Analytic combinatorics of connected graphs

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2016

Introduction

$n = \#$ vertices, $m = \#$ edges

Goal: counting (n, m)-connected graphs when $m = \Theta(n)$
(Bender Canfield McKay 1990, Pittel Wormald 2005, Hofstad Spencer 2006).

Tool: analytic combinatorics (Flajolet Sedgewick 2009).

Future extensions:

- investigate the "giant component" of random graphs with $m/n > 1/2$
  (Ederős Rényi 1959, Janson Knuth Łuczak Pittel 1993)
- extend the result to other graph-like models
  (degree constraints, inhomogeneous, hypergraphs).

Related works

$m = n - 1$	trees	Borchardt 1860 Cayley formula
$m = n$	unicycles	Rényi 1959
$m - n = O(1)$	fixed excess	Wright 1977, 1978, 1980 Flajolet Schaeffer Salvy 2004
$m - n \to \infty$	large excess	Bender Canfield McKay 1990 Pittel Wormald 2005 Hofstad Spencer 2006
$m = o(n)$		Łuczak 1990
$\frac{2m}{n} - \log(n) \to \infty$	almost all graphs	Erdős Rényi 1959

Structure of the proof

Multigraphs with degree constraints (all vertices have their degree in D)

remove loops and double edges

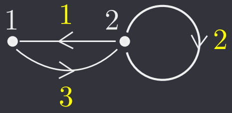
Graphs with degree constraints (E.d.P Ramos 2006)

counting graphs without trees
(same principle as in Pittel Wormald 2005)

Connected graphs

Multigraphs with degree constraints

Multigraphs: labelled vertices, labelled oriented edges



Degree constraints: all vertices have their degree in a given set D

$$\delta_d = \begin{cases} 1 & \text{if } d \in D \\ 0 & \text{otherwise} \end{cases} \qquad \Delta(x) = \sum_{d \geq 0} \delta_d \frac{x^d}{d!}.$$

Generating function:

$$\mathrm{MG}(z, w) = \sum_{\text{multigraph } G} \prod_{d \geq 0} \delta_d^{\deg_d(G)} \frac{w^{m(G)}}{2^{m(G)} m(G)!} \frac{z^{n(G)}}{n(G)!}$$

Half-edges representation:

$$\mathrm{MG}(z, w) = \sum_{m \geq 0} (2m)! [x^{2m}] e^{z\Delta(x)} \frac{w^m}{2^m m!}.$$

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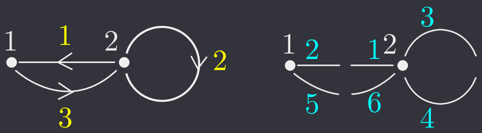

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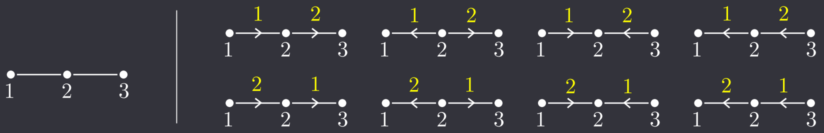
From multigraphs to simple graphs

Simple graphs: unlabelled unoriented edges, no loops nor multiple edges

$$\mathrm{SG}(z,w) = \sum_{\text{graph } G} \prod_{d \geq 0} \delta_d^{\deg_d(G)} w^{m(G)} \frac{z^{n(G)}}{n(G)!}$$

Any (n,m)-graph matches $2^m m!$ multigraphs



Corollary: set $\mathrm{MG}(z,w,u) :=$ gf of multigraphs with loops and double edges marked by u

$$\mathrm{MG}(z,w,0) = \sum_{\substack{\text{multigraph } G\\ \text{no loop, no double edge}}} \prod_{d \geq 0} \delta_d^{\deg_d(G)} \frac{w^{m(G)}}{2^{m(G)} m(G)!} \frac{z^{n(G)}}{n(G)!}$$

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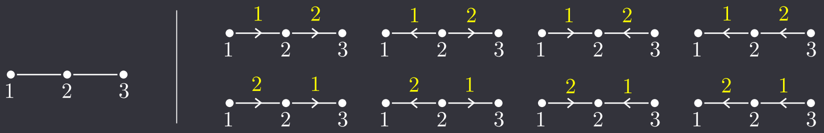

Corollary: set $\mathrm{MG}(z,w,u) :=$ gf of multigraphs with loops and double edges marked by u

$$\mathrm{MG}(z,w,0) = \sum_{\text{graph } G} 2^{m(G)} m(G)! \prod_{d \geq 0} \delta_d^{\deg_d(G)} \frac{w^{m(G)}}{2^{m(G)} m(G)!} \frac{z^{n(G)}}{n(G)!}$$

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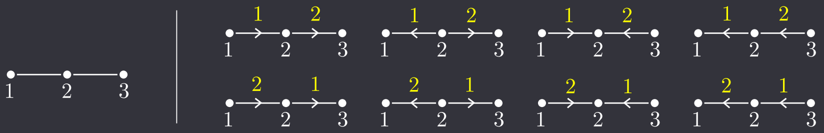

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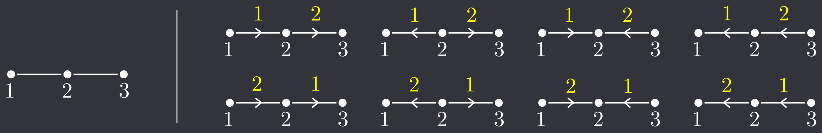
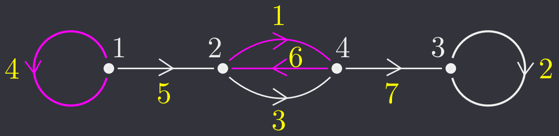


Corollary: set $\mathrm{MG}(z,w,u) :=$ gf of multigraphs with loops and double edges marked by u

$$\mathrm{MG}(z,w,0) = \mathrm{SG}(z,w)$$

Removing loops and double edges

Inclusion-exclusion: express $\mathrm{MG}(z, w, u+1)$: each loop and double edge is either marked or left unmarked. Then set $u = -1$.



Patchwork: set of marked loops and double edges

$$P(z, w, (\delta_d)_{d\geq 0}, u) = \sum_{\text{patchwork } P} u^{\mathrm{LD}(P)} \prod_{d\geq 0} \delta_d^{\deg_d(P)} \frac{w^{m(P)}}{2^{m(P)} m(P)!} \frac{z^{n(P)}}{n(P)!}$$

Building a multigraph from $\mathrm{MG}(z, w, u+1)$:

- start with a patchwork (the marked loops and double edges),
- add isolated vertices,
- add half edges on each vertex,
- match them to form edges and relabel the edges.

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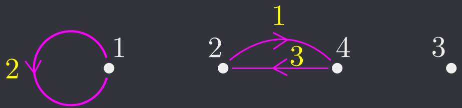

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$$P(z, w, \quad (\delta_d)_{d \geq 0} \quad, \; u \;) e^{z\delta_0}$$

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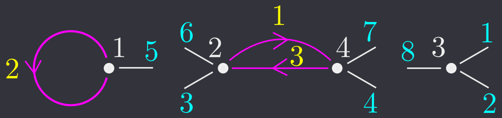

Patchwork: set of marked loops and double edges

$$P(z, w, (\partial^d \Delta(x))_{d \geq 0},\ u\ )e^{z\Delta(x)}$$

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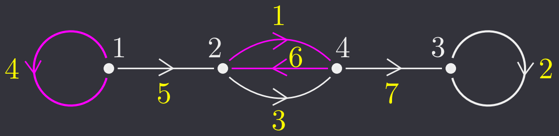

Patchwork: set of marked loops and double edges

$$\mathrm{MG}(z, w, u+1) = \sum_{m \geq 0} (2m)! [x^{2m}] P(z, w, (\partial^d \Delta(x))_{d \geq 0},\ u\ ) e^{z\Delta(x)} \frac{w^m}{2^m m!}$$

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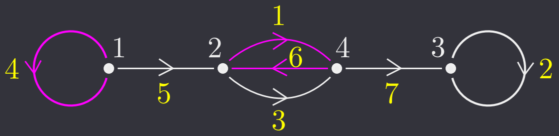

Patchwork: set of marked loops and double edges

$$\mathrm{SG}(z, w) = \sum_{m \geq 0} (2m)! [x^{2m}] P(z, w, (\partial^d \Delta(x))_{d \geq 0}, -1) e^{z\Delta(x)} \frac{w^m}{2^m m!}$$

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Asymptotic analysis

Gauss transform: $\sqrt{\frac{2}{\pi}} \int_0^{+\infty} t^{2m} e^{-t^2/2} dt = \frac{(2m)!}{2^m m!}$

$$\mathrm{SG}(z,w) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} P\Big(z, w, (\partial^d \Delta(\sqrt{w}t))_{d \geq 0}, -1\Big) e^{z\Delta(\sqrt{w}t)} e^{-t^2/2} dt$$

Changes of variables: $n![z^n w^m]\,\mathrm{SG}(z,w)$ becomes

$$u_n[z^n x^{2m}] \int_0^{+\infty} P\left(\frac{nz}{\Delta(x)}, \frac{x^2}{2mt^2}, (\partial^d \Delta(x))_{d \geq 0}, -1\right) \Delta(x)^n e^{nz} t^{2m} e^{-mt^2} dt$$

Saddle-point: for $m = \Theta(n)$ at $z = t = 1$ and $x = \zeta$ computable.

Each (p,q)-patchwork counts for $\left(\frac{nz}{\Delta(x)}\right)^p \left(\frac{x^2}{2mt^2}\right)^q = O\left(\frac{1}{n^{q-p}}\right)$.

Non-negligible only for $p = q$: disjoint loops and double edges.

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Changes of variables: $n![z^n w^m]\,\mathrm{SG}(z,w)$ becomes

$$u_n[z^n x^{2m}] \int_0^{+\infty} \qquad e^{-\frac{zx^2}{t^2\Delta(x)}\frac{n}{4m} - \left(\frac{zx^2}{t^2\Delta(x)}\frac{n}{4m}\right)^2} \qquad \Delta(x)^n e^{nz} t^{2m} e^{-mt^2} dt$$

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Connected multigraphs

2-Core: multigraph with mininum degree at least 2

$$\mathrm{MG}^{\geq 2}(z,w) = \sum_{k \geq 0} (2k)! [x^{2k}] \left(1 - zw \frac{e^x - 1 - x}{x^2/2} \right)^{-k-\frac{1}{2}} \frac{w^k}{2^k k!}$$

A connected multigraph is either

- a tree $U(z)$,
- a unicyclic component $V(z)$,
- a connected 2-Core where vertices are replaced by rooted trees

$$\mathrm{CMG}(z,w) = \frac{U(zw)}{w} + V(z) + \log \left(\mathrm{MG}^{\geq 2} \left(\frac{T(zw)}{w}, w \right) e^{-V(z)} \right)$$

Dominant coefficient for $m = \Theta(n)$:

$$n! [z^n w^m] \, \mathrm{CMG}(z,w) \approx n! [z^n w^m] \, \mathrm{MG}^{\geq 2} \left(\frac{T(zw)}{w}, w \right) e^{-V(z)}$$

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Dominant coefficient for $m = \Theta(n)$:

$$n![z^n w^m]\,\mathrm{CMG}(z,w) \approx \frac{n!(2(m-n))!}{2^{m-n}(m-n)!} [x^{2(m-n)}] \left(1 - T(z) \frac{e^x - 1 - x}{x^2/2}\right)^{-m+n-\frac{1}{2}}$$

Contributions and future extensions

Develop the analytic combinatorics of graphs.

Multigraphs: new interpretation of the model used by Flajolet Knuth Pittel 1989, Janson Knuth Łuczak Pittel 1993.

Degree constraints: we could also consider a different set of degrees for each vertex.

Remove loops and double edges: working on other subgraph families.

Structure of random graphs: beyond the birth of the giant component.

Other models: hypergraphs, inhomogeneous graphs.