Chiral symmetry in polytopes

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Chirality
The term “chiral” comes from the greek χειρ (kheir), which means hand.
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In 1893 Lord Kelvin used the term “chiral” in a scientific context for the first time:

“I call any geometrical figure, or group of points, 'chiral', and say that it has chirality if its image in a plane mirror, ideally realized, cannot be brought to coincide with itself”
Chirality in nature
Chirality in nature
Chirality in nature
Chirality in chemistry

(S)-(-)-limonene  (R)-(+) -limonene
Chirality in chemistry

Thalidomide

One is a sedative, the other one weakens the bones (and can produce birth defects)
Chirality in mathematics

Trefoil knot
Chirality in mathematics

Trefoil knot
Chirality in mathematics

Trefoil knot
Chirality in mathematics

Trefoil knot

Snob cube
Chirality in mathematics
Chirality in mathematics
Abstract polytopes
Abstract polytopes generalize the (face lattice) of convex (and some other “classic” geometric) polytopes to combinatorial structures.
Polyhedra

Platonic solids
Polyhedra

Kepler (~1620)
Polyhedra

Kepler (~1620)  Poinsot (~1810)
Polyhedra

In the 1920’s...
Polyhedra

In the 1920’s...

Petrie-Coxeter polyhedra
Polyhedra

In the 1920’s...

Brahana: maps in surfaces
(he was in algebra!)

Projective plane

Torus
Polyhedra

Coxeter
Polyhedra

Coxeter
Polyhedra

Coxeter
Polyhedra

Coxeter
Polyhedra

Coxeter
Polyhedra

Coxeter
Polyhedra
Coxeter
Polyhedra

Coxeter
Polyhedra
Higher dimensions

Convex polytopes
Ludwig Schläfli (1852)

Convex hull of a finite number of points
Higher dimensions

Polyhedra and polytopes:

Coxeter
(1907-2003)
Higher dimensions

Polyhedra and polytopes:

Geometry
Combinatorics
Group Theory
Topology

Coxeter
(1907-2003)
maps on surfaces

convex polytopes

1970's

Grünbaum
Proposes to study “polytopes” whose facets and vertex-figures are not spherical
maps on surfaces

convex polytopes

1970's

Grünbaum

Proposes to study “polytopes” whose facets and vertex-figures are not spherical

Tits

Develops the ideas of incidence geometries
Incidence polytopes, now called abstract polytopes
An abstract n-polytope is a partially ordered set endowed with a rank function to \{-1,0,...,n\} (dimension, in the convex case)
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Technical: Min and max element
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An abstract n-polytope is a partially ordered set endowed with a rank function to \([-1,0,...,n]\) (dimension, in the convex case)

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Flags: \(n+2\) elements

Strongly connected
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Strongly connected

Diamond
An abstract $n$-polytope is a partially ordered set endowed with a rank function to $\{-1,0,\ldots,n\}$ (dimension, in the convex case)

Technical: Min and max element

Flags: $n+2$ elements

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An abstract $n$-polytope is a partially ordered set endowed with a rank function to $\{-1,0,\ldots,n\}$ (dimension, in the convex case)

Technical: Min and max element

Flags: $n+2$ elements

Strongly connected

Diamond
Abstract polytope $P$

Vertices

Edges

$l$-skeleton

Vertices
Abstract polytope P
Abstract polytope $P$ is NOT determined by its 1-skeleton.
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Given an abstract polytope $P$

Two flags are adjacent if they differ in exactly one face.
Given an abstract polytope $P$

Two flags are \textbf{adjacent} if they differ in exactly on one face.
Given an abstract polytope $P$

Two flags are \textit{adjacent} if they differ in exactly one face.

\begin{align*}
\Phi & \Rightarrow \Phi^i \\
\text{Flag} & \Rightarrow \text{Its (unique!)} \text{ i-adjacent}
\end{align*}
Symmetries
An automorphism of a polytope $P$ is an order preserving bijection of $P$. 
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An **automorphism** of a polytope $P$ is a bijection of the set of flags of $P$ that preserves the incidences.
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An automorphism of a polytope $P$ is a bijection of the set of flags of $P$ that preserves the incidences.

We often study automorphisms through their action on the flags of the polytope.
A polytope is **regular** if its automorphism group acts transitively on the flags.
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If a polytope is regular, fixing a base flag $\Phi$, there exist automorphisms $\rho_i$, for each $i$, such that $\Phi \rho_i = \Phi^i$.
If a polytope is such that, fixing a base flag $\Phi$, there exist automorphisms $\rho_i$, for each $i$, satisfying

$$\Phi \rho_i = \Phi^i$$

then the polytope is regular.
Chirality in polytopes
A polytope is **chiral** if its automorphism group has two orbits on flags with adjacent flags in different orbits.
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Regular case:
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Snob cube
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All rank 2 polytopes are regular (easy to see)
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There are no finite chiral polytopes in Euclidian 3-space (Schulte)
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There are no finite chiral polytopes in Euclidian 3-space (Schulte)

There are no convex chiral polytopes (McMullen)
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During the 1990’s, Monson, Nostrand, Schulte, Weiss constructed infinite families
Higher ranks
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In the 1990’s Schulte and Weiss gave a construction in which, given a finite chiral n-polytope, constructed a (locally) infinite chiral (n+1)-polytope
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Rank 5
Higher ranks

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Rank 5

Ranks 6-8
2009 Conder, Devillers
Higher ranks
Higher ranks

2010 Pellicer.
Gave a recursive construction and showed that they exist for every rank...
Higher ranks

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Gave a recursive construction and showed that they exist for every rank...
Their groups are uncontrollable
Higher ranks

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Gave a recursive construction and showed that they exist for every rank...
Their groups are uncontrollable

2014 Cunningham, Pellicer.
Constructed chiral \((n+1)\)-polytopes provided they have chiral \(n\)-polytopes with regular facets
Why is it so difficult???
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Why is it so difficult???

The faces of rank $n-2$ are always regular
A polytope is **chiral** if its automorphism group has two orbits on flags with adjacent flags in different orbits.

The automorphism of a chiral n-polytope $P$ can be generated by $\sigma_1, \ldots, \sigma_{n-1}$ such that

$$(\sigma_i \ldots \sigma_j)^2 = \varepsilon \quad \text{for} \quad i < j$$
A polytope is chiral if its automorphism group has two orbits on flags with adjacent flags in different orbits.

The automorphism of a chiral n-polytope P can be generated by $\sigma_1, \ldots, \sigma_{n-1}$ such that

$$(\sigma_i \ldots \sigma_j)^2 = \varepsilon \text{ for } i < j$$

and the generators satisfy certain "intersection conditions"
Given a group $\Gamma$ with distinguished generators $\sigma_1, \ldots, \sigma_{n-1}$ such that:

$$(\sigma_i \ldots \sigma_j)^2 = e \text{ for } i < j$$

and satisfying certain “intersection conditions”, one can construct an $n$-polytope with $\Gamma$ acting on it.
1991. Schulte and Weiss
Given a group $\Gamma$ with distinguished generators $\sigma_1, \ldots, \sigma_{n-1}$ such that:

$$\left(\sigma_i \ldots \sigma_j\right)^2 = \varepsilon \quad \text{for} \quad i < j$$

and satisfying certain “intersection conditions”, one can construct an $n$-polytope with $\Gamma$ acting on it.

The resulting polytope is either chiral or regular
Some open questions

• Given a (finite) group $\Gamma$, is there a chiral polytope having $\Gamma$ as its automorphism group?

• Given a (finite/simple) group $\Gamma$, can one determine all chiral polytopes having $\Gamma$ as automorphism group?

• Given a (finite) regular $n$-polytope $P$, is there a (finite) chiral polytope whose facets are all isomorphic to $P$? Can one classify them all?
Some open questions

- For each dimension $n$, is there a finite “geometrically chiral” $n$-polytope in $\mathbb{R}^n$? Can one classify them all?

- Can one classify all chiral $(n-1)$-polytopes in $\mathbb{R}^n$?

- Given a graph $G$, is there a chiral polytope having $G$ as its 1-skeleton? Can one classify them all?
Some open questions

• The smallest chiral polytopes are known for ranks 3, 4 and 5. What are is the smallest chiral polytope of rank 6? Of rank n?

• How prevalent is chirality (vs. regularity) among n-chiral polytopes? (or among polytopes with certain properties, for example, with a given automorphism group or with a given 1-skeleton)
• For each dimension $n$, is there a finite “geometrically chiral” $n$-polytope in $\mathbb{R}^n$?

In a work with Javier Bracho and Daniel Pellicer, we found the first example of a chiral 4-polytope in $\mathbb{R}^4$. (The one on the video!)

The polytope is combinatorially regular, but geometrically chiral.

It’s 1-skeleton is the hypercube.

The facets are double covers of a cube.

The automorphism group is the rotational group of the hyper-cube.
Given a (finite) group $\Gamma$, is there a chiral polytope having $\Gamma$ as its automorphism group?
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For $\Gamma = A_n$ or $\Gamma = S_n$

(work with Marston Conder, Eugenia O’Reilly and Daniel Pellicer)

Recently we showed that:
For all but finitely many $n$, both $S_n$ and $A_n$ are the automorphism group of a chiral 4-polytope
For $\Gamma = A_n$ or $\Gamma = S_n$

(work with Marston Conder, Eugenia O’Reilly and Daniel Pellicer)

Recently we showed that:
For all but finitely many $n$, both $S_n$ and $A_n$ are the automorphism group of a chiral 4-polytope

We are working on showing that that:
Given $d>4$, for infinitely many $n$, both $S_n$ and $A_n$ are the automorphism group of a chiral $d$-polytope
How prevalent is chirality (vs. regularity) among n-chiral polytopes with Suzuki simple groups $Sz(q)$?

In a work with Dimitri Leemans we showed that:

• there are no chiral n-polytopes for $n>4$, with automorphism group $Sz(q)$.

• if $a(q)$ is the number of regular 3-polytopes with $Sz(q)$, and $b(q)$ the number of chiral ones, then

$$b(q) = O(q \cdot a(q))$$
THANK YOU!