Lattice polytopes: width and enumeration

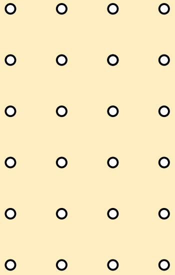
Mónica Blanco, Francisco Santos
(partially with C. Haase and J. Hofmann)

Universidad de Cantabria
Freie Universität Berlin

LIX, École Polytechnique — April 20, 2016

Definition

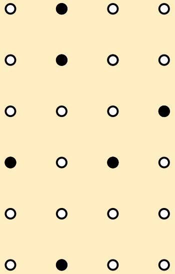
- **Lattice polytope** $P :=$
  convex hull of a finite set of points in $\mathbb{Z}^d$
  (or in a d-dimensional lattice).



Definition

- **Lattice polytope** $P :=$

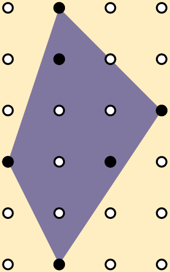
  convex hull of a finite set of points in $\mathbb{Z}^d$ (or in a d-dimensional lattice).



Definition

- **Lattice polytope** $P :=$

  convex hull of a finite set of points in $\mathbb{Z}^d$ (or in a d-dimensional lattice).



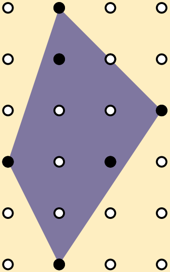
Definition

- **Lattice polytope** $P :=$

  convex hull of a finite set of points in $\mathbb{Z}^d$ (or in a d-dimensional lattice).

- **Size** of $P :=$

  number of lattice points in P: $|P \cap \mathbb{Z}^d|$



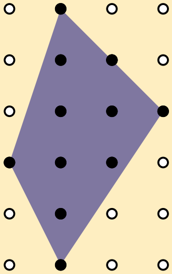
Definition

- **Lattice polytope** $P :=$

  convex hull of a finite set of points in $\mathbb{Z}^d$ (or in a d-dimensional lattice).

- **Size** of $P :=$

  number of lattice points in P: $|P \cap \mathbb{Z}^d|$



Definition

- **Lattice polytope** $P :=$

  convex hull of a finite set of points in $\mathbb{Z}^d$ (or in a d-dimensional lattice).

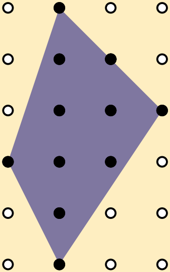
- **Size** of $P :=$

  number of lattice points in P: $|P \cap \mathbb{Z}^d|$

- **Volume** of $P :=$

  volume "normalized to the lattice" $=$

  $d! \times$ Euclidean volume.



Definition

- **Lattice polytope** $P :=$

  convex hull of a finite set of points in $\mathbb{Z}^d$ (or in a d-dimensional lattice).

- **Size** of $P :=$

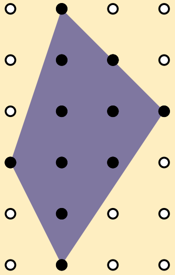
  number of lattice points in P: $|P \cap \mathbb{Z}^d|$

- **Volume** of $P :=$

  volume "normalized to the lattice" $=$

  $d! \times$ Euclidean volume.

  The volume of a lattice polytope is always an integer.



Definition

- **Lattice polytope** $P :=$

  convex hull of a finite set of points in $\mathbb{Z}^d$ (or in a d-dimensional lattice).

- **Size** of $P :=$

  number of lattice points in P: $|P \cap \mathbb{Z}^d|$

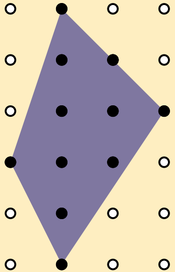
- **Volume** of $P :=$

  volume "normalized to the lattice" $=$

  $d! \times$ Euclidean volume.

  The volume of a lattice polytope is always an integer.

  $d+1$ points form a simplex of volume 1 (called UNIMODULAR) if and only if they affinely span the lattice



Definition

- **Lattice polytope** $P :=$

  convex hull of a finite set of points in $\mathbb{Z}^d$ (or in a d-dimensional lattice).

- **Size** of $P :=$

  number of lattice points in P: $|P \cap \mathbb{Z}^d|$

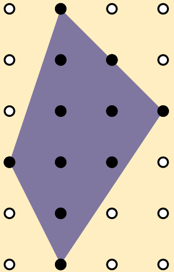
- **Volume** of $P :=$

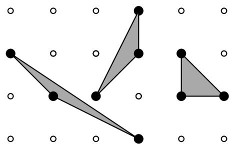
  volume "normalized to the lattice" $=$

  $d! \times$ Euclidean volume.

  The volume of a lattice polytope is always an integer.

  $d+1$ points form a simplex of volume 1 (called UNIMODULAR) if and only if they affinely span the lattice





Definition

- **Lattice polytope** $P :=$

  convex hull of a finite set of points in $\mathbb{Z}^d$ (or in a d-dimensional lattice).

- **Size** of $P :=$

  number of lattice points in P: $|P \cap \mathbb{Z}^d|$

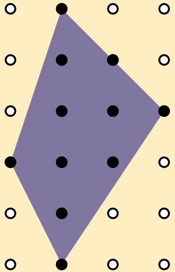
- **Volume** of $P :=$

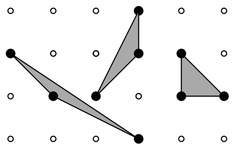
  volume "normalized to the lattice" $=$

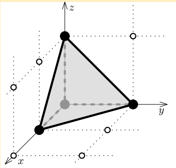
  $d! \times$ Euclidean volume.

  The volume of a lattice polytope is always an integer.

  $d + 1$ points form a simplex of volume 1 (called UNIMODULAR) if and only if they affinely span the lattice







Unimodular equivalence

A **unimodular transformation** is a linear integer map $t : \mathbb{R}^d \to \mathbb{R}^d$ that preserves the lattice. That is,

$$t(x) = A \cdot x + b, \; x \in \mathbb{R}^d$$

for $A \in \mathbb{Z}^{d \times d}$, $\det(A) = \pm 1$ and $b \in \mathbb{Z}^d$.
Two lattice polytopes P and Q are said **unimodularly equivalent** (or simply **equivalent**) if there is an affine unimodular transformation t such that $t(P) = Q$.

Unimodular equivalence

A **unimodular transformation** is a linear integer map $t : \mathbb{R}^d \to \mathbb{R}^d$ that preserves the lattice. That is,

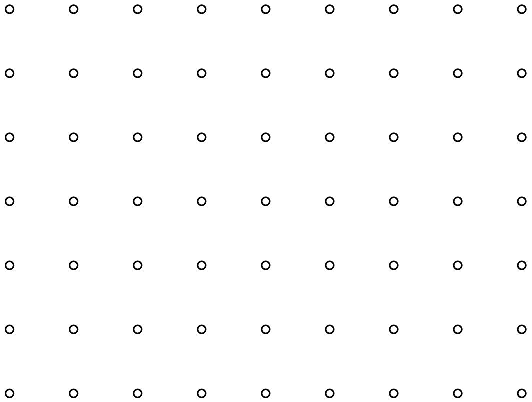
$$t(x) = A \cdot x + b, \; x \in \mathbb{R}^d$$

for $A \in \mathbb{Z}^{d \times d}$, $\det(A) = \pm 1$ and $b \in \mathbb{Z}^d$.
Two lattice polytopes P and Q are said **unimodularly equivalent** (or simply **equivalent**) if there is an affine unimodular transformation t such that $t(P) = Q$.

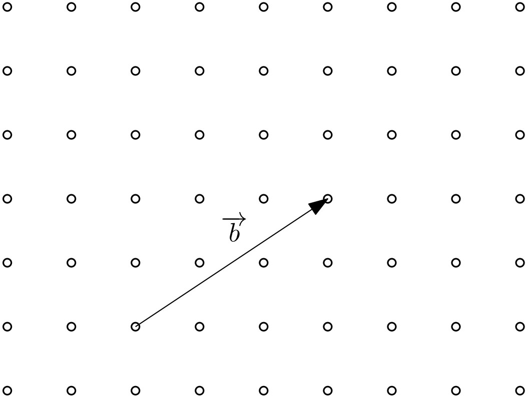
Remark

Size, volume, combinatorial type, ... are invariant modulo unimodular equivalence.

Examples of unimodular transformations

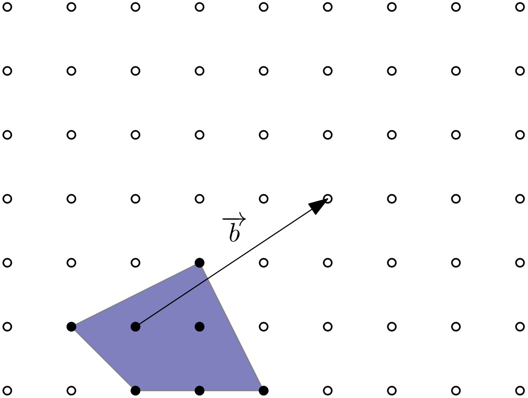


Examples of unimodular transformations



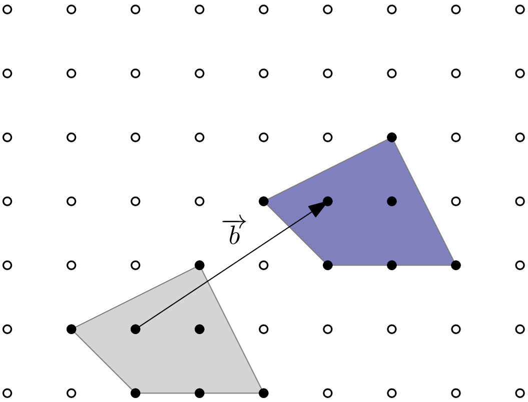
- $\overrightarrow{x} \mapsto \overrightarrow{x} + \overrightarrow{b}$

Examples of unimodular transformations



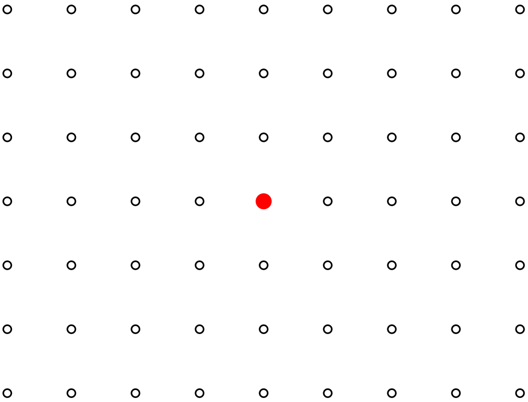
- $\overrightarrow{x} \mapsto \overrightarrow{x} + \overrightarrow{b}$

Examples of unimodular transformations



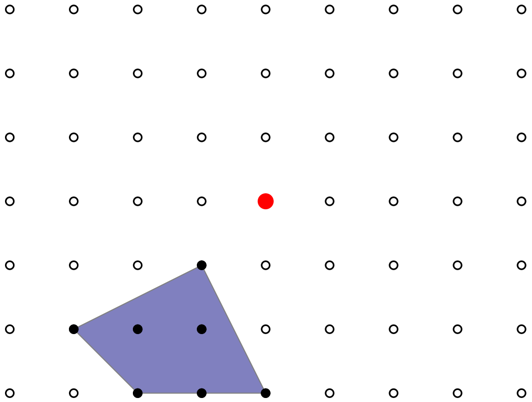
- $\overrightarrow{x} \mapsto \overrightarrow{x} + \overrightarrow{b}$

Examples of unimodular transformations



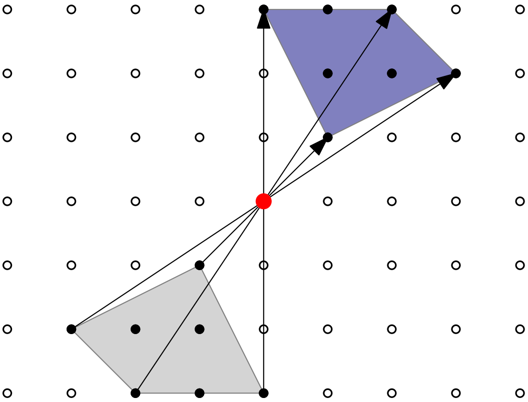
- $\overrightarrow{x} \mapsto \overrightarrow{x} + \overrightarrow{b}$
- $\overrightarrow{x} \mapsto -\overrightarrow{x}$

Examples of unimodular transformations



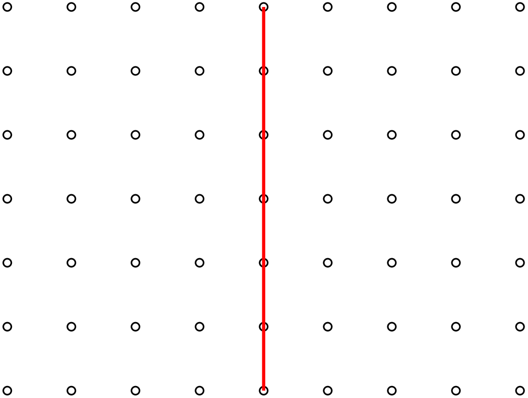
- $\overrightarrow{x} \mapsto \overrightarrow{x} + \overrightarrow{b}$
- $\overrightarrow{x} \mapsto -\overrightarrow{x}$

Examples of unimodular transformations



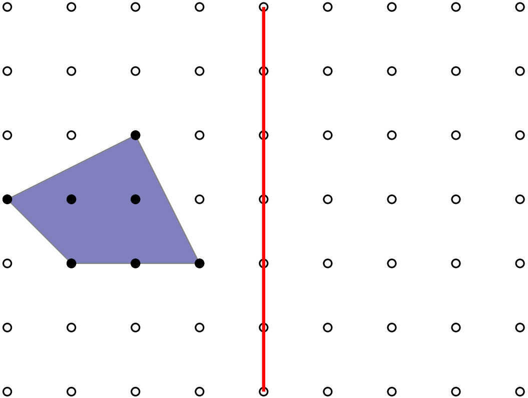
- $\overrightarrow{x} \mapsto \overrightarrow{x} + \overrightarrow{b}$
- $\overrightarrow{x} \mapsto -\overrightarrow{x}$

Examples of unimodular transformations



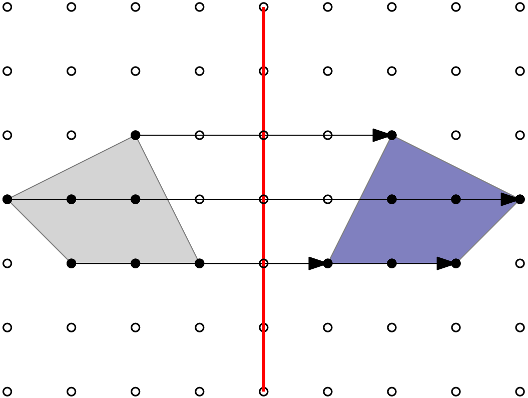
- $\overrightarrow{x} \mapsto \overrightarrow{x} + \overrightarrow{b}$
- $\overrightarrow{x} \mapsto -\overrightarrow{x}$
- $(x, y) \mapsto (-x, y)$

Examples of unimodular transformations



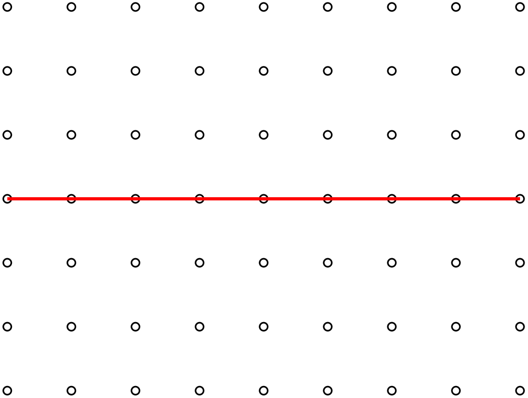
- $\overrightarrow{x} \mapsto \overrightarrow{x} + \overrightarrow{b}$
- $\overrightarrow{x} \mapsto -\overrightarrow{x}$
- $(x, y) \mapsto (-x, y)$

Examples of unimodular transformations



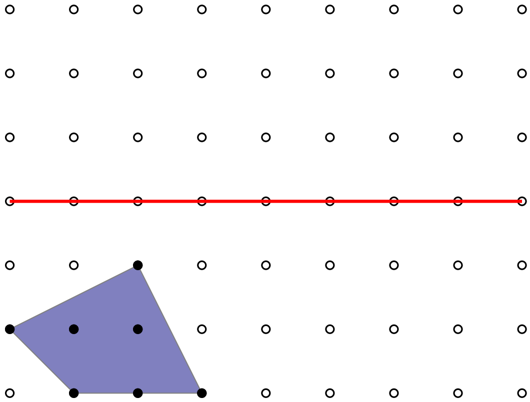
- $\overrightarrow{x} \mapsto \overrightarrow{x} + \overrightarrow{b}$
- $\overrightarrow{x} \mapsto -\overrightarrow{x}$
- $(x, y) \mapsto (-x, y)$

Examples of unimodular transformations



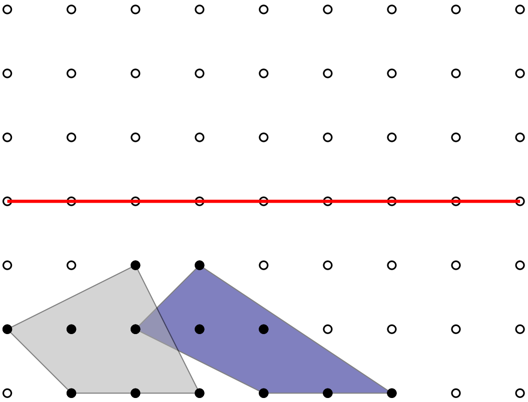
- $\overrightarrow{x} \mapsto \overrightarrow{x} + \overrightarrow{b}$
- $\overrightarrow{x} \mapsto -\overrightarrow{x}$
- $(x, y) \mapsto (-x, y)$
- $(x, y) \mapsto (x + (-1)y, y)$

Examples of unimodular transformations



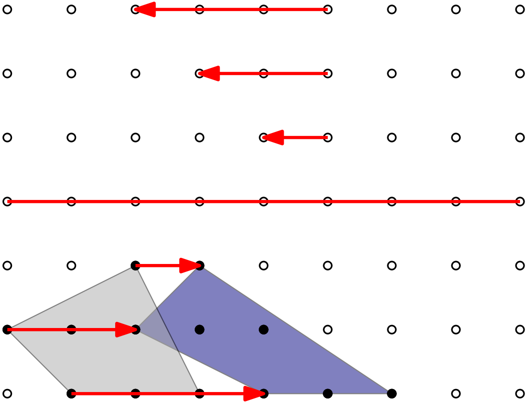
- $\overrightarrow{x} \mapsto \overrightarrow{x} + \overrightarrow{b}$
- $\overrightarrow{x} \mapsto -\overrightarrow{x}$
- $(x, y) \mapsto (-x, y)$
- $(x, y) \mapsto (x + (-1)y, y)$

Examples of unimodular transformations



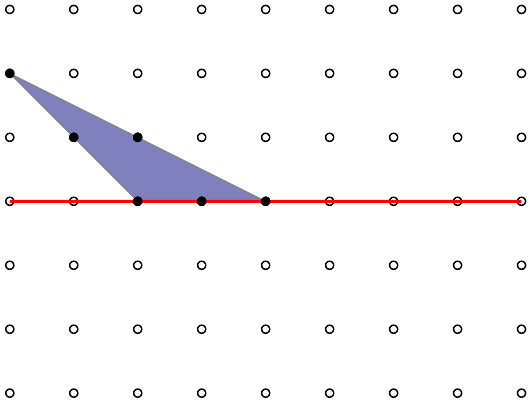
- $\vec{x} \mapsto \vec{x} + \vec{b}$
- $\vec{x} \mapsto -\vec{x}$
- $(x, y) \mapsto (-x, y)$
- $(x, y) \mapsto$
  $(x + (-1)y, y)$

Examples of unimodular transformations



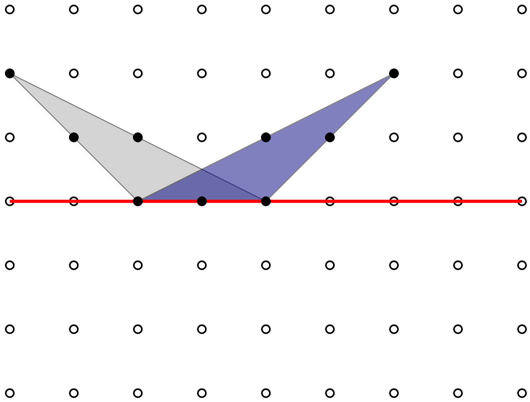
- $\overrightarrow{x} \mapsto \overrightarrow{x} + \overrightarrow{b}$
- $\overrightarrow{x} \mapsto -\overrightarrow{x}$
- $(x, y) \mapsto (-x, y)$
- $(x, y) \mapsto$
  $(x + (-1)y, y)$

Examples of unimodular transformations



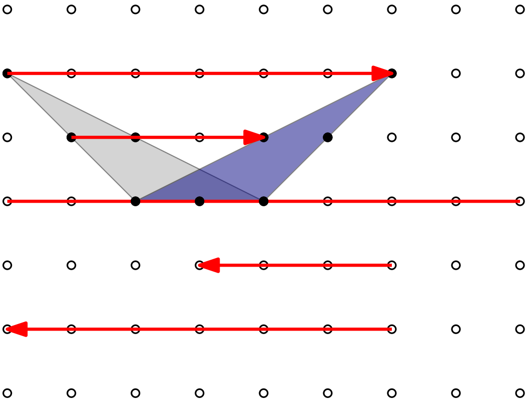
- $\vec{x} \mapsto \vec{x} + \vec{b}$
- $\vec{x} \mapsto -\vec{x}$
- $(x, y) \mapsto (-x, y)$
- $(x, y) \mapsto$
  $(x + (-1)y, y)$
- $(x, y) \mapsto$
  $(x + 3y, y)$

Examples of unimodular transformations



- $\overrightarrow{x} \mapsto \overrightarrow{x} + \overrightarrow{b}$
- $\overrightarrow{x} \mapsto -\overrightarrow{x}$
- $(x, y) \mapsto (-x, y)$
- $(x, y) \mapsto$ $(x + (-1)y, y)$
- $(x, y) \mapsto$ $(x + 3y, y)$

Examples of unimodular transformations



- $\vec{x} \mapsto \vec{x} + \vec{b}$
- $\vec{x} \mapsto -\vec{x}$
- $(x, y) \mapsto (-x, y)$
- $(x, y) \mapsto$
  $(x + (-1)y, y)$
- $(x, y) \mapsto$
  $(x + 3y, y)$

Goal

WE WANT TO CLASSIFY (classes of) LATTICE d-POLYTOPES

Goal

WE WANT TO CLASSIFY (classes of) LATTICE d-POLYTOPES

For this, we separate by size: let $n \geq d+1$

$$\mathcal{P}_d(n) := \{(\text{classes of}) \text{ lattice } d\text{-polytopes of size } n\}$$

Goal

WE WANT TO CLASSIFY (classes of) LATTICE d-POLYTOPES

For this, we separate by size: let $n \geq d+1$

$$\mathcal{P}_d(n) := \{(\text{classes of}) \text{ lattice } d\text{-polytopes of size } n\}$$

- **Dimension** 1: for each $n \geq 2$, there is one lattice 1-polytope of size n, a segment of length $n-1$: $|\mathcal{P}_1(n)| = 1$

Goal

WE WANT TO CLASSIFY (classes of) LATTICE d-POLYTOPES

For this, we separate by size: let $n \geq d + 1$

$$\mathcal{P}_d(n) := \{(\text{classes of}) \text{ lattice } d\text{-polytopes of size } n\}$$

- **Dimension** 1: for each $n \geq 2$, there is one lattice 1-polytope of size n, a segment of length $n - 1$: $|\mathcal{P}_1(n)| = 1$

Goal

WE WANT TO CLASSIFY (classes of) LATTICE d-POLYTOPES

For this, we separate by size: let $n \geq d+1$

$$\mathcal{P}_d(n) := \{(\text{classes of}) \text{ lattice } d\text{-polytopes of size } n\}$$

- **Dimension** 1: for each $n \geq 2$, there is one lattice 1-polytope of size n, a segment of length $n-1$: $|\mathcal{P}_1(n)| = 1$

- **Dimension** 2: for each n, the cardinal of $\mathcal{P}_2(n)$ is **finite** $|\mathcal{P}_2(n)| < \infty$

Goal

WE WANT TO CLASSIFY (classes of) LATTICE d-POLYTOPES

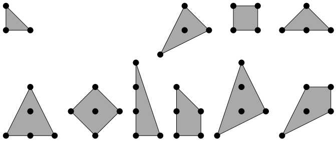
For this, we separate by size: let $n \geq d+1$

$$\mathcal{P}_d(n) := \{(\text{classes of}) \text{ lattice } d\text{-polytopes of size } n\}$$

- **Dimension** 1: for each $n \geq 2$, there is one lattice 1-polytope of size n, a segment of length $n-1$: $|\mathcal{P}_1(n)| = 1$

- **Dimension** 2: for each n, the cardinal of $\mathcal{P}_2(n)$ is **finite** $|\mathcal{P}_2(n)| < \infty$

  All polygons of sizes 3, 4 and 5:



Goal

WE WANT TO CLASSIFY (classes of) LATTICE d-POLYTOPES

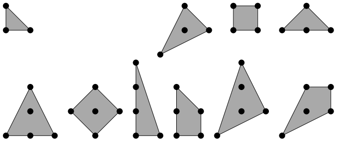
For this, we separate by size: let $n \geq d+1$

$$\mathcal{P}_d(n) := \{(\text{classes of}) \text{ lattice } d\text{-polytopes of size } n\}$$

- **Dimension** 1: for each $n \geq 2$, there is one lattice 1-polytope of size n, a segment of length $n-1$: $|\mathcal{P}_1(n)| = 1$

- **Dimension** 2: for each n, the cardinal of $\mathcal{P}_2(n)$ is **finite** $|\mathcal{P}_2(n)| < \infty$

  All polygons of sizes 3, 4 and 5:



Goal

WE WANT TO CLASSIFY (classes of) LATTICE d-POLYTOPES

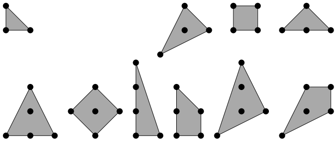
For this, we separate by size: let $n \geq d+1$

$$\mathcal{P}_d(n) := \{(\text{classes of}) \text{ lattice } d\text{-polytopes of size } n\}$$

- **Dimension** 1: for each $n \geq 2$, there is one lattice 1-polytope of size n, a segment of length $n-1$: $|\mathcal{P}_1(n)| = 1$

- **Dimension** 2: for each n, the cardinal of $\mathcal{P}_2(n)$ is **finite** $|\mathcal{P}_2(n)| < \infty$

  All polygons of sizes 3, 4 and 5:



Goal

WE WANT TO CLASSIFY (classes of) LATTICE d-POLYTOPES

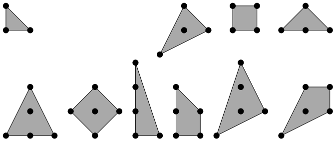
For this, we separate by size: let $n \geq d+1$

$$\mathcal{P}_d(n) := \{\text{(classes of) lattice } d\text{-polytopes of size } n\}$$

- **Dimension** 1: for each $n \geq 2$, there is one lattice 1-polytope of size n, a segment of length $n-1$: $|\mathcal{P}_1(n)| = 1$

- **Dimension** 2: for each n, the cardinal of $\mathcal{P}_2(n)$ is **finite** $|\mathcal{P}_2(n)| < \infty$

  All polygons of sizes 3, 4 and 5:



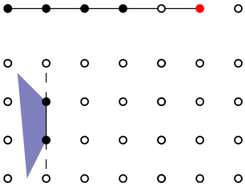
Goal

WE WANT TO CLASSIFY (classes of) LATTICE d-POLYTOPES

For this, we separate by size: let $n \geq d+1$

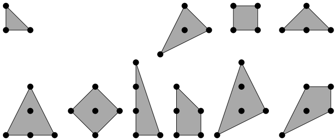
$$\mathcal{P}_d(n) := \{(\text{classes of})\text{ lattice } d\text{-polytopes of size } n\}$$

- **Dimension** 1: for each $n \geq 2$, there is one lattice 1-polytope of size n, a segment of length $n-1$: $|\mathcal{P}_1(n)| = 1$



- **Dimension** 2: for each n, the cardinal of $\mathcal{P}_2(n)$ is **finite** $|\mathcal{P}_2(n)| < \infty$

All polygons of sizes 3, 4 and 5:

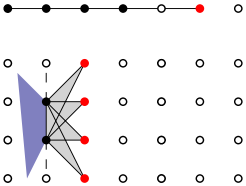


Goal

WE WANT TO CLASSIFY (classes of) LATTICE d-POLYTOPES

For this, we separate by size: let $n \geq d+1$

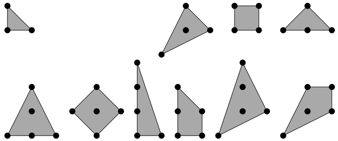
$$\mathcal{P}_d(n) := \{(\text{classes of}) \text{ lattice } d\text{-polytopes of size } n\}$$



- **Dimension** 1: for each $n \geq 2$, there is one lattice 1-polytope of size n, a segment of length $n-1$: $|\mathcal{P}_1(n)| = 1$

- **Dimension** 2: for each n, the cardinal of $\mathcal{P}_2(n)$ is **finite** $|\mathcal{P}_2(n)| < \infty$

  All polygons of sizes 3, 4 and 5:



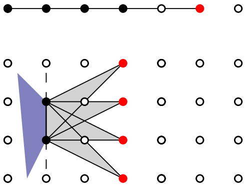
Goal

WE WANT TO CLASSIFY (classes of) LATTICE d-POLYTOPES

For this, we separate by size: let $n \geq d+1$

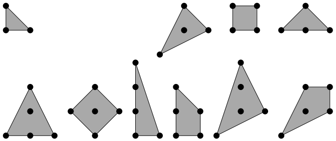
$$\color{red}{\mathcal{P}_d(n)} := \{(\text{classes of})\text{ lattice } d\text{-polytopes of size } n\}$$

- **Dimension** 1: for each $n \geq 2$, there is one lattice 1-polytope of size n, a segment of length $n-1$: $\color{red}{|\mathcal{P}_1(n)| = 1}$



- **Dimension** 2: for each n, the cardinal of $\mathcal{P}_2(n)$ is **finite** $\color{red}{|\mathcal{P}_2(n)| < \infty}$

  All polygons of sizes 3, 4 and 5:



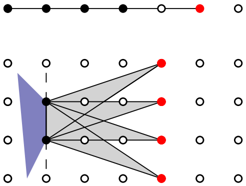
Goal

WE WANT TO CLASSIFY (classes of) LATTICE d-POLYTOPES

For this, we separate by size: let $n \geq d+1$

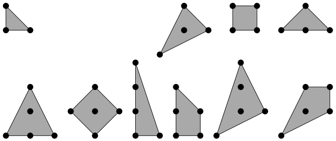
$$\mathcal{P}_d(n) := \{(\text{classes of}) \text{ lattice } d\text{-polytopes of size } n\}$$

- **Dimension** 1: for each $n \geq 2$, there is one lattice 1-polytope of size n, a segment of length $n-1$: $|\mathcal{P}_1(n)| = 1$



- **Dimension** 2: for each n, the cardinal of $\mathcal{P}_2(n)$ is **finite** $|\mathcal{P}_2(n)| < \infty$

  All polygons of sizes 3, 4 and 5:



Goal

WE WANT TO CLASSIFY (classes of) LATTICE d-POLYTOPES

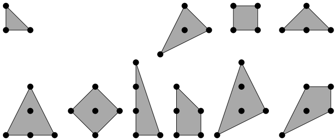
For this, we separate by size: let $n \geq d+1$

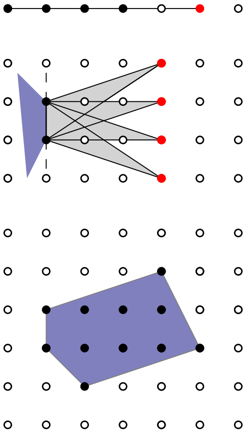
$$\mathcal{P}_d(n) := \{(\text{classes of}) \text{ lattice } d\text{-polytopes of size } n\}$$

- **Dimension** 1: for each $n \geq 2$, there is one lattice 1-polytope of size n, a segment of length $n-1$: $|\mathcal{P}_1(n)| = 1$

- **Dimension** 2: for each n, the cardinal of $\mathcal{P}_2(n)$ is **finite** $|\mathcal{P}_2(n)| < \infty$

All polygons of sizes 3, 4 and 5:





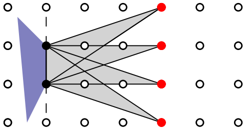
Goal

WE WANT TO CLASSIFY (classes of) LATTICE d-POLYTOPES

For this, we separate by size: let $n \geq d+1$

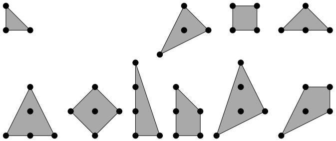
$$\mathcal{P}_d(n) := \{(\text{classes of}) \text{ lattice } d\text{-polytopes of size } n\}$$

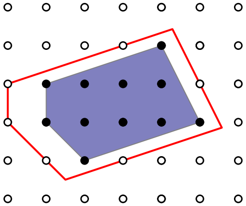
- **Dimension** 1: for each $n \geq 2$, there is one lattice 1-polytope of size n, a segment of length $n-1$: $|\mathcal{P}_1(n)| = 1$



- **Dimension** 2: for each n, the cardinal of $\mathcal{P}_2(n)$ is **finite** $|\mathcal{P}_2(n)| < \infty$

  All polygons of sizes 3, 4 and 5:



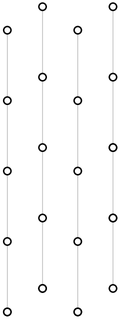


Dimension 3, $n = 4$ (empty tetrahedra)

In dimension 3, Reeve tetrahedra are infinitely many lattice 3-polytopes with 4 lattice points $|\mathcal{P}_3(4)| = \infty$

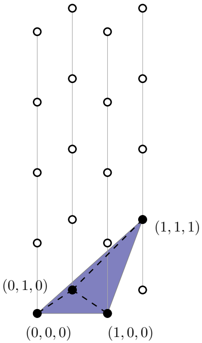
Dimension 3, $n = 4$ (empty tetrahedra)

In dimension 3, Reeve tetrahedra are infinitely many lattice 3-polytopes with 4 lattice points $|\mathcal{P}_3(4)| = \infty$



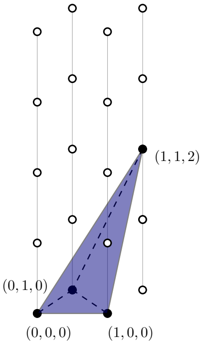
Dimension 3, $n = 4$ (empty tetrahedra)

In dimension 3, Reeve tetrahedra are infinitely many lattice 3-polytopes with 4 lattice points $|\mathcal{P}_3(4)| = \infty$



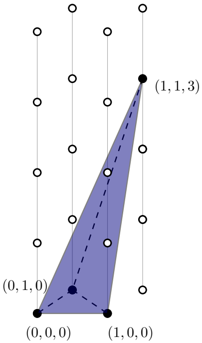
Dimension 3, $n = 4$ (empty tetrahedra)

In dimension 3, Reeve tetrahedra are infinitely many lattice 3-polytopes with 4 lattice points $|\mathcal{P}_3(4)| = \infty$



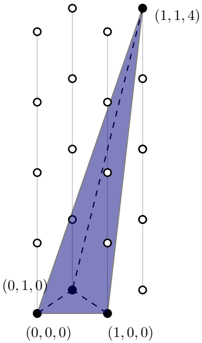
Dimension 3, $n = 4$ (empty tetrahedra)

In dimension 3, Reeve tetrahedra are infinitely many lattice 3-polytopes with 4 lattice points $|\mathcal{P}_3(4)| = \infty$



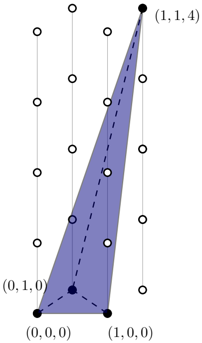
Dimension 3, $n = 4$ (empty tetrahedra)

In dimension 3, Reeve tetrahedra are infinitely many lattice 3-polytopes with 4 lattice points $|\mathcal{P}_3(4)| = \infty$



Dimension 3, $n = 4$ (empty tetrahedra)

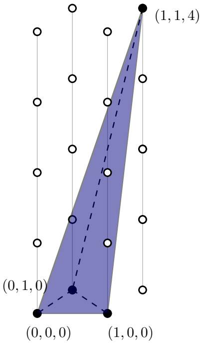
In dimension 3, Reeve tetrahedra are infinitely many lattice 3-polytopes with 4 lattice points $|\mathcal{P}_3(4)| = \infty$



Elements in $\mathcal{P}_3(4)$ are called **empty tetrahedra**: tetrahedra in which the only lattice points are the four vertices. Their classification is classical (White 1964):

Dimension 3, $n = 4$ (empty tetrahedra)

In dimension 3, Reeve tetrahedra are infinitely many lattice 3-polytopes with 4 lattice points $|\mathcal{P}_3(4)| = \infty$



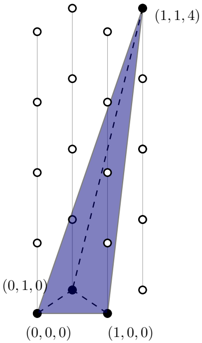
Elements in $\mathcal{P}_3(4)$ are called **empty tetrahedra**: tetrahedra in which the only lattice points are the four vertices. Their classification is classical (White 1964):

$$\mathcal{P}_3(4) = \{\, T(p,q) \mid p, q \in \mathbb{Z},\ 0 < p \le q,\ \gcd(p,q) = 1 \,\},$$

where $T(p,q) := \operatorname{conv}\{(0,0,0),(1,0,0),(0,0,1),(p,q,1)\}$.

Dimension 3, $n = 4$ (empty tetrahedra)

In dimension 3, Reeve tetrahedra are infinitely many lattice 3-polytopes with 4 lattice points $|\mathcal{P}_3(4)| = \infty$



Elements in $\mathcal{P}_3(4)$ are called **empty tetrahedra**: tetrahedra in which the only lattice points are the four vertices. Their classification is classical (White 1964):

$$\mathcal{P}_3(4) = \{\, T(p,q) \mid p, q \in \mathbb{Z},\ 0 < p \le q,\ \gcd(p,q) = 1 \,\},$$

where $T(p,q) := \text{conv}\{(0,0,0), (1,0,0), (0,0,1), (p,q,1)\}$.

Remark

All empty tetrahedra have *width 1*.

(Lattice) Width

Definition

- **Width of P with respect to**

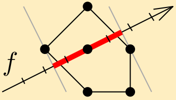
  a linear functional $f : \mathbb{R}^d \to \mathbb{R}$

  $=$ length of the interval $f(P)$

(Lattice) Width

Definition

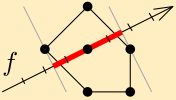
- **Width of P with respect to**
  a linear functional $f : \mathbb{R}^d \to \mathbb{R}$
  $=$ length of the interval $f(P)$



(Lattice) Width

Definition

- **Width of P with respect to** a linear functional $f : \mathbb{R}^d \to \mathbb{R}$ $=$ length of the interval $f(P)$

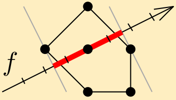


- **Width** of P:$=$ Minimum width of P with respect to a linear NON-CONSTANT, INTEGER functional

(Lattice) Width

Definition

- **Width of P with respect to** a linear functional $f : \mathbb{R}^d \to \mathbb{R}$ $=$ length of the interval $f(P)$

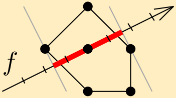


- **Width** of P:= Minimum width of P with respect to a linear NON-CONSTANT, INTEGER functional = minimum lattice distance between two parallel lattice hyperplanes enclosing P

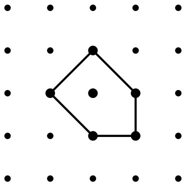
(Lattice) Width

Definition

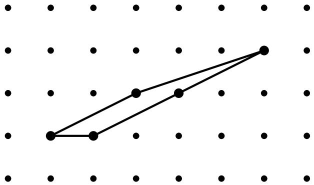
- **Width of P with respect to** a linear functional $f : \mathbb{R}^d \to \mathbb{R}$ $=$ length of the interval $f(P)$



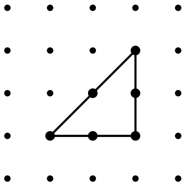
- **Width** of P:= Minimum width of P with respect to a linear NON-CONSTANT, INTEGER functional = minimum lattice distance between two parallel lattice hyperplanes enclosing P



Width: 2



Width: 1

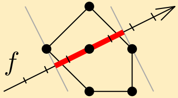


Width: 2

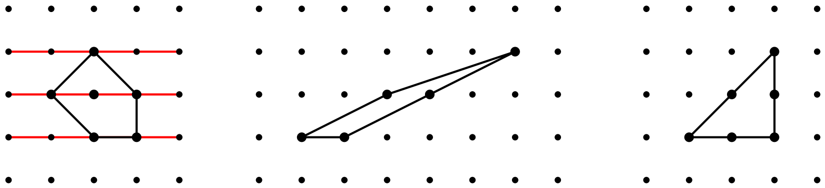
(Lattice) Width

Definition

- **Width of P with respect to** a linear functional $f : \mathbb{R}^d \to \mathbb{R}$ $=$ length of the interval $f(P)$



- **Width** of P:= Minimum width of P with respect to a linear NON-CONSTANT, INTEGER functional = minimum lattice distance between two parallel lattice hyperplanes enclosing P

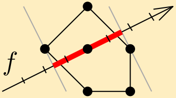


Width: 2 | Width: 1 | Width: 2

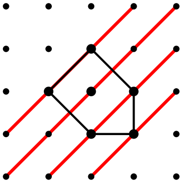
(Lattice) Width

Definition

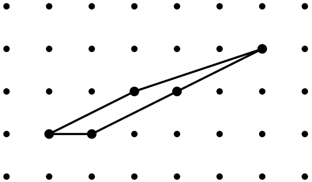
- **Width of P with respect to** a linear functional $f : \mathbb{R}^d \to \mathbb{R}$ = length of the interval $f(P)$



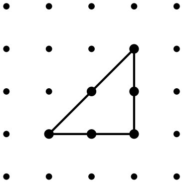
- **Width** of P:= Minimum width of P with respect to a linear NON-CONSTANT, INTEGER functional = minimum lattice distance between two parallel lattice hyperplanes enclosing P



Width: 2



Width: 1



Width: 2

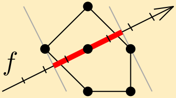
(Lattice) Width

Definition

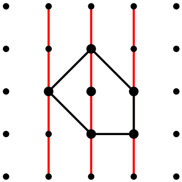
- **Width of P with respect to**

  a linear functional $f : \mathbb{R}^d \to \mathbb{R}$

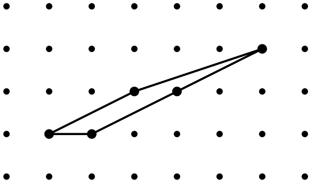
  $=$ length of the interval $f(P)$



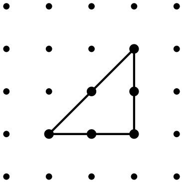
- **Width** of P:= Minimum width of P with respect to a linear NON-CONSTANT, INTEGER functional = minimum lattice distance between two parallel lattice hyperplanes enclosing P



Width: 2



Width: 1

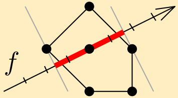


Width: 2

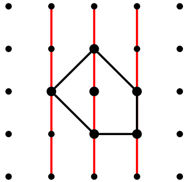
(Lattice) Width

Definition

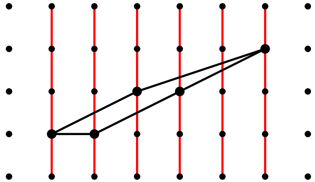
- **Width of P with respect to** a linear functional $f : \mathbb{R}^d \to \mathbb{R}$ $=$ length of the interval $f(P)$



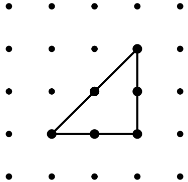
- **Width** of P:= Minimum width of P with respect to a linear NON-CONSTANT, INTEGER functional $=$ minimum lattice distance between two parallel lattice hyperplanes enclosing P



Width: 2



Width: 1

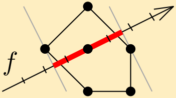


Width: 2

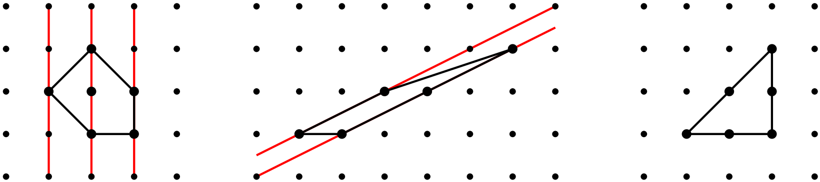
(Lattice) Width

Definition

- **Width of P with respect to** a linear functional $f : \mathbb{R}^d \to \mathbb{R}$ $=$ length of the interval $f(P)$



- **Width** of P:= Minimum width of P with respect to a linear NON-CONSTANT, INTEGER functional = minimum lattice distance between two parallel lattice hyperplanes enclosing P

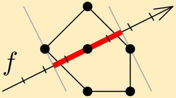


Width: 2 | Width: 1 | Width: 2

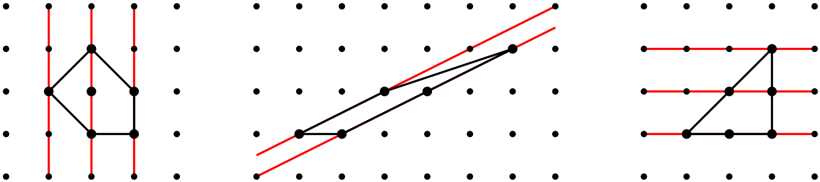
(Lattice) Width

Definition

- **Width of P with respect to** a linear functional $f : \mathbb{R}^d \to \mathbb{R}$ $=$ length of the interval $f(P)$



- **Width** of $P :=$ Minimum width of P with respect to a linear NON-CONSTANT, INTEGER functional $=$ minimum lattice distance between two parallel lattice hyperplanes enclosing P

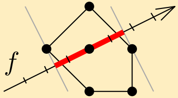


Width: 2 Width: 1 Width: 2

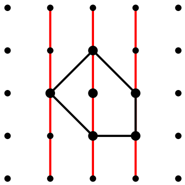
(Lattice) Width

Definition

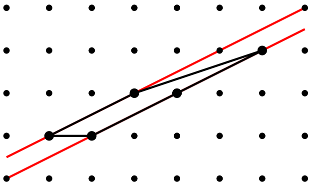
- **Width of P with respect to**
  a linear functional $f : \mathbb{R}^d \to \mathbb{R}$
  $=$ length of the interval $f(P)$



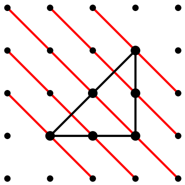
- **Width** of P:= Minimum width of P with respect to a linear NON-CONSTANT, INTEGER functional $=$ minimum lattice distance between two parallel lattice hyperplanes enclosing P



Width: 2



Width: 1

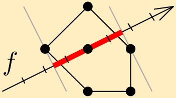


Width: 2

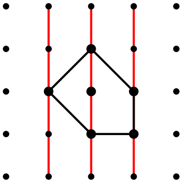
(Lattice) Width

Definition

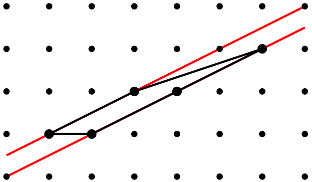
- **Width of P with respect to** a linear functional $f : \mathbb{R}^d \to \mathbb{R}$ = length of the interval $f(P)$



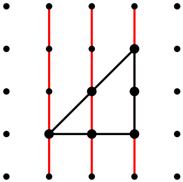
- **Width** of P:= Minimum width of P with respect to a linear NON-CONSTANT, INTEGER functional = minimum lattice distance between two parallel lattice hyperplanes enclosing P



Width: 2



Width: 1



Width: 2

Width $> 1 \implies$ **finite** number of classes (for $d = 3$)

$$\mathcal{P}_d^*(n) := \{P \in \mathcal{P}_d(n) \mid \text{width}(P) > 1\}$$

Width $> 1 \implies$ **finite** number of classes (for $d = 3$)

$$\mathcal{P}_d^*(n) := \{P \in \mathcal{P}_d(n) \mid \text{width}(P) > 1\}$$

For each $n \geq 4$:

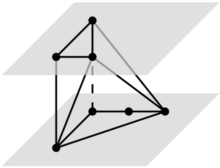
- There are **infinitely** many equivalence classes of **width** 1:
  $$|\mathcal{P}_d(n) \setminus \mathcal{P}_d^*(n)| = \infty.$$

Width $> 1 \implies$ **finite** number of classes (for $d = 3$)

$$\mathcal{P}_d^*(n) := \{P \in \mathcal{P}_d(n) \mid \text{width}(P) > 1\}$$

For each $n \geq 4$:

- There are **infinitely** many equivalence classes of **width** 1:
  $$|\mathcal{P}_d(n) \setminus \mathcal{P}_d^*(n)| = \infty.$$

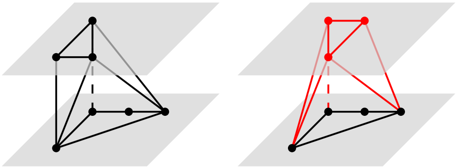


Width $> 1 \implies$ **finite** number of classes (for $d = 3$)

$$\mathcal{P}_d^*(n) := \{P \in \mathcal{P}_d(n) \mid \text{width}(P) > 1\}$$

For each $n \geq 4$:

- There are **infinitely** many equivalence classes of **width** 1:
  $$|\mathcal{P}_d(n) \setminus \mathcal{P}_d^*(n)| = \infty.$$

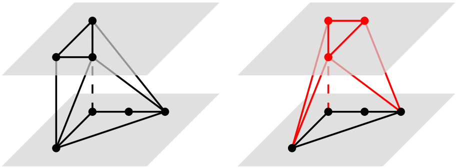


Width $> 1 \implies$ **finite** number of classes (for $d = 3$)

$$\mathcal{P}_d^*(n) := \{P \in \mathcal{P}_d(n) \mid \text{width}(P) > 1\}$$

For each $n \geq 4$:

- There are **infinitely** many equivalence classes of **width** 1:
  $$|\mathcal{P}_d(n) \setminus \mathcal{P}_d^*(n)| = \infty.$$
- But for width > 1:



Lemma (Blanco-Santos, 2016)

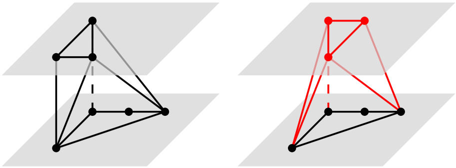
For each $n \geq 4$, there are **finitely** *many lattice 3-polytopes of width greater than one and size n. That is,*

$$|\mathcal{P}_3^*(n)| < \infty, \textit{ for each } n \geq 4$$

Width $> 1 \implies$ **finite** number of classes (for $d = 3$)

$$\mathcal{P}_d^*(n) := \{P \in \mathcal{P}_d(n) \mid \text{width}(P) > 1\}$$

For each $n \geq 4$:



- There are **infinitely** many equivalence classes of **width** 1:
  $$|\mathcal{P}_d(n) \setminus \mathcal{P}_d^*(n)| = \infty.$$
- But for width > 1:

Lemma (Blanco-Santos, 2016)

For each $n \geq 4$, there are **finitely** *many lattice 3-polytopes of width greater than one and size n. That is,*

$$|\mathcal{P}_3^*(n)| < \infty, \text{ for each } n \geq 4$$

WE CAN (a priori) ENUMERATE the complete list $\mathcal{P}_3^*(n)$ of lattice 3-polytopes of size n AND WIDTH > 1, for each n

Dimension 3, $n = 5, 6$

- For sizes $n = 5, 6$, we previously classified all lattice 3-polytopes of those sizes.

Dimension 3, $n = 5, 6$

- For sizes $n = 5, 6$, we previously classified all lattice 3-polytopes of those sizes.

Size	4	5	6
width 1	∞	∞	∞
width 2	–	9	74
width 3	–	–	2

Dimension 3, $n = 5, 6$

- For sizes $n = 5, 6$, we previously classified all lattice 3-polytopes of those sizes.

Size	4	5	6
width 1	∞	∞	∞
width 2	–	9	74
width 3	–	–	2

- The classification for $n = 5, 6$ was done via *oriented matroids*, a.k.a. *order types* (information on the position of the set of points in the space). For 5 points, there are 5 posible oriented matroids, for 6 points there are 55.

Dimension 3, $n = 5, 6$

- For sizes $n = 5, 6$, we previously classified all lattice 3-polytopes of those sizes.

Size	4	5	6
width 1	∞	∞	∞
width 2	–	9	74
width 3	–	–	2

- The classification for $n = 5, 6$ was done via *oriented matroids*, a.k.a. *order types* (information on the position of the set of points in the space). For 5 points, there are 5 posible oriented matroids, for 6 points there are 55.
- But for 7 points there are already 5000...

Dimension 3, $n = 5, 6$

- For sizes $n = 5, 6$, we previously classified all lattice 3-polytopes of those sizes.

Size	4	5	6
width 1	∞	∞	∞
width 2	–	9	74
width 3	–	–	2

- The classification for $n = 5, 6$ was done via *oriented matroids*, a.k.a. *order types* (information on the position of the set of points in the space). For 5 points, there are 5 posible oriented matroids, for 6 points there are 55.
- But for 7 points there are already 5000... and for 8 points the number of them is around the 10 millions!!!!!

Dimension 3, $n = 5, 6$

- For sizes $n = 5, 6$, we previously classified all lattice 3-polytopes of those sizes.

Size	4	5	6
width 1	∞	∞	∞
width 2	–	9	74
width 3	–	–	2

- The classification for $n = 5, 6$ was done via *oriented matroids*, a.k.a. *order types* (information on the position of the set of points in the space). For 5 points, there are 5 posible oriented matroids, for 6 points there are 55.
- But for 7 points there are already 5000... and for 8 points the number of them is around the 10 millions!!!!! So another approach is required.

Essential vertices

From now on, let $P \in \mathcal{P}_d^*(n)$. For each vertex $v \in \mathrm{vert}(P)$, we denote by P^v the polytope $\mathrm{conv}(P \setminus \{v\} \cap \mathbb{Z}^d) \subset \mathbb{R}^d$. This polytope has size $n-1$ but it is not necessarily full-dimensional.

Essential vertices

From now on, let $P \in \mathcal{P}_d^*(n)$. For each vertex $v \in \mathrm{vert}(P)$, we denote by P^v the polytope $\mathrm{conv}(P \setminus \{v\} \cap \mathbb{Z}^d) \subset \mathbb{R}^d$. This polytope has size $n-1$ but it is not necessarily full-dimensional.

Definition (Essential vertex)

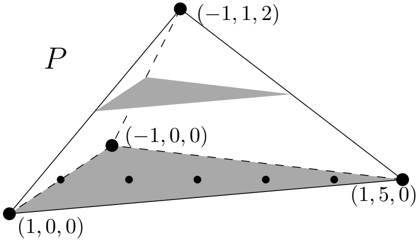
We say that v is an **essential vertex** of P if $P^v \notin \mathcal{P}_d^*(n-1)$. That is, if P^v is either $(d-1)$-dimensional or has width one.

Essential vertices

From now on, let $P \in \mathcal{P}^*_d(n)$. For each vertex $v \in \mathrm{vert}(P)$, we denote by P^v the polytope $\mathrm{conv}(P \setminus \{v\} \cap \mathbb{Z}^d) \subset \mathbb{R}^d$. This polytope has size $n-1$ but it is not necessarily full-dimensional.

Definition (Essential vertex)

We say that v is an **essential vertex** of P if $P^v \notin \mathcal{P}^*_d(n-1)$. That is, if P^v is either $(d-1)$-dimensional or has width one.

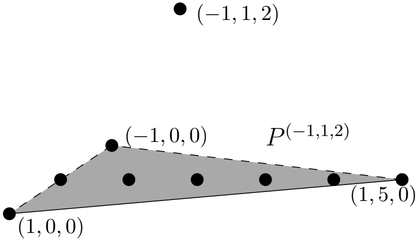


Essential vertices

From now on, let $P \in \mathcal{P}_d^*(n)$. For each vertex $v \in \mathrm{vert}(P)$, we denote by P^v the polytope $\mathrm{conv}(P \setminus \{v\} \cap \mathbb{Z}^d) \subset \mathbb{R}^d$. This polytope has size $n-1$ but it is not necessarily full-dimensional.

Definition (Essential vertex)

We say that v is an **essential vertex** of P if $P^v \notin \mathcal{P}_d^*(n-1)$. That is, if P^v is either $(d-1)$-dimensional or has width one.



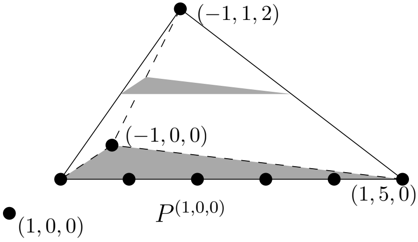
$P^{(-1,1,2)}$ is 2-dimensional

Essential vertices

From now on, let $P \in \mathcal{P}_d^*(n)$. For each vertex $v \in \mathrm{vert}(P)$, we denote by P^v the polytope $\mathrm{conv}(P \setminus \{v\} \cap \mathbb{Z}^d) \subset \mathbb{R}^d$. This polytope has size $n-1$ but it is not necessarily full-dimensional.

Definition (Essential vertex)

We say that v is an **essential vertex** of P if $P^v \notin \mathcal{P}_d^*(n-1)$. That is, if P^v is either $(d-1)$-dimensional or has width one.



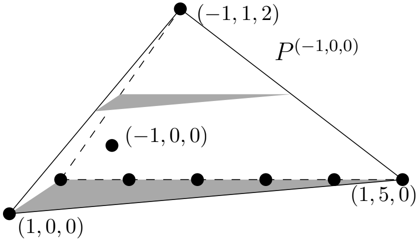
$P^{(1,0,0)}$ has width one

Essential vertices

From now on, let $P \in \mathcal{P}_d^*(n)$. For each vertex $v \in \text{vert}(P)$, we denote by P^v the polytope $\text{conv}(P \setminus \{v\} \cap \mathbb{Z}^d) \subset \mathbb{R}^d$. This polytope has size $n-1$ but it is not necessarily full-dimensional.

Definition (Essential vertex)

We say that v is an **essential vertex** of P if $P^v \notin \mathcal{P}_d^*(n-1)$. That is, if P^v is either $(d-1)$-dimensional or has width one.



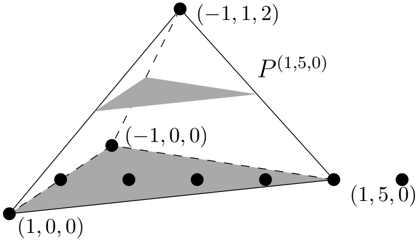
$P^{(-1,0,0)}$ has width one

Essential vertices

From now on, let $P \in \mathcal{P}^*_d(n)$. For each vertex $v \in \text{vert}(P)$, we denote by P^v the polytope $\text{conv}(P \setminus \{v\} \cap \mathbb{Z}^d) \subset \mathbb{R}^d$. This polytope has size $n-1$ but it is not necessarily full-dimensional.

Definition (Essential vertex)

We say that v is an **essential vertex** of P if $P^v \notin \mathcal{P}^*_d(n-1)$. That is, if P^v is either $(d-1)$-dimensional or has width one.



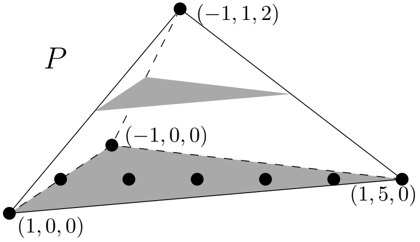
$P^{(0,5,0)}$ has width larger than one

Essential vertices

From now on, let $P \in \mathcal{P}_d^*(n)$. For each vertex $v \in \mathrm{vert}(P)$, we denote by P^v the polytope $\mathrm{conv}(P \setminus \{v\} \cap \mathbb{Z}^d) \subset \mathbb{R}^d$. This polytope has size $n-1$ but it is not necessarily full-dimensional.

Definition (Essential vertex)

We say that v is an **essential vertex** of P if $P^v \notin \mathcal{P}_d^*(n-1)$. That is, if P^v is either $(d-1)$-dimensional or has width one.



$P^{(-1,1,2)}$ is 2-dimensional
$P^{(1,0,0)}$ has width one
$P^{(-1,0,0)}$ has width one

$\Bigg\}$ *essential* vertices

$P^{(0,5,0)}$ has width larger than one

Quasiminimal vs. Merged

Definition

Let $P \in \mathcal{P}_d^*(n)$.

- We say that P is **quasiminimal** if it has ≤ 1 NON-essential vertices. That is, if there is at most one vertex v such that $P^v \in \mathcal{P}_d^*(n-1)$.

Quasiminimal vs. Merged

Definition

Let $P \in \mathcal{P}_d^*(n)$.

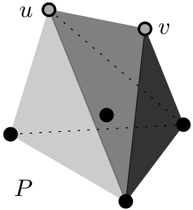
- We say that P is **quasiminimal** if it has ≤ 1 NON-essential vertices. That is, if there is at most one vertex v such that $P^v \in \mathcal{P}_d^*(n-1)$.
- We say that P is **merged** if there are ≥ 2 NON-essential vertices (i.e. $u, v \in \mathrm{vert}(P)$ with $P^u, P^v \in \mathcal{P}_d^*(n-1)$) AND the polytope $P^{u,v} := \mathrm{conv}(P^u \cap P^u \cap \mathbb{Z}^d)$ is still d-dimensional.

Quasiminimal vs. Merged

Definition

Let $P \in \mathcal{P}^*_d(n)$.

- We say that P is **quasiminimal** if it has ≤ 1 NON-essential vertices. That is, if there is at most one vertex v such that $P^v \in \mathcal{P}^*_d(n-1)$.
- We say that P is **merged** if there are ≥ 2 NON-essential vertices (i.e. $u, v \in \mathrm{vert}(P)$ with $P^u, P^v \in \mathcal{P}^*_d(n-1)$) AND the polytope $P^{u,v} := \mathrm{conv}(P^u \cap P^u \cap \mathbb{Z}^d)$ is still d-dimensional.

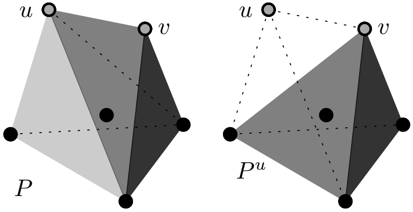


Quasiminimal vs. Merged

Definition

Let $P \in \mathcal{P}_d^*(n)$.

- We say that P is **quasiminimal** if it has ≤ 1 NON-essential vertices. That is, if there is at most one vertex v such that $P^v \in \mathcal{P}_d^*(n-1)$.
- We say that P is **merged** if there are ≥ 2 NON-essential vertices (i.e. $u, v \in \text{vert}(P)$ with $P^u, P^v \in \mathcal{P}_d^*(n-1)$) AND the polytope $P^{u,v} := \text{conv}(P^u \cap P^u \cap \mathbb{Z}^d)$ is still d-dimensional.

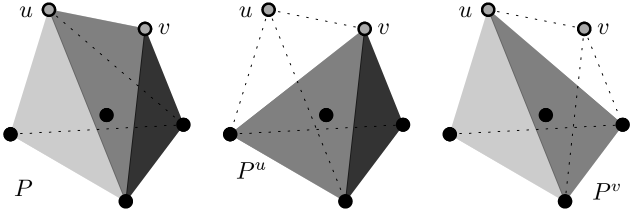


Quasiminimal vs. Merged

Definition

Let $P \in \mathcal{P}^*_d(n)$.

- We say that P is **quasiminimal** if it has ≤ 1 NON-essential vertices. That is, if there is at most one vertex v such that $P^v \in \mathcal{P}^*_d(n-1)$.
- We say that P is **merged** if there are ≥ 2 NON-essential vertices (i.e. $u, v \in \mathrm{vert}(P)$ with $P^u, P^v \in \mathcal{P}^*_d(n-1)$) AND the polytope $P^{u,v} := \mathrm{conv}(P^u \cap P^u \cap \mathbb{Z}^d)$ is still d-dimensional.

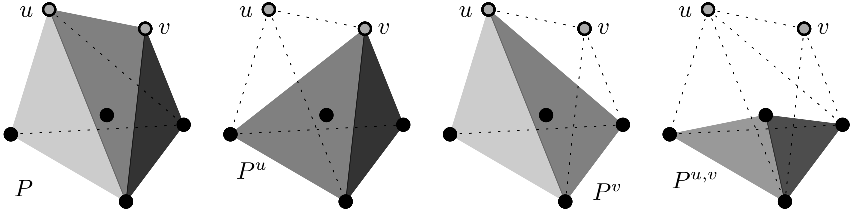


Quasiminimal vs. Merged

Definition

Let $P \in \mathcal{P}_d^*(n)$.

- We say that P is **quasiminimal** if it has ≤ 1 NON-essential vertices. That is, if there is at most one vertex v such that $P^v \in \mathcal{P}_d^*(n-1)$.
- We say that P is **merged** if there are ≥ 2 NON-essential vertices (i.e. $u, v \in \mathrm{vert}(P)$ with $P^u, P^v \in \mathcal{P}_d^*(n-1)$) AND the polytope $P^{u,v} := \mathrm{conv}(P^u \cap P^u \cap \mathbb{Z}^d)$ is still d-dimensional.

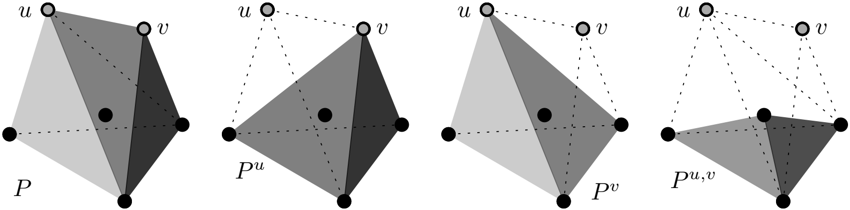


Quasiminimal vs. Merged

Definition

Let $P \in \mathcal{P}_d^*(n)$.

- We say that P is **quasiminimal** if it has ≤ 1 NON-essential vertices. That is, if there is at most one vertex v such that $P^v \in \mathcal{P}_d^*(n-1)$.
- We say that P is **merged** if there are ≥ 2 NON-essential vertices (i.e. $u, v \in \mathrm{vert}(P)$ with $P^u, P^v \in \mathcal{P}_d^*(n-1)$) AND the polytope $P^{u,v} := \mathrm{conv}(P^u \cap P^u \cap \mathbb{Z}^d)$ is still d-dimensional.



We will denote by $\mathcal{Q}_d(n)$ and $\mathcal{M}_d(n)$ the sets of quasiminimal and merged d-polytopes of size n, respectively.

Exceptions

Theorem (Blanco and Santos, 2016)

There is a single lattice 3*-dimensional lattice polytope that is neither* quasiminimal *nor* merged, *and it is of size $n = 6$:*

$|\mathcal{P}_3^*(6) \setminus (\mathcal{Q}_3(6) \cup \mathcal{M}_3(6))| = 1, \qquad \mathcal{P}_3^*(n) = \mathcal{Q}_3(n) \cup \mathcal{M}_3(n)$, *for all $n \geq 7$.*

- That is, this polytope has ≥ 2 NON-essential vertices, AND for all pairs $u, v \in \text{vert}(P)$ of non-essential vertices, $P^u, P^v \in \mathcal{P}_d^*(n-1)$ are such that $P^{u,v} := \text{conv}(P^u \cap P^u \cap \mathbb{Z}^d)$ is $(d-1)$-dimensional.

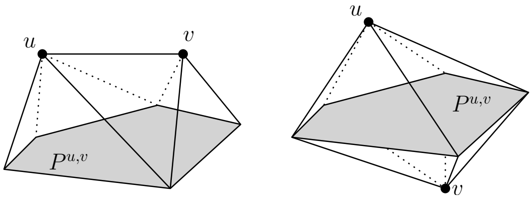
Exceptions

Theorem (Blanco and Santos, 2016)

There is a single lattice 3-dimensional lattice polytope that is neither quasiminimal *nor* merged, *and it is of size $n = 6$:*

$|\mathcal{P}_3^*(6) \setminus (\mathcal{Q}_3(6) \cup \mathcal{M}_3(6))| = 1, \qquad \mathcal{P}_3^*(n) = \mathcal{Q}_3(n) \cup \mathcal{M}_3(n)$, *for all $n \geq 7$.*

- That is, this polytope has ≥ 2 NON-essential vertices, AND for all pairs $u, v \in \mathrm{vert}(P)$ of non-essential vertices, $P^u, P^v \in \mathcal{P}_d^*(n-1)$ are such that $P^{u,v} := \mathrm{conv}(P^u \cap P^u \cap \mathbb{Z}^d)$ is $(d-1)$-dimensional.



Merging algorithm

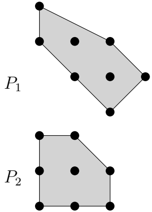
INPUT: a finite list L of lattice d-polytopes of size $n-1$ and width > 1.

Merging algorithm

INPUT: a finite list L of lattice d-polytopes of size $n-1$ and width >1.
OUTPUT: the list $L' = \texttt{Merging}(L)$ of all lattice d-polytopes of size n obtained merging polytopes of L.

Merging algorithm

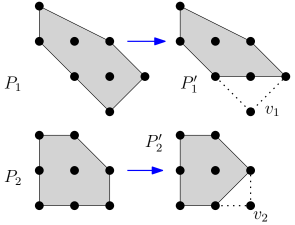
INPUT: a finite list L of lattice d-polytopes of size $n-1$ and width > 1.
OUTPUT: the list $L' = \texttt{Merging}(L)$ of all lattice d-polytopes of size n obtained merging polytopes of L.
For each $P_1, P_2 \in L$, and for each vertex v_1 of P_1 and v_2 of P_2:



Merging algorithm

INPUT: a finite list L of lattice d-polytopes of size $n-1$ and width > 1.
OUTPUT: the list $L' = \texttt{Merging}(L)$ of all lattice d-polytopes of size n obtained merging polytopes of L.
For each $P_1, P_2 \in L$, and for each vertex v_1 of P_1 and v_2 of P_2:

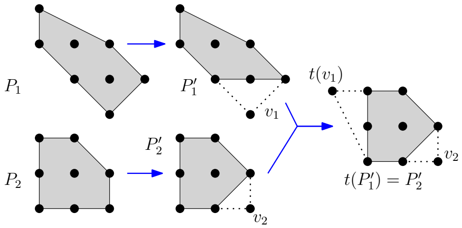
1. Let $P_1' = \mathrm{conv}(\mathbb{Z}^d \cap P_1 \setminus \{v_1\})$ and $P_2' = \mathrm{conv}(\mathbb{Z}^d \cap P_2 \setminus \{v_2\})$.



Merging algorithm

INPUT: a finite list L of lattice d-polytopes of size $n-1$ and width >1.
OUTPUT: the list $L' = \texttt{Merging}(L)$ of all lattice d-polytopes of size n obtained merging polytopes of L.
For each $P_1, P_2 \in L$, and for each vertex v_1 of P_1 and v_2 of P_2:

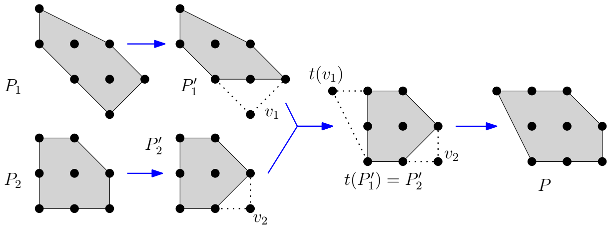
1. Let $P_1' = \operatorname{conv}(\mathbb{Z}^d \cap P_1 \setminus \{v_1\})$ and $P_2' = \operatorname{conv}(\mathbb{Z}^d \cap P_2 \setminus \{v_2\})$.
2. Check if P_1' and P_2' are d-dimensional and equivalent. If they are, let $t : \mathbb{Z}^d \to \mathbb{Z}^d$ be an equivalence sending P_1' to P_2' (there are finitely many possibilities for t; do step 3 for each).



Merging algorithm

INPUT: a finite list L of lattice d-polytopes of size $n-1$ and width >1.
OUTPUT: the list $L' = \texttt{Merging}(L)$ of all lattice d-polytopes of size n obtained merging polytopes of L.
For each $P_1, P_2 \in L$, and for each vertex v_1 of P_1 and v_2 of P_2:

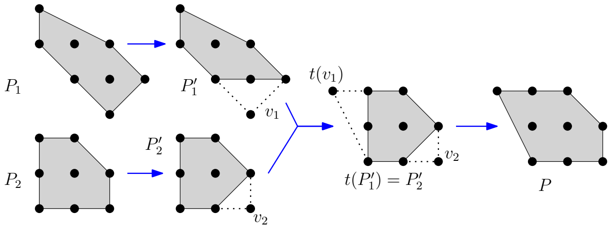
1. Let $P_1' = \operatorname{conv}(\mathbb{Z}^d \cap P_1 \setminus \{v_1\})$ and $P_2' = \operatorname{conv}(\mathbb{Z}^d \cap P_2 \setminus \{v_2\})$.
2. Check if P_1' and P_2' are d-dimensional and equivalent. If they are, let $t : \mathbb{Z}^d \to \mathbb{Z}^d$ be an equivalence sending P_1' to P_2' (there are finitely many possibilities for t; do step 3 for each).
3. If the size of $P := \operatorname{conv}(t(P_1) \cup P_2)$ equals n, add P to L'.



Merging algorithm

INPUT: a finite list L of lattice d-polytopes of size $n-1$ and width >1.
OUTPUT: the list $L' = \mathtt{Merging}(L)$ of all lattice d-polytopes of size n obtained merging polytopes of L.
For each $P_1, P_2 \in L$, and for each vertex v_1 of P_1 and v_2 of P_2:

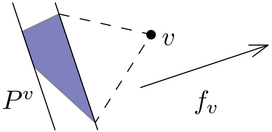
1. Let $P_1' = \mathrm{conv}(\mathbb{Z}^d \cap P_1 \setminus \{v_1\})$ and $P_2' = \mathrm{conv}(\mathbb{Z}^d \cap P_2 \setminus \{v_2\})$.
2. Check if P_1' and P_2' are d-dimensional and equivalent. If they are, let $t : \mathbb{Z}^d \to \mathbb{Z}^d$ be an equivalence sending P_1' to P_2' (there are finitely many possibilities for t; do step 3 for each).
3. If the size of $P := \mathrm{conv}(t(P_1) \cup P_2)$ equals n, add P to L'.



Dimension 3: By definition, and since $\mathcal{P}_3^*(n-1)$ is a finite list:
$\mathcal{M}_3(n) = \mathtt{Merging}(\mathcal{P}_3^*(n-1))$, for all n.

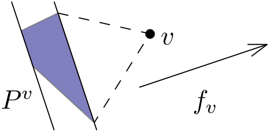
Quasiminimal polytopes

Let $P \in \mathcal{Q}_d(n)$ and, for each essential vertex $v \in \mathrm{vert}(P)$, let $f_v : \mathbb{R}^d \to \mathbb{R}$ be an integer linear functional that gives width one (or zero) to P^v.



Quasiminimal polytopes

Let $P \in \mathcal{Q}_d(n)$ and, for each essential vertex $v \in \mathrm{vert}(P)$, let $f_v : \mathbb{R}^d \to \mathbb{R}$ be an integer linear functional that gives width one (or zero) to P^v.



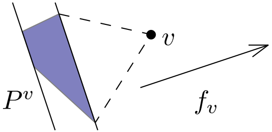
We distinguish two cases:

Definition (Boxed vs. spiked)

- If the set $\{f_v : v \text{ is essential vertex of } P\}$ linearly spans $(\mathbb{R}^d)^*$, then we can find d linearly independent f_v. We call these polytopes **boxed**, because *all except d of its lattice points lie in a d-parallelepiped of facet-width one*.

Quasiminimal polytopes

Let $P \in \mathcal{Q}_d(n)$ and, for each essential vertex $v \in \mathrm{vert}(P)$, let $f_v : \mathbb{R}^d \to \mathbb{R}$ be an integer linear functional that gives width one (or zero) to P^v.



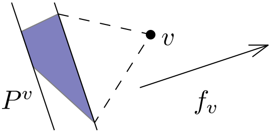
We distinguish two cases:

Definition (Boxed vs. spiked)

- If the set $\{f_v : v \text{ is essential vertex of } P\}$ linearly spans $(\mathbb{R}^d)^*$, then we can find d linearly independent f_v. We call these polytopes **boxed**, because *all except d of its lattice points lie in a d-parallelepiped of facet-width one*.
- If the set $\{f_v : v \text{ is essential vertex of } P\}$ does not linearly span $(\mathbb{R}^d)^*$, then there is a projection that respects all f_v. We call these polytopes **spiked**, because *most of their lattice points lie in a lattice segment*.

Quasiminimal polytopes

Let $P \in \mathcal{Q}_d(n)$ and, for each essential vertex $v \in \mathrm{vert}(P)$, let $f_v : \mathbb{R}^d \to \mathbb{R}$ be an integer linear functional that gives width one (or zero) to P^v.

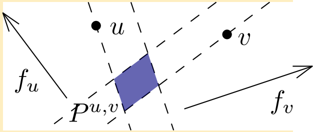


We distinguish two cases:

Definition (Boxed vs. spiked)

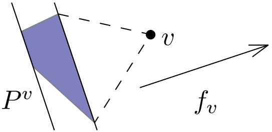
- If the set $\{f_v : v \text{ is essential vertex of } P\}$ linearly spans $(\mathbb{R}^d)^*$, then we can find d linearly independent f_v. We call these polytopes **boxed**, because *all except d of its lattice points lie in a d-parallelepiped of facet-width one*.

- If the set $\{f_v : v \text{ is essential vertex of } P\}$ does not linearly span $(\mathbb{R}^d)^*$, then there is a projection that respects all f_v. We call these polytopes **spiked**, because *most of their lattice points lie in a lattice segment*.



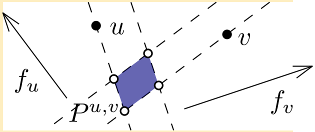
Quasiminimal polytopes

Let $P \in \mathcal{Q}_d(n)$ and, for each essential vertex $v \in \mathrm{vert}(P)$, let $f_v : \mathbb{R}^d \to \mathbb{R}$ be an integer linear functional that gives width one (or zero) to P^v.
We distinguish two cases:



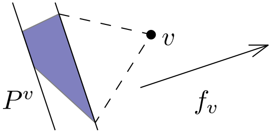
Definition (Boxed vs. spiked)

- If the set $\{f_v : v \text{ is essential vertex of } P\}$ linearly spans $(\mathbb{R}^d)^*$, then we can find d linearly independent f_v. We call these polytopes **boxed**, because *all except d of its lattice points lie in a d-parallelepiped of facet-width one*.
- If the set $\{f_v : v \text{ is essential vertex of } P\}$ does not linearly span $(\mathbb{R}^d)^*$, then there is a projection that respects all f_v. We call these polytopes **spiked**, because *most of their lattice points lie in a lattice segment*.



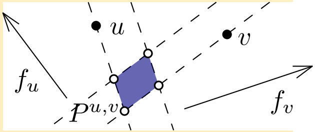
Quasiminimal polytopes

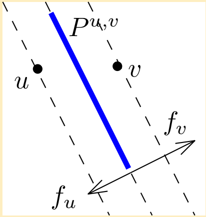
Let $P \in \mathcal{Q}_d(n)$ and, for each essential vertex $v \in \mathrm{vert}(P)$, let $f_v : \mathbb{R}^d \to \mathbb{R}$ be an integer linear functional that gives width one (or zero) to P^v.
We distinguish two cases:



Definition (Boxed vs. spiked)

- If the set $\{f_v : v \text{ is essential vertex of } P\}$ linearly spans $(\mathbb{R}^d)^*$, then we can find d linearly independent f_v. We call these polytopes **boxed**, because *all except d of its lattice points lie in a d-parallelepiped of facet-width one*.
- If the set $\{f_v : v \text{ is essential vertex of } P\}$ does not linearly span $(\mathbb{R}^d)^*$, then there is a projection that respects all f_v. We call these polytopes **spiked**, because *most of their lattice points lie in a lattice segment*.

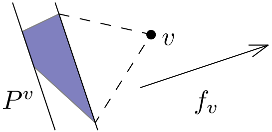




Quasiminimal polytopes

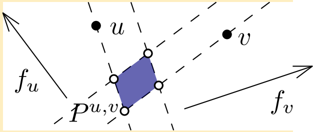
Let $P \in \mathcal{Q}_d(n)$ and, for each essential vertex $v \in \mathrm{vert}(P)$, let $f_v : \mathbb{R}^d \to \mathbb{R}$ be an integer linear functional that gives width one (or zero) to P^v.

We distinguish two cases:

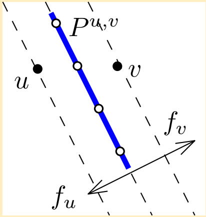


Definition (Boxed vs. spiked)

- If the set $\{f_v : v \text{ is essential vertex of } P\}$ linearly spans $(\mathbb{R}^d)^*$, then we can find d linearly independent f_v. We call these polytopes **boxed**, because *all except d of its lattice points lie in a d-parallelepiped of facet-width one*.



- If the set $\{f_v : v \text{ is essential vertex of } P\}$ does not linearly span $(\mathbb{R}^d)^*$, then there is a projection that respects all f_v. We call these polytopes **spiked**, because *most of their lattice points lie in a lattice segment*.



Boxed polytopes

It is easy to see that boxed d-polytopes have size at most $2^d + d$: apart from d vertices, the only possible lattice points are the 2^d vertices of *the* d-parallelepiped.

Boxed polytopes

It is easy to see that boxed d-polytopes have size at most $2^d + d$: apart from d vertices, the only possible lattice points are the 2^d vertices of *the* d-parallelepiped.

In dimension three this implies (by our Lemma that almost all lattice 3-polytopes of fixed size have width one) that there are finitely many.

We have enumerated those of dimension 3 with computer help. Let the list of them, for each size $n \in \{7, \dots, 11\}$, be denoted $\texttt{Boxed}_3(n)$.

Spiked polytopes

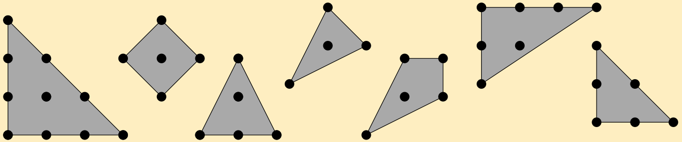
Spiked d-polytopes can, however, have arbitrarily large size. Still, it turns out that the projection of spiked d-polytopes in the direction of *the* lattice segment has very specific properties:

Spiked polytopes

Spiked d-polytopes can, however, have arbitrarily large size. Still, it turns out that the projection of spiked d-polytopes in the direction of *the* lattice segment has very specific properties:

Theorem (Blanco and Santos, 2016)

Every spiked 3-polytope of size $n \geq 7$ projects to one of the following 2-polytopes in such a way that all the vertices in the projection have a unique element in the preimage.



This allows us to explicitly list spiked 3-polytopes for each given size $n \geq 7$. We denote this list by $\texttt{Spiked}_3(n)$.

Quasiminimal polytopes

Putting these things together, we present the full classification of quasiminimal 3-polytopes:

Theorem (Blanco and Santos, 2016)

For $7 \leq n \leq 11$, $\mathcal{Q}_3(n) = \texttt{Boxed}_3(n) \cup \texttt{Spiked}_3(n)$, *and it has* 50, 42, 44, 46 *and* 49 *elements, respectively.*
For $n > 11$, $\mathcal{Q}_3(n) = \texttt{Spiked}_3(n)$ *and it has* $4n + 7$ *elements if* $n \equiv 0 \pmod 3$, *and* $4n + 5$ *otherwise.*

Summary & computational results

- $\mathcal{P}_3^*(5)$ and $\mathcal{P}_3^*(6)$ (explicitly classified previously by us).
- $\mathcal{Q}_3(n)$, for $n \geq 7$, can be computed explicitly.
- $\mathcal{M}_3(n) = \texttt{Merging}(\mathcal{P}_3^*(n-1))$, for $n \geq 7$.

Summary & computational results

- $\mathcal{P}_3^*(5)$ and $\mathcal{P}_3^*(6)$ (explicitly classified previously by us).
- $\mathcal{Q}_3(n)$, for $n \geq 7$, can be computed explicitly.
- $\mathcal{M}_3(n) = \texttt{Merging}(\mathcal{P}_3^*(n-1))$, for $n \geq 7$.

Assume $\mathcal{P}_3^*(n-1)$ previously computed. Then:

$$\mathcal{P}_3^*(n) = \texttt{Merging}(\mathcal{P}_3^*(n-1)) \ \cup \ \mathcal{Q}_3(n).$$

Summary & computational results

- $\mathcal{P}_3^*(5)$ and $\mathcal{P}_3^*(6)$ (explicitly classified previously by us).
- $\mathcal{Q}_3(n)$, for $n \geq 7$, can be computed explicitly.
- $\mathcal{M}_3(n) = \texttt{Merging}(\mathcal{P}_3^*(n-1))$, for $n \geq 7$.

Assume $\mathcal{P}_3^*(n-1)$ previously computed. Then:

$$\mathcal{P}_3^*(n) = \texttt{Merging}(\mathcal{P}_3^*(n-1)) \ \cup \ \mathcal{Q}_3(n).$$

Size	4	5	6	7	8	9	10	11
width 2	0	9	74	477	2524	10862	40885	137803
width 3	0	0	2	19	151	836	4148	18635
width 4	0	0	0	0	0	0	2	26
quasiminimal	0	9	35	50	42	44	46	49
merged	0	0	40	446	2633	11654	44989	156415
exceptions	0	0	1	0	0	0	0	0
total	0	9	76	496	2675	11698	45035	156464

Infinitely many polytopes of width larger than one

What about dimension 4 and higher?

Infinitely many polytopes of width larger than one

What about dimension 4 and higher?

Theorem (Haase-Ziegler 2000)

Let $\Delta(v)$ for $v \in \mathbb{Z}^4$ denote the simplex with vertices e_1, e_2, e_3, e_4 and v:

- $\Delta(2, 2, 3, D-6)$ *has determinant D for all D, width two for all $D \geq 8$, and is empty for infinitely many values of D.*

Infinitely many polytopes of width larger than one

What about dimension 4 and higher?

Theorem (Haase-Ziegler 2000)

Let $\Delta(v)$ for $v \in \mathbb{Z}^4$ denote the simplex with vertices e_1, e_2, e_3, e_4 and v:

- $\Delta(2, 2, 3, D-6)$ *has determinant D for all D, width two for all $D \geq 8$, and is empty for infinitely many values of D.*
- $\Delta(6, 14, 17, 65)$ *is empty and has width four.*

Infinitely many polytopes of width larger than one

What about dimension 4 and higher?

Theorem (Haase-Ziegler 2000)

Let $\Delta(v)$ for $v \in \mathbb{Z}^4$ denote the simplex with vertices e_1, e_2, e_3, e_4 and v:

- $\Delta(2, 2, 3, D - 6)$ *has determinant D for all D, width two for all $D \geq 8$, and is empty for infinitely many values of D.*
- $\Delta(6, 14, 17, 65)$ *is empty and has width four.*

The classification of empty 4-simplices is unknown. What we know is:

Infinitely many polytopes of width larger than one

What about dimension 4 and higher?

Theorem (Haase-Ziegler 2000)

Let $\Delta(v)$ for $v \in \mathbb{Z}^4$ denote the simplex with vertices e_1, e_2, e_3, e_4 and v:

- $\Delta(2, 2, 3, D-6)$ *has determinant D for all D, width two for all $D \geq 8$, and is empty for infinitely many values of D.*
- $\Delta(6, 14, 17, 65)$ *is empty and has width four.*

The classification of empty 4-simplices is unknown. What we know is:

Theorem

- *(Mori-Morrison-Morrison 1988, Sankaran 1990, Bover 2009) There are only finitely many empty 4-simplices not falling into an explicitly described set of empty 4-simplices (consisting of 29 one-parameter families and two two-parameter families).*

Infinitely many polytopes of width larger than one

What about dimension 4 and higher?

Theorem (Haase-Ziegler 2000)

Let $\Delta(v)$ for $v \in \mathbb{Z}^4$ denote the simplex with vertices e_1, e_2, e_3, e_4 and v:

- $\Delta(2, 2, 3, D - 6)$ *has determinant D for all D, width two for all $D \geq 8$, and is empty for infinitely many values of D.*
- $\Delta(6, 14, 17, 65)$ *is empty and has width four.*

The classification of empty 4-simplices is unknown. What we know is:

Theorem

- *(Mori-Morrison-Morrison 1988, Sankaran 1990, Bover 2009) There are only finitely many empty 4-simplices not falling into an explicitly described set of empty 4-simplices (consisting of 29 one-parameter families and two two-parameter families).*
- *(Barile-Bernardi-Borisov-Kantor, 2011) In particular, there are only finitely many empty 4-simplices of width larger than two.*

Threshold width

Theorem (Blanco-Haase-Hofmann-Santos, 16+)

For each dimension d there is a threshold $w^\infty(d) \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ all but finitely many d-polytopes of size n have width $\leq w^\infty(d)$.

Threshold width

Theorem (Blanco-Haase-Hofmann-Santos, 16+)

For each dimension d there is a threshold $w^\infty(d) \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ all but finitely many d-polytopes of size n have width $\leq w^\infty(d)$.

E.g.: $w^\infty(2) = 0$ (folklore + Pick's Theorem) and $w^\infty(3) = 1$ (Blanco-Santos, 2016).

Threshold width

Theorem (Blanco-Haase-Hofmann-Santos, 16+)

For each dimension d there is a threshold $w^\infty(d) \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ all but finitely many d-polytopes of size n have width $\leq w^\infty(d)$.

E.g.: $w^\infty(2) = 0$ (folklore + Pick's Theorem) and $w^\infty(3) = 1$ (Blanco-Santos, 2016).

Proof.

- (Nill-Ziegler, 2011) There is only a finite number of hollow lattice d-polytopes that do not admit a lattice projection onto a hollow lattice $(d-1)$-polytope.

Threshold width

Theorem (Blanco-Haase-Hofmann-Santos, 16+)

For each dimension d there is a threshold $w^\infty(d) \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ all but finitely many d-polytopes of size n have width $\leq w^\infty(d)$.

E.g.: $w^\infty(2) = 0$ (folklore + Pick's Theorem) and $w^\infty(3) = 1$ (Blanco-Santos, 2016).

Proof.

- (Nill-Ziegler, 2011) There is only a finite number of hollow lattice d-polytopes that do not admit a lattice projection onto a hollow lattice $(d-1)$-polytope. (Hollow = "no interior lattice points").

Threshold width

Theorem (Blanco-Haase-Hofmann-Santos, 16+)

For each dimension d there is a threshold $w^\infty(d) \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ all but finitely many d-polytopes of size n have width $\leq w^\infty(d)$.

E.g.: $w^\infty(2) = 0$ (folklore + Pick's Theorem) and $w^\infty(3) = 1$ (Blanco-Santos, 2016).

Proof.

- (Nill-Ziegler, 2011) There is only a finite number of hollow lattice d-polytopes that do not admit a lattice projection onto a hollow lattice $(d-1)$-polytope. (Hollow = "no interior lattice points").
- (Kannan-Lovász, 1988) Hollow *convex bodies* have width in $O(d^{5/2})$.

Threshold width

Theorem (Blanco-Haase-Hofmann-Santos, 16+)

For each dimension d there is a threshold $w^\infty(d) \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ all but finitely many d-polytopes of size n have width $\leq w^\infty(d)$.

E.g.: $w^\infty(2) = 0$ (folklore + Pick's Theorem) and $w^\infty(3) = 1$ (Blanco-Santos, 2016).

Proof.

- (Nill-Ziegler, 2011) There is only a finite number of hollow lattice d-polytopes that do not admit a lattice projection onto a hollow lattice $(d-1)$-polytope. (Hollow = "no interior lattice points").
- (Kannan-Lovász, 1988) Hollow *convex bodies* have width in $O(d^{5/2})$. (Better bound $O(d^{3/2})$ for *lattice polytopes*, Banaszczyk et al. 1999).

Threshold width

Theorem (Blanco-Haase-Hofmann-Santos, 16+)

For each dimension d there is a threshold $w^\infty(d) \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ all but finitely many d-polytopes of size n have width $\leq w^\infty(d)$.

E.g.: $w^\infty(2) = 0$ (folklore + Pick's Theorem) and $w^\infty(3) = 1$ (Blanco-Santos, 2016).

Proof.

- (Nill-Ziegler, 2011) There is only a finite number of hollow lattice d-polytopes that do not admit a lattice projection onto a hollow lattice $(d-1)$-polytope. (Hollow ="no interior lattice points").
- (Kannan-Lovász, 1988) Hollow *convex bodies* have width in $O(d^{5/2})$. (Better bound $O(d^{3/2})$ for *lattice polytopes*, Banaszczyk et al. 1999).
- (Hensley, 1983) For each positive $k \in \mathbb{Z}_{\leq 0}$ there is a bound $V(k) \in \mathbb{N}$ for the volume of lattice d-polytopes with exactly k interior points.

Threshold width

Theorem (Blanco-Haase-Hofmann-Santos, 16+)

For each dimension d there is a threshold $w^\infty(d) \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ all but finitely many d-polytopes of size n have width $\leq w^\infty(d)$.

E.g.: $w^\infty(2) = 0$ (folklore + Pick's Theorem) and $w^\infty(3) = 1$ (Blanco-Santos, 2016).

Proof.

- (Nill-Ziegler, 2011) There is only a finite number of hollow lattice d-polytopes that do not admit a lattice projection onto a hollow lattice $(d-1)$-polytope. (Hollow ="no interior lattice points").
- (Kannan-Lovász, 1988) Hollow *convex bodies* have width in $O(d^{5/2})$. (Better bound $O(d^{3/2})$ for *lattice polytopes*, Banaszczyk et al. 1999).
- (Hensley, 1983) For each positive $k \in \mathbb{Z}_{\leq 0}$ there is a bound $V(k) \in \mathbb{N}$ for the volume of lattice d-polytopes with exactly k interior points.
- (Lagarias-Ziegler, 1991) A family of lattice d-polytopes with bounded volume contains only a finite number of integral equivalence classes.

Threshold width

Theorem (Blanco-Haase-Hofmann-Santos, 16+)

For each dimension d there is a threshold $w^\infty(d) \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ all but finitely many d-polytopes of size n have width $\leq w^\infty(d)$.

E.g.: $w^\infty(2) = 0$ (folklore + Pick's Theorem) and $w^\infty(3) = 1$ (Blanco-Santos, 2016).

Proof.

- (Nill-Ziegler, 2011) There is only a finite number of hollow lattice d-polytopes that do not admit a lattice projection onto a hollow lattice $(d-1)$-polytope. (Hollow ="no interior lattice points").
- (Kannan-Lovász, 1988) Hollow *convex bodies* have width in $O(d^{5/2})$. (Better bound $O(d^{3/2})$ for *lattice polytopes*, Banaszczyk et al. 1999).
- (Hensley, 1983) For each positive $k \in \mathbb{Z}_{\leq 0}$ there is a bound $V(k) \in \mathbb{N}$ for the volume of lattice d-polytopes with exactly k interior points.
- (Lagarias-Ziegler, 1991) A family of lattice d-polytopes with bounded volume contains only a finite number of integral equivalence classes.

Threshold width via lifts of $(d-1)$-polytopes

Let $w_H(d)$:= maximum width of a *hollow* lattice d-polytope (which is finite by Kannan-Lovász). As a by-product of the previous proof we have:

Corollary

$$w^\infty(d) \leq w_H(d-1) \leq O(d^{3/2}).$$

Threshold width via lifts of $(d-1)$-polytopes

Let $w_H(d)$:= maximum width of a *hollow* lattice d-polytope (which is finite by Kannan-Lovász). As a by-product of the previous proof we have:

Corollary

$$w^\infty(d) \leq w_H(d-1) \leq O(d^{3/2}).$$

For lower bounds, there is the following observation: if there is a lattice $(d-1)$-polytope Q that can be *lifted* to infinitely many lattice d-polytopes of the same size n, then $w^\infty(d) \geq \text{width}(Q)$.

Threshold width via lifts of $(d-1)$-polytopes

Let $w_H(d)$:= maximum width of a *hollow* lattice d-polytope (which is finite by Kannan-Lovász). As a by-product of the previous proof we have:

Corollary

$$w^\infty(d) \le w_H(d-1) \le O(d^{3/2}).$$

For lower bounds, there is the following observation: if there is a lattice $(d-1)$-polytope Q that can be *lifted* to infinitely many lattice d-polytopes of the same size n, then $w^\infty(d) \ge \text{width}(Q)$.

E.g.: $w^\infty(3) \ge 1$ because the unit square has infinitely many lifts of size 4 (Reeve tetrahedra).

Threshold width via lifts of $(d-1)$-polytopes

Let $w_H(d)$:= maximum width of a *hollow* lattice d-polytope (which is finite by Kannan-Lovász). As a by-product of the previous proof we have:

Corollary

$$w^\infty(d) \le w_H(d-1) \le O(d^{3/2}).$$

For lower bounds, there is the following observation: if there is a lattice $(d-1)$-polytope Q that can be *lifted* to infinitely many lattice d-polytopes of the same size n, then $w^\infty(d) \ge \mathrm{width}(Q)$.

E.g.: $w^\infty(3) \ge 1$ because the unit square has infinitely many lifts of size 4 (Reeve tetrahedra).

Threshold width via lifts of $(d-1)$-polytopes

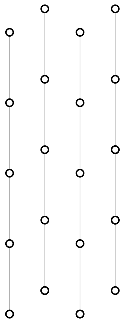
Let $w_H(d)$:= maximum width of a *hollow* lattice d-polytope (which is finite by Kannan-Lovász). As a by-product of the previous proof we have:

Corollary

$$w^{\infty}(d) \leq w_H(d-1) \leq O(d^{3/2}).$$

For lower bounds, there is the following observation: if there is a lattice $(d-1)$-polytope Q that can be *lifted* to infinitely many lattice d-polytopes of the same size n, then $w^{\infty}(d) \geq \mathrm{width}(Q)$.

E.g.: $w^{\infty}(3) \geq 1$ because the unit square has infinitely many lifts of size 4 (Reeve tetrahedra).



Threshold width via lifts of $(d-1)$-polytopes

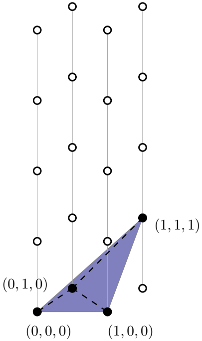
Let $w_H(d)$:= maximum width of a *hollow* lattice d-polytope (which is finite by Kannan-Lovász). As a by-product of the previous proof we have:

Corollary

$$w^\infty(d) \leq w_H(d-1) \leq O(d^{3/2}).$$

For lower bounds, there is the following observation: if there is a lattice $(d-1)$-polytope Q that can be *lifted* to infinitely many lattice d-polytopes of the same size n, then $w^\infty(d) \geq \text{width}(Q)$.

E.g.: $w^\infty(3) \geq 1$ because the unit square has infinitely many lifts of size 4 (Reeve tetrahedra).



Threshold width via lifts of $(d-1)$-polytopes

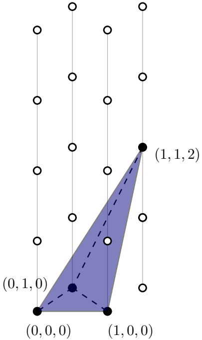
Let $w_H(d)$:= maximum width of a *hollow* lattice d-polytope (which is finite by Kannan-Lovász). As a by-product of the previous proof we have:

Corollary

$$w^\infty(d) \le w_H(d-1) \le O(d^{3/2}).$$

For lower bounds, there is the following observation: if there is a lattice $(d-1)$-polytope Q that can be *lifted* to infinitely many lattice d-polytopes of the same size n, then $w^\infty(d) \ge \mathrm{width}(Q)$.

E.g.: $w^\infty(3) \ge 1$ because the unit square has infinitely many lifts of size 4 (Reeve tetrahedra).



Threshold width via lifts of $(d-1)$-polytopes

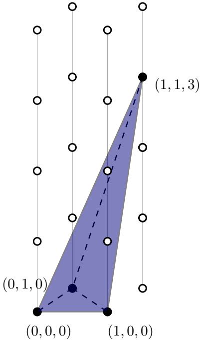
Let $w_H(d)$:= maximum width of a *hollow* lattice d-polytope (which is finite by Kannan-Lovász). As a by-product of the previous proof we have:

Corollary

$$w^\infty(d) \le w_H(d-1) \le O(d^{3/2}).$$

For lower bounds, there is the following observation: if there is a lattice $(d-1)$-polytope Q that can be *lifted* to infinitely many lattice d-polytopes of the same size n, then $w^\infty(d) \ge \mathrm{width}(Q)$.

E.g.: $w^\infty(3) \ge 1$ because the unit square has infinitely many lifts of size 4 (Reeve tetrahedra).



Threshold width via lifts of $(d-1)$-polytopes

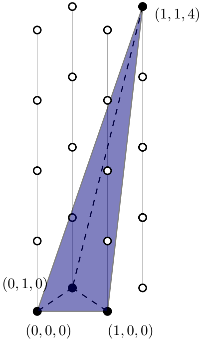
Let $w_H(d)$:= maximum width of a *hollow* lattice d-polytope (which is finite by Kannan-Lovász). As a by-product of the previous proof we have:

Corollary

$$w^\infty(d) \le w_H(d-1) \le O(d^{3/2}).$$

For lower bounds, there is the following observation: if there is a lattice $(d-1)$-polytope Q that can be *lifted* to infinitely many lattice d-polytopes of the same size n, then $w^\infty(d) \ge \text{width}(Q)$.

E.g.: $w^\infty(3) \ge 1$ because the unit square has infinitely many lifts of size 4 (Reeve tetrahedra).



Threshold width via lifts of $(d-1)$-polytopes

In fact this is an if and only if:

Threshold width via lifts of $(d-1)$-polytopes

In fact this is an if and only if:

Theorem (Blanco-Haase-Hofmann-Santos, 16+)

$w^\infty(d)$ is the maximum width of a lattice $(d-1)$-polytope Q that has infinitely many lifts of the same size. Moreover, any such Q must be hollow.

Threshold width via lifts of $(d-1)$-polytopes

In fact this is an if and only if:

Theorem (Blanco-Haase-Hofmann-Santos, 16+)

$w^\infty(d)$ is the maximum width of a lattice $(d-1)$-polytope Q that has infinitely many lifts of the same size. Moreover, any such Q must be hollow.

We also show that every hollow $(d-2)$-polytope can be lifted to a hollow $(d-1)$-polytope Q that has infinitely many lifts of constant size to dimension d, which implies:

Corollary

$w^\infty(d) \in [w_E(d-2), w_E(d-1)]$.

Threshold width via lifts of $(d-1)$-polytopes

In fact this is an if and only if:

Theorem (Blanco-Haase-Hofmann-Santos, 16+)

$w^\infty(d)$ is the maximum width of a lattice $(d-1)$-polytope Q that has infinitely many lifts of the same size. Moreover, any such Q must be hollow.

We also show that every hollow $(d-2)$-polytope can be lifted to a hollow $(d-1)$-polytope Q that has infinitely many lifts of constant size to dimension d, which implies:

Corollary

$w^\infty(d) \in [w_E(d-2), w_E(d-1)]$. *To compute $w^\infty(d)$ one can:*

Threshold width via lifts of $(d-1)$-polytopes

In fact this is an if and only if:

Theorem (Blanco-Haase-Hofmann-Santos, 16+)

$w^\infty(d)$ is the maximum width of a lattice $(d-1)$-polytope Q that has infinitely many lifts of the same size. Moreover, any such Q must be hollow.

We also show that every hollow $(d-2)$-polytope can be lifted to a hollow $(d-1)$-polytope Q that has infinitely many lifts of constant size to dimension d, which implies:

Corollary

$w^\infty(d) \in [w_E(d-2), w_E(d-1)]$. To compute $w^\infty(d)$ one can:

1. *Compute the (finitely many, by Nill-Ziegler) hollow $(d-1)$-polytopes that* **do not project** *to a hollow $(d-2)$-polytope.*

Threshold width via lifts of $(d-1)$-polytopes

In fact this is an if and only if:

Theorem (Blanco-Haase-Hofmann-Santos, 16+)

$w^\infty(d)$ is the maximum width of a lattice $(d-1)$-polytope Q that has infinitely many lifts of the same size. Moreover, any such Q must be hollow.

We also show that every hollow $(d-2)$-polytope can be lifted to a hollow $(d-1)$-polytope Q that has infinitely many lifts of constant size to dimension d, which implies:

Corollary

$w^\infty(d) \in [w_E(d-2), w_E(d-1)]$. *To compute $w^\infty(d)$ one can:*

1. *Compute the (finitely many, by Nill-Ziegler) hollow $(d-1)$-polytopes that* **do not project** *to a hollow $(d-2)$-polytope.*
2. *For each of them check whether it has infinitely many lifts of some fixed size. $w^\infty(d)$ equals the maximal width of one that does.*

Threshold width in dimensions three and four

$d = 3$: There is a unique hollow 2-polytope of width larger than one, the second dilation 2Δ of a unimodular triangle. By the corollary, $w^{\infty}(3)$ equals $2 = w_E(2)$ or $1 = w_E(1)$ depending solely on whether 2Δ has infinitely many lifts of some constant size.

Threshold width in dimensions three and four

$d = 3$: There is a unique hollow 2-polytope of width larger than one, the second dilation 2Δ of a unimodular triangle. By the corollary, $w^\infty(3)$ equals $2 = w_E(2)$ or $1 = w_E(1)$ depending solely on whether 2Δ has infinitely many lifts of some constant size. It is easy to check that it does not, so $w^\infty(3) = 1$ (Blanco-Santos, 2016).

Threshold width in dimensions three and four

$d = 3$: There is a unique hollow 2-polytope of width larger than one, the second dilation 2Δ of a unimodular triangle. By the corollary, $w^\infty(3)$ equals $2 = w_E(2)$ or $1 = w_E(1)$ depending solely on whether 2Δ has infinitely many lifts of some constant size. It is easy to check that it does not, so $w^\infty(3) = 1$ (Blanco-Santos, 2016).

$d = 4$: The list of hollow 3-polytopes that do not project to a hollow 2 polytope has been computed in (Averkov-Krümpelmann-Weltge, 2015). There are five of width three, and the rest have width two.

Threshold width in dimensions three and four

$d = 3$: There is a unique hollow 2-polytope of width larger than one, the second dilation 2Δ of a unimodular triangle. By the corollary, $w^\infty(3)$ equals $2 = w_E(2)$ or $1 = w_E(1)$ depending solely on whether 2Δ has infinitely many lifts of some constant size. It is easy to check that it does not, so $w^\infty(3) = 1$ (Blanco-Santos, 2016).

$d = 4$: The list of hollow 3-polytopes that do not project to a hollow 2 polytope has been computed in (Averkov-Krümpelmann-Weltge, 2015). There are five of width three, and the rest have width two. Thus, $w^\infty(4)$ equals $3 = w_E(3)$ or $2 = w_E(2)$ depending solely on whether some of those five has infinitely many lifts of some constant size. We (Blanco-Haase-Hofmann-Santos, 16+) show that they do not, so:

Theorem (BHSS)

$w^\infty(4) = 2$.

Summary of known values

d	$w_H(d-2)$	$w^\infty(d)$	$w_H(d-1)$
1	–	0	–
2	–	0	1
3	1	1	2
4	2	2	3
5	3	≥ 4	≥ 4

Summary of known values

d	$w_H(d-2)$	$w^\infty(d)$	$w_H(d-1)$
1	–	0	–
2	–	0	1
3	1	1	2
4	2	2	3
5	3	≥ 4	≥ 4

Final remarks

Summary of known values

d	$w_H(d-2)$	$w^\infty(d)$	$w_H(d-1)$
1	–	0	–
2	–	0	1
3	1	1	2
4	2	2	3
5	3	≥ 4	≥ 4

Final remarks

1. We know $w^\infty(5) \geq 4 > w_H(3)$ because we prove that $w^\infty(d)$ is at least the maximum width of an *empty*$(d-1)$-polytope (empty = all lattice points are vertices) and there are empty 4-simplices of width 4.

Summary of known values

d	$w_H(d-2)$	$w^\infty(d)$	$w_H(d-1)$
1	–	0	–
2	–	0	1
3	1	1	2
4	2	2	3
5	3	≥ 4	≥ 4

Final remarks

1. We know $w^\infty(5) \geq 4 > w_H(3)$ because we prove that $w^\infty(d)$ is at least the maximum width of an *empty*$(d-1)$-polytope (empty = all lattice points are vertices) and there are empty 4-simplices of width 4.
2. As a by-product we have an independent proof of:

Corollary (Barile-Bernardi-Borisov-Kantor, 2011)

There are only finitely many empty 4-simplices of width larger than two.

Thank you for your attention

http://personales.unican.es/santosf
http://personales.unican.es/blancogm/latticepoints.html