F(G) = \sum_{c=1}^{d} F_{0c}(G)$

- Classify graphs according to $F(G)$ at fixed $b(G)$

$$\mathcal{G}_b(B) = \bigcup_{F} \mathcal{G}_{b}^{(F)}(B)$$

- Focus on $\mathcal{G}_{b}^{(F)}(B)$
  How to maximize $F(G)$ at fixed number of bubbles $b(G)$?
Gurau’s degree theorem

Bound on $F(G)$

There exists $\omega(G) \geq 0$

$$F(G) - (d - 1)(p(B) - 1)b(G) = d - \omega(G) \leq d$$

- $d = 2$
  $$F(G) - (p(B) - 1)b(G) = 2 - \omega(G) \Rightarrow \omega(G) = 2g(G)$$
- For $d \geq 3$, bound can be saturated only for certain type of bubbles
- Maximizing graphs (melonic) are series–parallel
  - Bijection with trees
  - Expected from numerics
- Gurau–Schaeffer classification according to the degree
- Need to investigate more generic bubbles
Colored triangulations and colored graphs

Bijection with (stuffed colored Walsh) maps
Mechanism

- How to control faces?

Cycle of graph to star–maps

$V$ $e_1$ $e_2$ $e_3$ $e_5$ $e_4$

$O_1 O_2 O_3 O_5 O_4$
Mechanism

- How to control faces? Maps!
- Same mechanism as Tutte’s bijection between bipartite quadrangulations and generic maps, in the dual picture

**Cycle of graph to star–maps**

Cyclically ordered list of objects \((o_1, \ldots, o_n)\)

Edge \((o_k, o_{k+1})\) maps to corner between \(e_k\) and \(e_{k+1}\)
From bubble to map

- Choose a pairing of $B$
From bubble to map

- Choose a pairing of $B$
- Orient edges from white to black; merge pairs to blue vertices
- Oriented cycle of color $j \rightarrow$ counter–clockwise star–map
Choose a pairing of $B$

Orient edges from white to black; merge pairs to blue vertices

Oriented cycle of color $j \rightarrow$ counter-clockwise star-map

Edge of color $j \rightarrow$ counter-clockwise corner of color $j$
Example

\[ B = \begin{array}{c}
1 & 2 \\
1 & 2 \\
1 & 2 \\
\end{array} \]

\[ \pi \rightarrow \begin{array}{c}
1 & 2 \\
1 & 2 \\
1 & 2 \\
\end{array} \]

\[ = M(B, \pi) \]
Universal part

Cycles of color 0 and pairs of vertices $\rightarrow$ counter-clockwise star-map

Use $M(B, \pi)$ as “colored hyper-edge”
Faces

- Black vertices of arbitrary degree
- Blue vertices and box-vertices form

Cycle of colors \((0c)\) of graph \(\rightarrow\) face of color \(c\) in map
Quartic case, $d = 4$

Simplification!
Quartic case, $d = 4$

- Maps of arbitrary degree
- Monocolored edges, colors 1, \ldots, $d$
- Bicolored edges, colors $1c$ for $c = 2, 3, 4$

Maximizing faces $\sum_{c=1}^{d} F_{0c}$

- Monocolored edges are bridges
- Bicolored form planar components
- Bicolored types $1c$ and $1c'$ touch on cut–vertices (similar to $O(n)$ model on planar maps)
The quartic case

- Generating function of (rooted) maps for $k$ types of bicolored edges

\[ f_k(t, \lambda) = \sum_{M} t^{|\text{edges}|} \lambda^{|\text{monocol. edges}|} \]

- Algebraicity

\[ f_k(t, \lambda) = 1 - k + t\lambda f_k(t, \lambda)^2 + kP(tf_k(t, \lambda)^2) \]

implies

\[
\begin{aligned}
    tf^2 &= u(1 - u)^2 \\
    f &= k(1 - u)(1 + 3 u) - k + 1 + \lambda u(1 - u)^2
\end{aligned}
\]

- Generic planar maps for $\lambda = 0$ and $k = 1$

\[ 27t^2 A(t)^2 + (1 - 18t)A(t) + 16t - 1 = 0 \]
Explicit singularity analysis for $k = 1$

- Quartic eq on $f(t, \lambda)$
- For $\lambda < 3$, singularity at $t_1(\lambda) = \frac{27}{4(\lambda+9)^2}$

$$f(t, \lambda) = \frac{4}{27}(\lambda + 9) + \frac{16(\lambda + 3)(\lambda + 9)^3}{729(\lambda - 3)}(t_1(\lambda) - t)$$

$$+ \frac{64(\lambda + 9)^{11/2}}{6561(3 - \lambda)^{5/2}}(t_1(\lambda) - t)^{3/2} + o((t_1(\lambda) - t)^{3/2})$$

- For $\lambda > 3$, singularity at $t_2(\lambda) = \frac{\lambda}{4(1+\lambda)^2}$

$$f(t, \lambda) = 2\frac{\lambda^2 - 1}{\lambda^2} - \frac{4(1 + \lambda)^2}{\lambda^{5/2}} \sqrt{\lambda^2 - 2\lambda - 3} \ (t_2(\lambda) - t)^{1/2} + o((t_2(\lambda) - t)^{1/2})$$

- $\lambda = 3$, proliferation of baby universes

$$f(t, \lambda = 3) = \frac{16}{9} \left( \frac{3}{64} - t \right)^{2/3} + o\left( \left( \frac{3}{64} - t \right)^{2/3} \right)$$
Same results with respect to $k$

- $k$ small enough: universality class of maps
- $k$ large enough: branching process and square–root singularity
- $k$ critical: singularity exponent $2/3$
(At least some) Enumeration is feasible in dim $d > 2$!
To appear with L. Lionni: enumeration of gluings of octahedra which maximize the number of edges
Beyond maximizing number of faces in quartic case  
→ Topological recursion! [to appear w/ S. Dartois]
More to be studied
Harer–Zagier formula equivalent for unicellular maps?