Analytic combinatorics of connected graphs

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Introduction

\[ n = \# \text{ vertices}, \ m = \# \text{ edges} \]

**Goal:** counting \((n, m)\)-connected graphs when \(m = \Theta(n)\)

**Tool:** analytic combinatorics (Flajolet Sedgewick 2009).

**Future extensions:**
- investigate the “giant component” of random graphs with \(m/n > 1/2\)
  (Ederős Rényi 1959, Janson Knuth Łuczak Pittel 1993)
- extend the result to other graph-like models
  (degree constraints, inhomogeneous, hypergraphs).
<table>
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<tr>
<th>Condition</th>
<th>Structure</th>
<th>Reference</th>
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<tr>
<td>$m = n - 1$</td>
<td>trees</td>
<td>Borchardt 1860</td>
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<td></td>
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<td>Cayley formula</td>
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<td>$m = n$</td>
<td>unicycles</td>
<td>Rényi 1959</td>
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<td>$m - n = O(1)$</td>
<td>fixed excess</td>
<td>Wright 1977, 1978, 1980</td>
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<td></td>
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<td>Flajolet Schaeffer Salvy 2004</td>
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<td>$m - n \to \infty$</td>
<td>large excess</td>
<td>Bender Canfield McKay 1990</td>
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<td></td>
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<td>Pittel Wormald 2005</td>
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<td></td>
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<td>Hofstad Spencer 2006</td>
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<td>$m = o(n)$</td>
<td></td>
<td>Łuczak 1990</td>
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<tr>
<td>$\frac{2m}{n} - \log(n) \to \infty$</td>
<td>almost all graphs</td>
<td>Erdős Rényi 1959</td>
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</table>
Structure of the proof

Multigraphs with degree constraints (all vertices have their degree in $D$)

remove loops and double edges

Graphs with degree constraints (E.d.P Ramos 2006)

counting graphs without trees
(same principle as in Pittel Wormald 2005)

Connected graphs
Multigraphs with degree constraints

Multigraphs: labelled vertices, labelled oriented edges

Degree constraints: all vertices have their degree in a given set $D$

$$\delta_d = \begin{cases} 1 & \text{if } d \in D \\ 0 & \text{otherwise} \end{cases}$$

$$\Delta(x) = \sum_{d \geq 0} \delta_d \frac{x^d}{d!}.$$ 

Generating function:

$$MG(z, w) = \sum_{\text{multigraph } G} \prod_{d \geq 0} \delta_d^{\deg_d(G)} \frac{w^m(G)}{2^m(G)m(G)!} \frac{z^n(G)}{n(G)!}.$$ 

Half-edges representation:

$$MG(z, w) = \sum_{m \geq 0} (2m)! [x^{2m}] e^{2z\Delta(x)} \frac{w^m}{2^m m!}.$$
**Multigraphs with degree constraints**

**Multigraphs:** labelled vertices, labelled oriented edges

![Multigraph Example]

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From multigraphs to simple graphs

Simple graphs: unlabelled unoriented edges, no loops nor multiple edges

\[ \text{SG}(z, w) = \sum_{\text{graph } G} \prod_{d \geq 0} \delta_{d}^{\deg d(G)} w^{m(G)} \frac{z^{n(G)}}{n(G)!} \]

Any \((n, m)\)-graph matches \(2^{m} m!\) multigraphs

Corollary: set \(\text{MG}(z, w, u) := \) generating function of multigraphs with loops and double edges marked by \(u\)

\[ \text{MG}(z, w, 0) = \sum_{\text{multigraph } G} \prod_{d \geq 0} \delta_{d}^{\deg d(G)} \frac{w^{m(G)}}{2^{m(G)} m(G)!} \frac{z^{n(G)}}{n(G)!} \]
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Any \((n, m)\)-graph matches \(2^m m!\) multigraphs

Corollary: set \(MG(z, w, u) := \) gf of multigraphs with loops and double edges marked by \(u\)

\[
MG(z, w, 0) = \sum_{\text{graph } G} 2^{m(G)} m(G)! \prod_{d \geq 0} \delta_{d}^{\text{deg}_d(G)} \frac{w^{m(G)}}{2^{m(G)} m(G)!} \frac{z^{n(G)}}{n(G)!}
\]
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SG(z, w) = \sum_{\text{graph } G} \prod_{d \geq 0} \delta_d^\deg_d(G) w^m(G) z^n(G) \frac{z^n(G)}{n(G)!}
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**Corollary:** set \(MG(z, w, u) := \) gf of multigraphs with loops and double edges marked by \(u\)

\(MG(z, w, 0) = SG(z, w)\)
Removing loops and double edges

**Inclusion-exclusion:** express $\text{MG}(z, w, u + 1)$: each loop and double edge is either marked or left unmarked. Then set $u = -1$.

**Patchwork:** set of marked loops and double edges

$$P(z, w, (\delta_d)_{d \geq 0}, u) = \sum_{\text{patchwork } P} u^{\text{LD}(P)} \prod_{d \geq 0} \delta_d^{\text{deg}_d(P)} \frac{w^{m(P)}}{2^{m(P)} m(P)!} \frac{z^{n(P)}}{n(P)!}$$

**Building a multigraph from $\text{MG}(z, w, u + 1)$:**

- start with a patchwork (the marked loops and double edges),
- add isolated vertices,
- add half edges on each vertex,
- match them to form edges and relabel the edges.
Removing loops and double edges

**Inclusion-exclusion:** express $\text{MG}(z, w, u + 1)$: each loop and double edge is either marked or left unmarked. Then set $u = -1$.

![Diagram](image)

**Patchwork:** set of marked loops and double edges

$$P(z, w, (\delta_d)_{d \geq 0}, u) e^{z\delta_0}$$

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$$P(z, w, (\partial^d \Delta(x))_{d \geq 0}, u) e^{z \Delta(x)}$$

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$$MG(z, w, u + 1) = \sum_{m \geq 0} (2m)! [x^{2m}] P(z, w, (\partial^d \Delta(x))_{d \geq 0}, u) e^{z \Delta(x)} \frac{w^m}{2^m m!}$$

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![Diagram of a multigraph with loops and double edges marked and unmarked.]

Patchwork: set of marked loops and double edges

$$SG(z, w) = \sum_{m \geq 0} (2m)! [x^{2m}] P(z, w, (\partial^d \Delta(x))_{d \geq 0}, -1) e^{z \Delta(x)} \frac{w^m}{2^m m!}$$

Building a multigraph from $MG(z, w, u + 1)$:
- start with a patchwork (the marked loops and double edges),
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Asymptotic analysis

Gauss transform: \[
\sqrt{\frac{2}{\pi}} \int_0^{+\infty} t^{2m} e^{-t^2/2} \, dt = \frac{(2m)!}{2^m m!}
\]

\[
\text{SG}(z, w) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} P\left(z, w, (\partial^d \Delta(\sqrt{w} t))_{d \geq 0}, -1\right) e^{z \Delta(\sqrt{w} t)} e^{-t^2/2} \, dt
\]

Changes of variables: \[n! [z^n w^m] \text{SG}(z, w) \text{ becomes}
\]

\[
u_n[z^n x^{2m}] \int_0^{+\infty} P \left( \frac{nz}{\Delta(x)}, \frac{x^2}{2mt^2}, (\partial^d \Delta(x))_{d \geq 0}, -1\right) \Delta(x)^n e^{nz} t^{2m} e^{-mt^2} \, dt
\]

Saddle-point: for \(m = \Theta(n)\) at \(z = t = 1\) and \(x = \zeta\) computable.
Each \((p, q)\)-patchwork counts for \((\frac{nz}{\Delta(x)})^p (\frac{x^2}{2mt^2})^q = O\left(\frac{1}{n^{q-p}}\right)\).
Non-negligible only for \(p = q\): disjoint loops and double edges.
Asymptotic analysis

Gauss transform: \[ \sqrt{\frac{2}{\pi}} \int_{0}^{+\infty} t^{2m} e^{-t^2/2} dt = \frac{(2m)!}{2^m m!} \]

\[ \text{SG}(z, w) = \sqrt{\frac{2}{\pi}} \int_{0}^{+\infty} P\left(z, w, \left(\partial^d \Delta(\sqrt{w}t)\right)_{d \geq 0}, -1\right) e^{z\Delta(\sqrt{w}t)} e^{-t^2/2} dt \]

Changes of variables: \( n![z^n w^m] \text{SG}(z, w) \) becomes

\[ u_n[z^n x^{2m}] \int_{0}^{+\infty} e^{-\frac{zx^2}{t^2\Delta(x)} \frac{n}{4m} - \left(\frac{zx^2}{t^2\Delta(x)} \frac{n}{4m}\right)^2} \Delta(x)^n e^{nz} t^{2m} e^{-mt^2} dt \]

Saddle-point: for \( m = \Theta(n) \) at \( z = t = 1 \) and \( x = \zeta \) computable.

Each \((p, q)\)-patchwork counts for \( \left(\frac{nz}{\Delta(x)}\right)^p \left(\frac{x^2}{2mt^2}\right)^q = O\left(\frac{1}{n^{q-p}}\right) \).

Non-negligible only for \( p = q \): disjoint loops and double edges.
Connected multigraphs

2-Core: multigraph with minimum degree at least 2

\[ MG^{\geq 2}(z, w) = \sum_{k \geq 0} (2k)! [x^{2k}] \left( 1 - zw \frac{e^x - 1 - x}{x^2/2} \right)^{-k - \frac{1}{2}} \frac{w^k}{2^k k!} \]

A connected multigraph is either

- a tree \( U(z) \),
- a unicyclic component \( V(z) \),
- a connected 2-Core where vertices are replaced by rooted trees

\[ CMG(z, w) = \frac{U(zw)}{w} + V(z) + \log \left( MG^{\geq 2} \left( \frac{T(zw)}{w}, w \right) e^{-V(z)} \right) \]

Dominant coefficient for \( m = \Theta(n) \):

\[ n! [z^n w^m] CMG(z, w) \approx n! [z^n w^m] MG^{\geq 2} \left( \frac{T(zw)}{w}, w \right) e^{-V(z)} \]
Connected multigraphs

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- a tree \( U(z) \),
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\[ \text{CMG}(z, w) = \frac{U(zw)}{w} + V(z) + \log \left( \text{MG}^{\geq 2} \left( \frac{T(zw)}{w}, w \right) e^{-V(z)} \right) \]

Dominant coefficient for \( m = \Theta(n) \):

\[ n! [z^n w^m] \text{CMG}(z, w) \approx \frac{n!(2(m - n))!}{2^{m-n}(m - n)!} [x^{2(m-n)}] \left(1 - T(z) \frac{e^x - 1 - x}{x^2/2}\right)^{-m+n-\frac{1}{2}} \]
Contributions and future extensions

Develop the analytic combinatorics of graphs.

**Multigraphs:** new interpretation of the model used by Flajolet Knuth Pittel 1989, Janson Knuth Łuczak Pittel 1993.

**Degree constraints:** we could also consider a different set of degrees for each vertex.

**Remove loops and double edges:** working on other subgraph families.

**Structure of random graphs:** beyond the birth of the giant component.

**Other models:** hypergraphs, inhomogeneous graphs.