Chiral symmetry in polytopes

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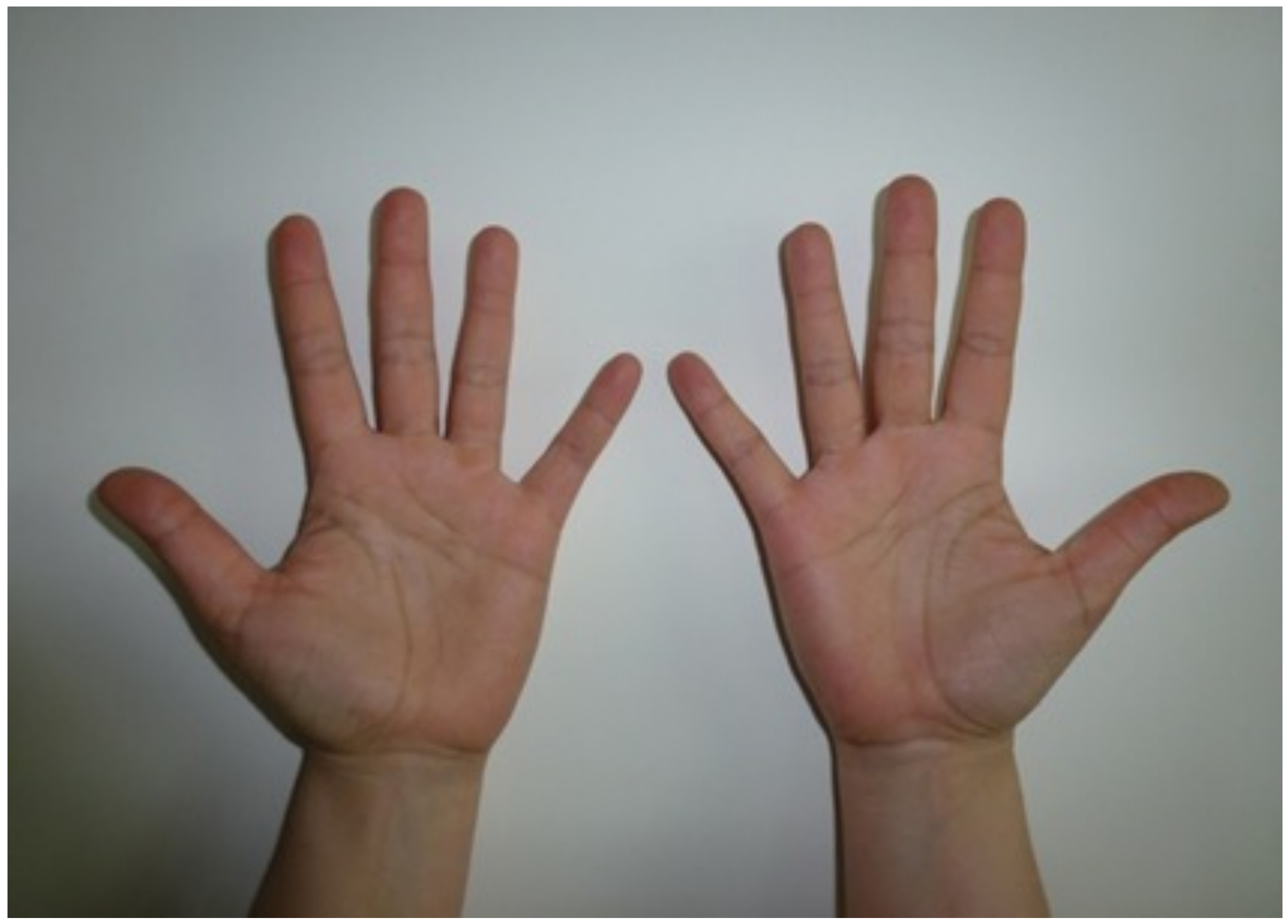
September 2015

Chirality

The term “chiral” comes from the greek χειρ (kheir), which means hand.



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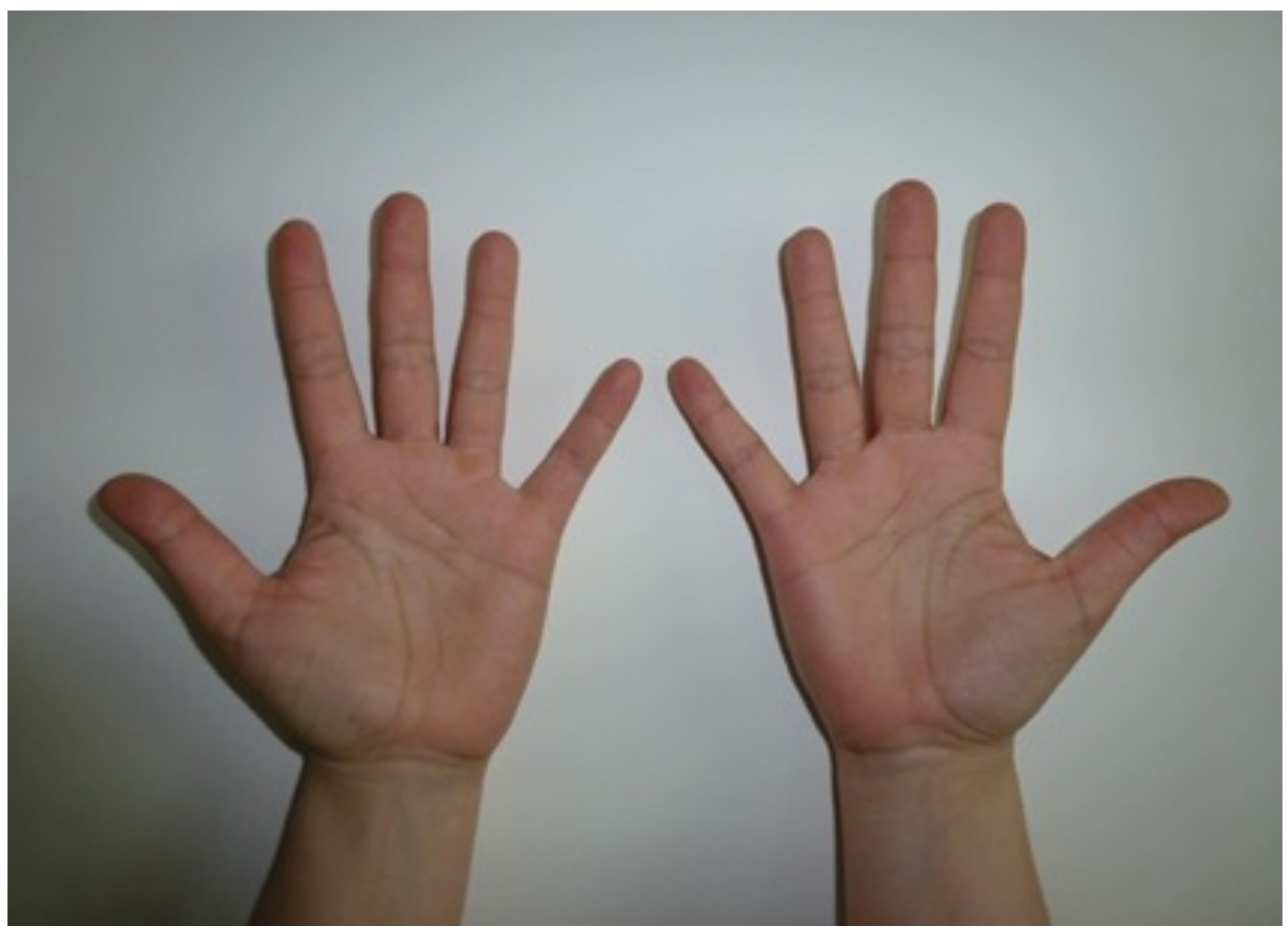
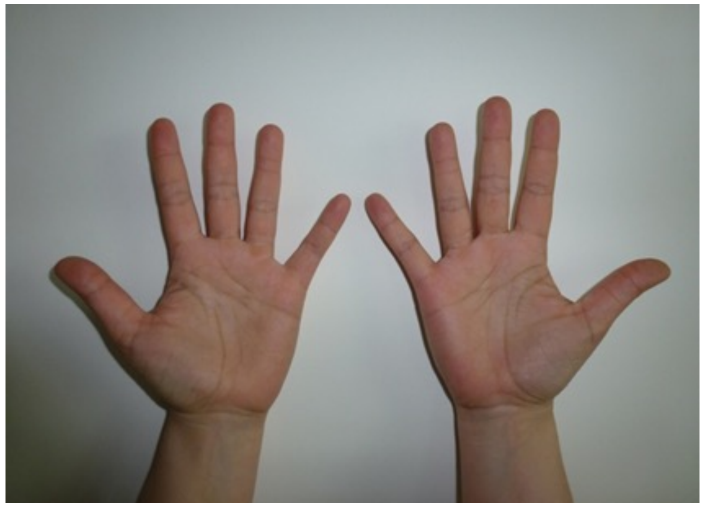
In 1893 Lord Kelvin use the term "chiral" in a scientific context for the first time:



The term “chiral” comes from the greek χειρ (kheir), which means hand.

In 1893 Lord Kelvin use the term “chiral” in a scientific context for the first time:

“I call any geometrical figure, or group of points, 'chiral', and say that it has chirality if its image in a plane mirror, ideally realized, cannot be brought to coincide with itself”



Chirality in nature

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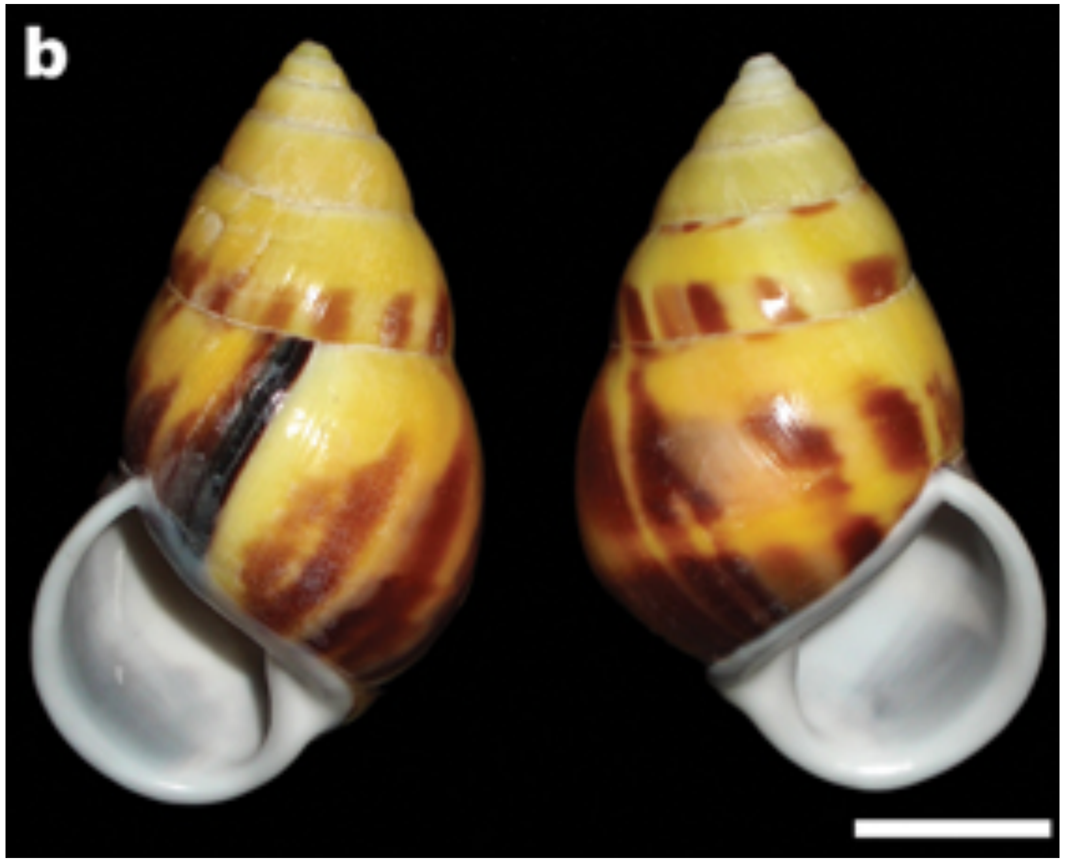
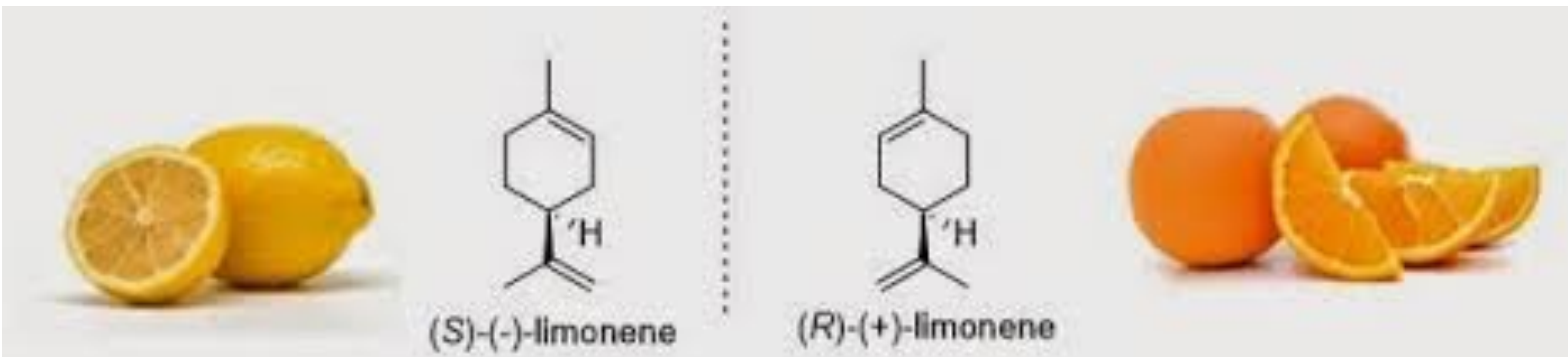
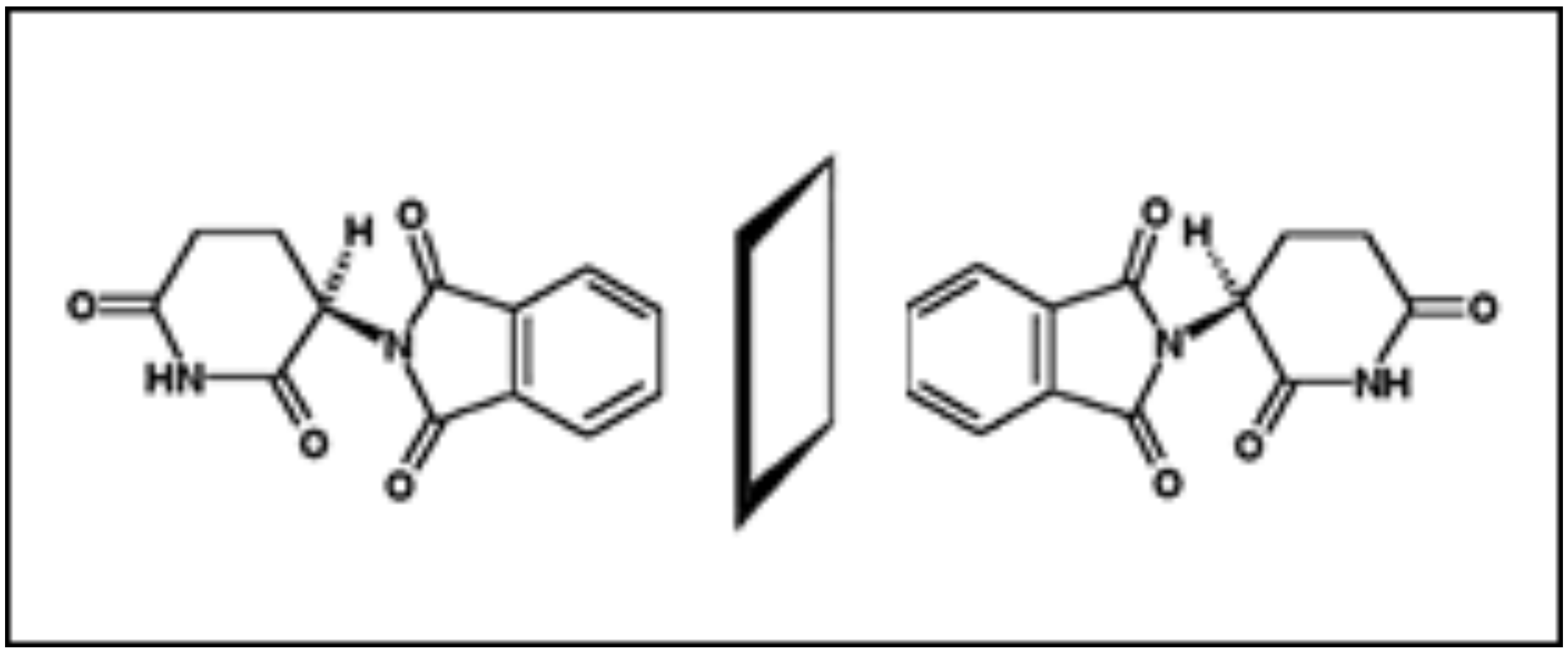


Chirality in chemistry



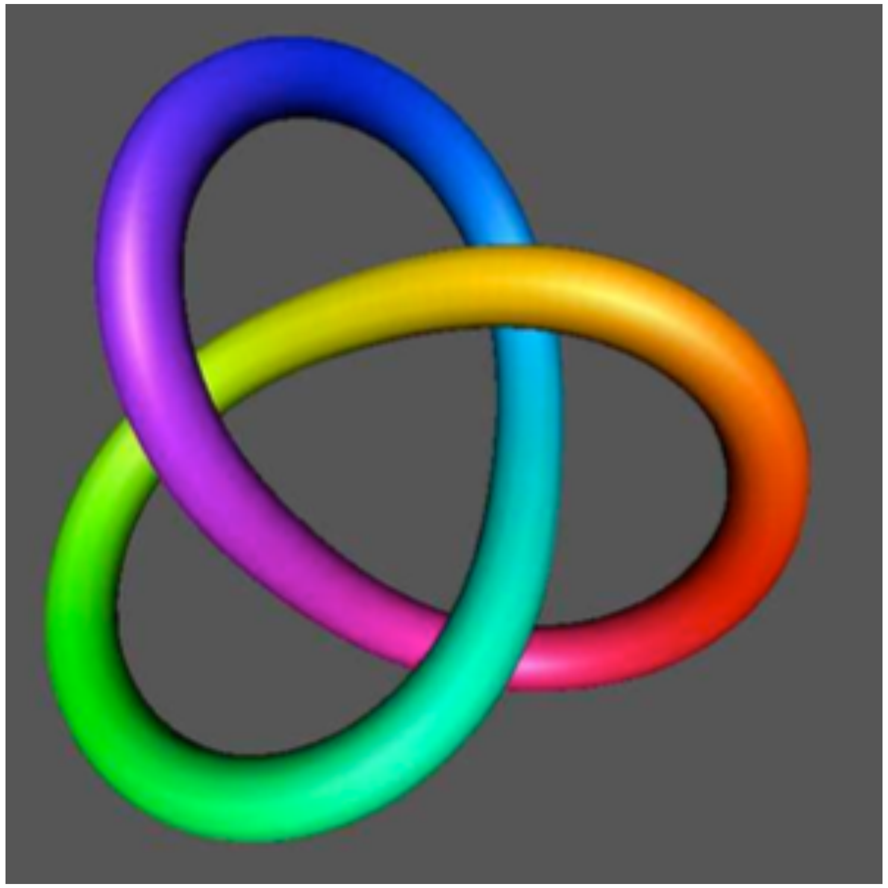
Chirality in chemistry



Thalidomide

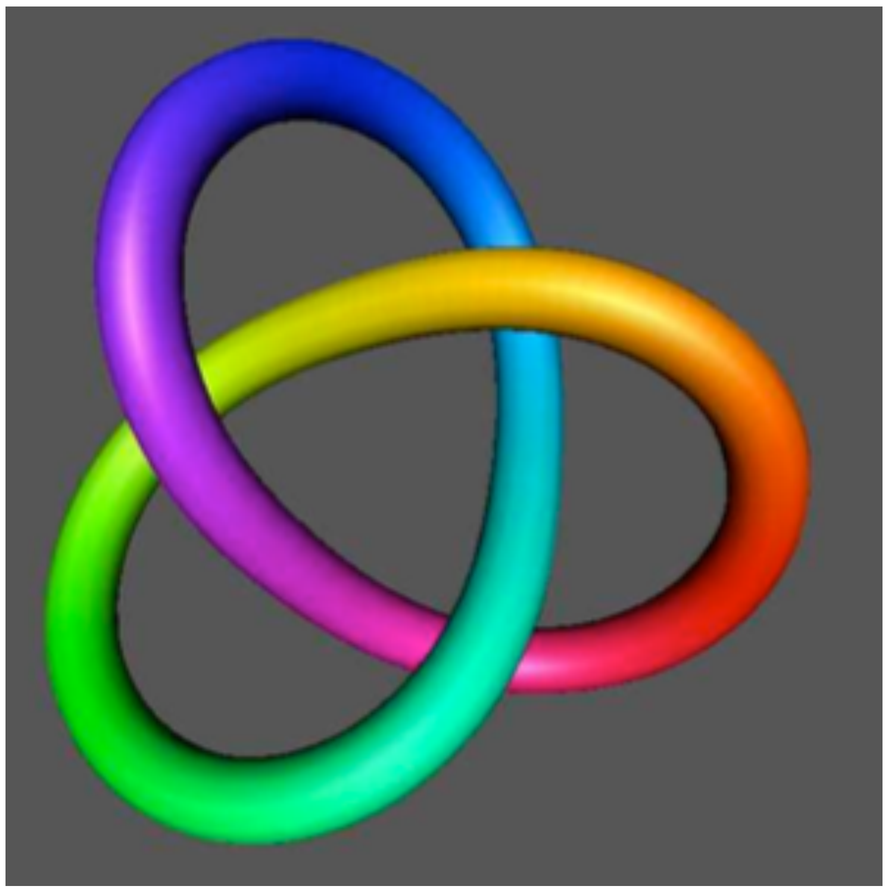
One is a sedative, the other one weakens the bones (and can produce birth defects)

Chirality in mathematics



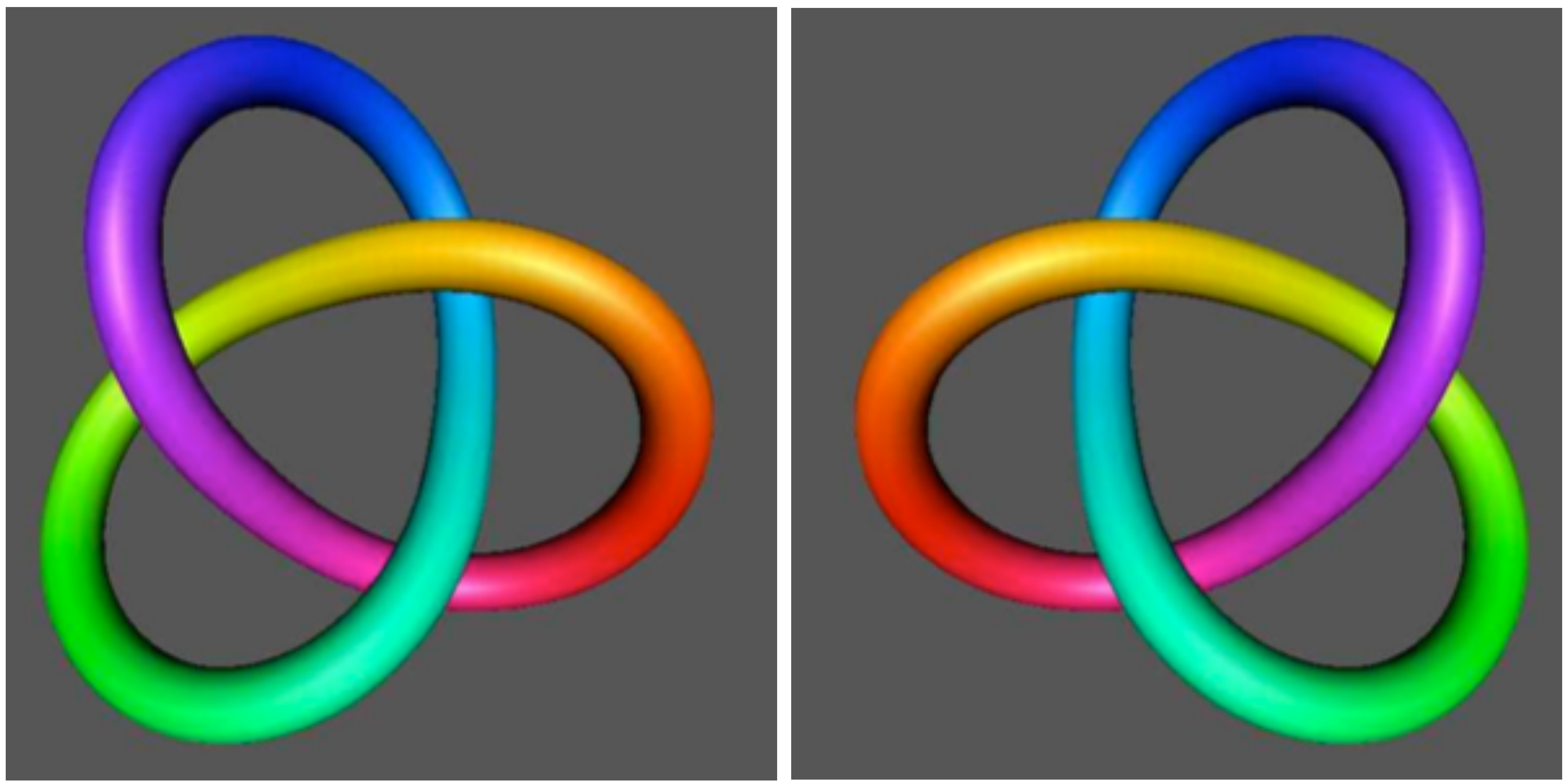
Trefoil knot

Chirality in mathematics



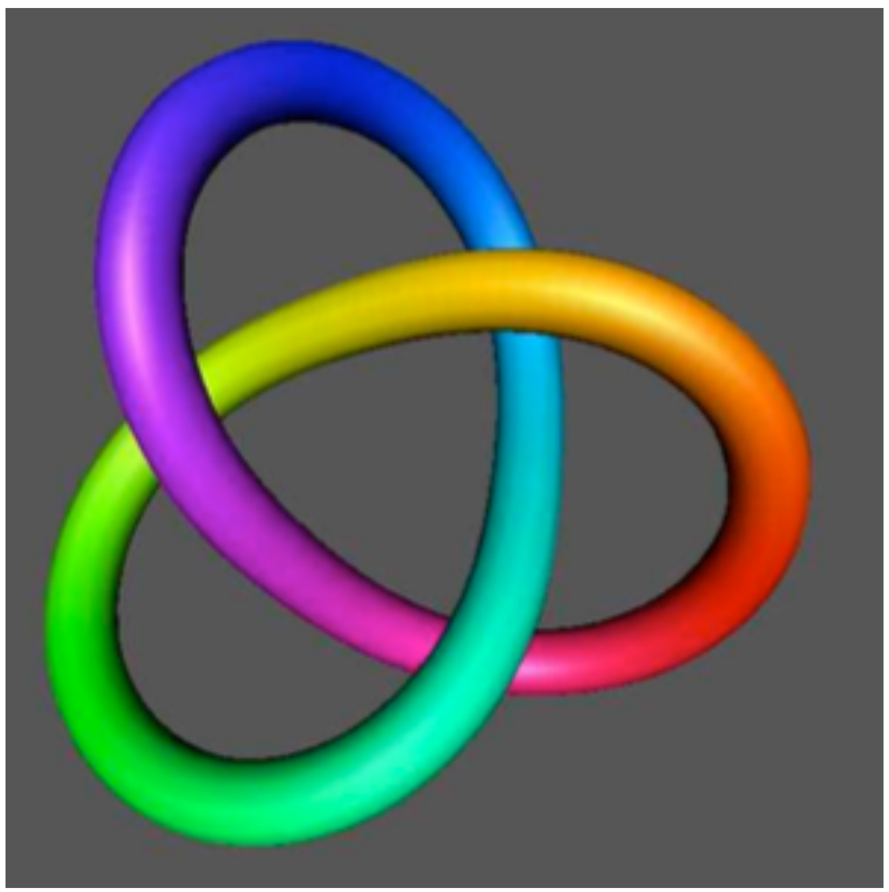
Trefoil knot

Chirality in mathematics

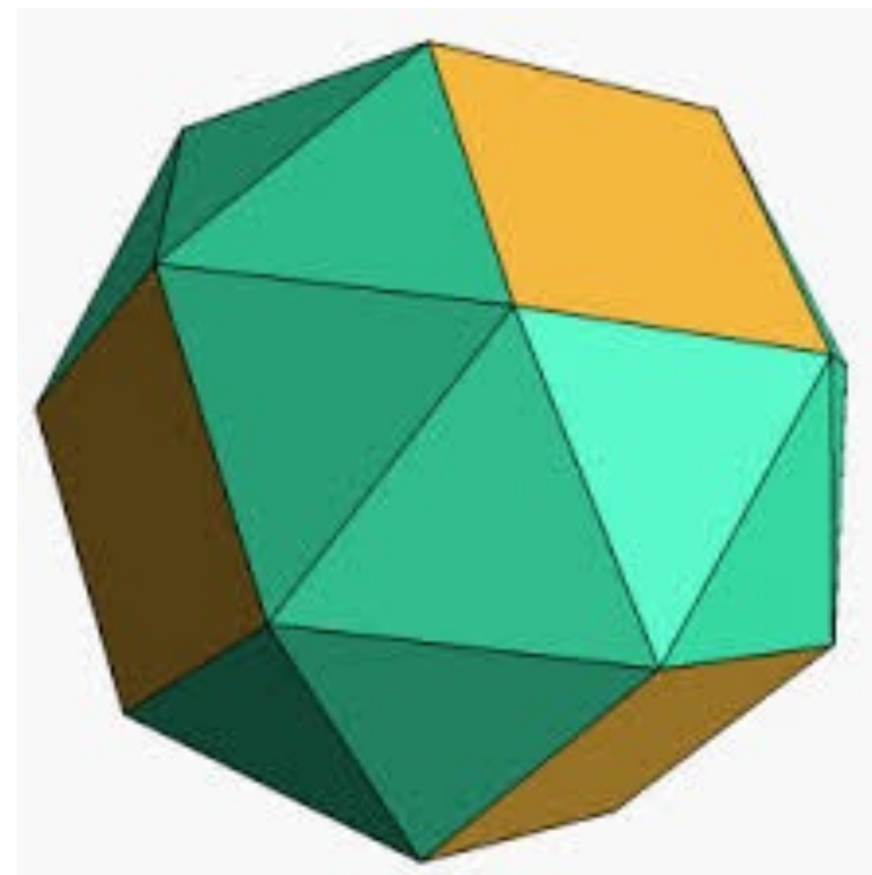


Trefoil knot

Chirality in mathematics



Trefoil knot



Snob cube

Chirality in mathematics

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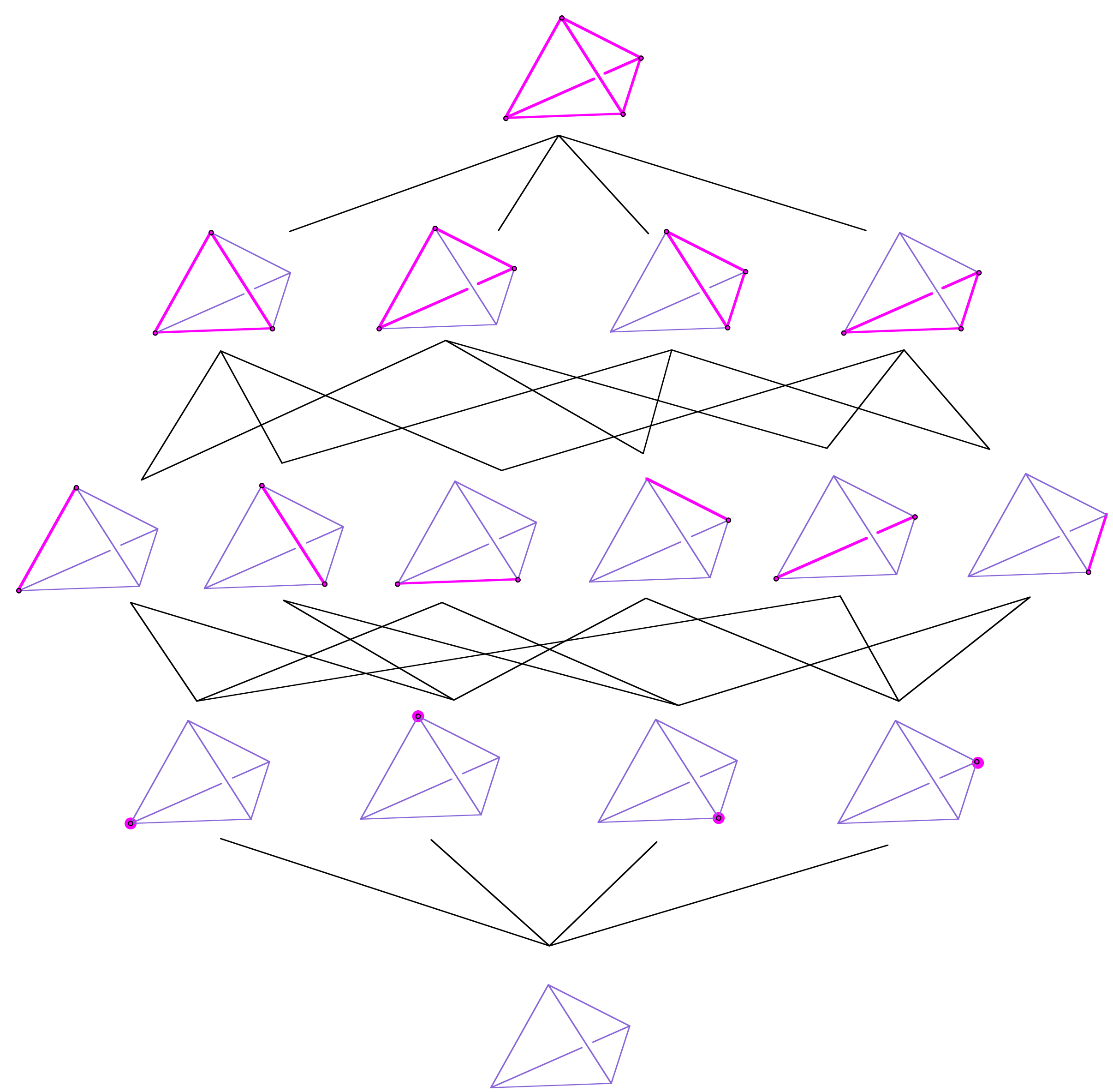
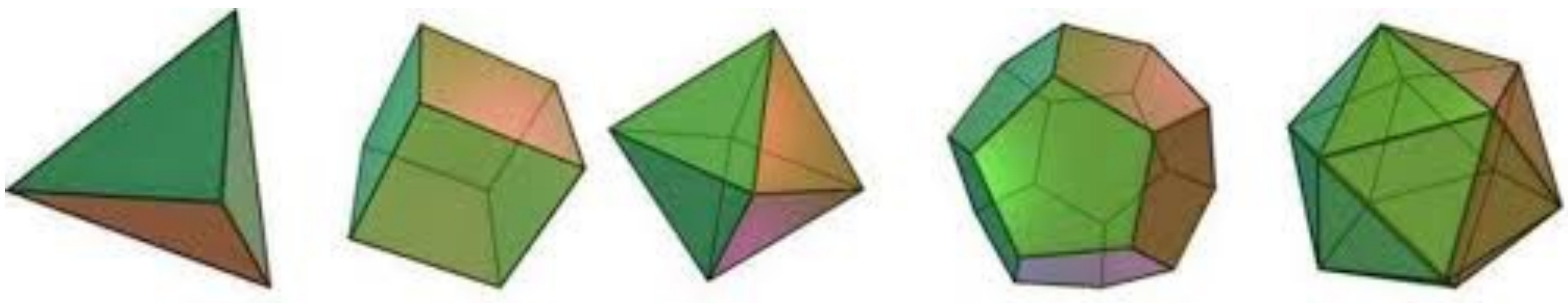
Chirality in mathematics

Abstract polytopes

Abstract polytopes generalize the (face lattice) of convex (and some other “classic” geometric) polytopes to combinatorial structures.



Polyhedra

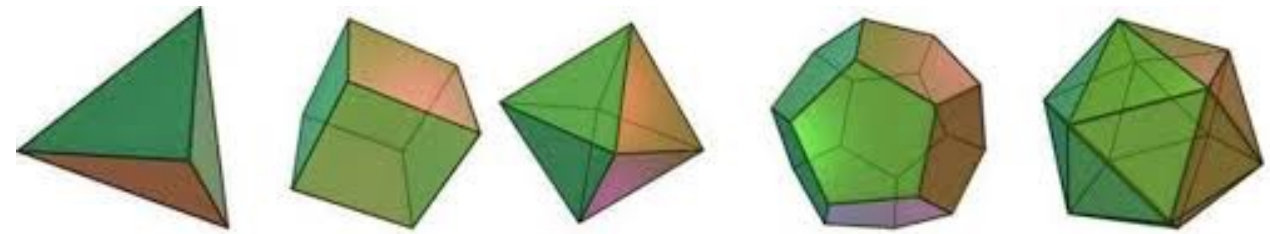


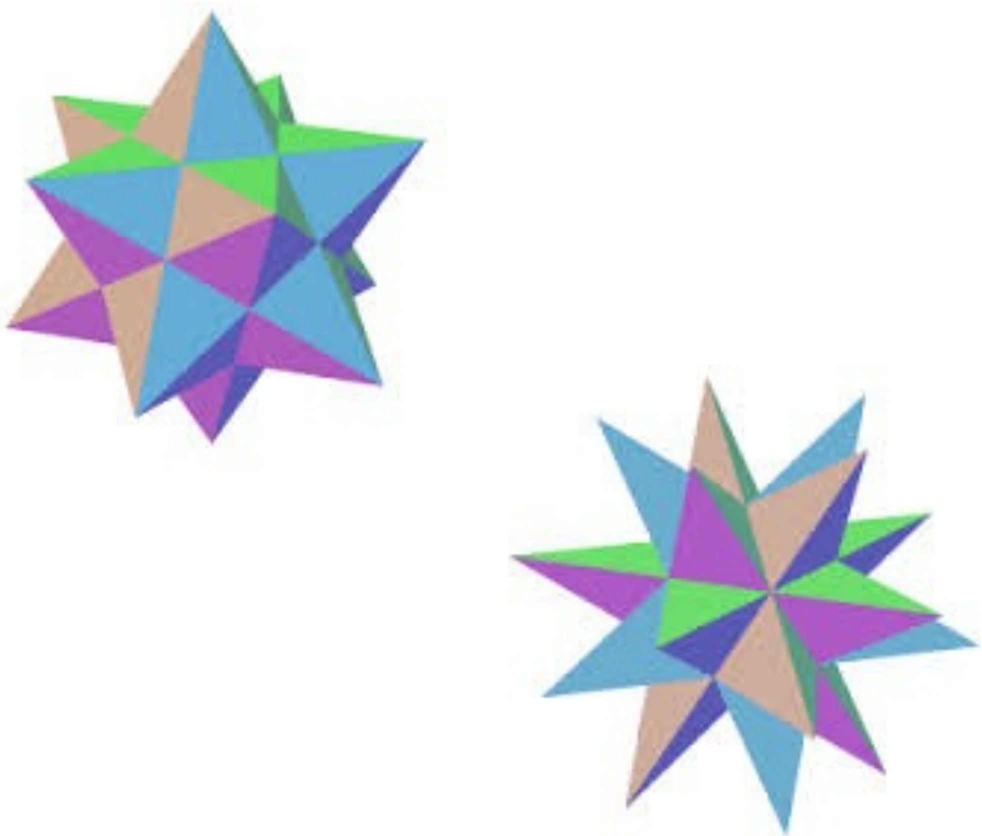
Platonic solids

Polyhedra

Kepler (~1620)

Polyhedra





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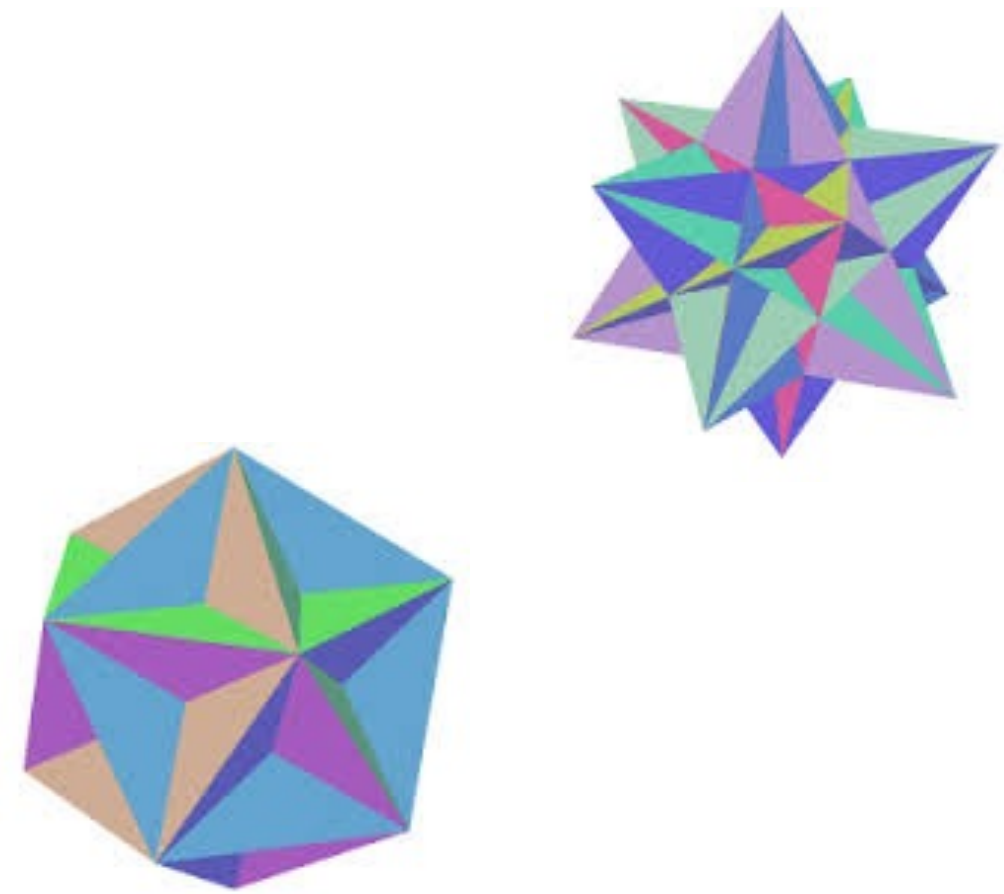
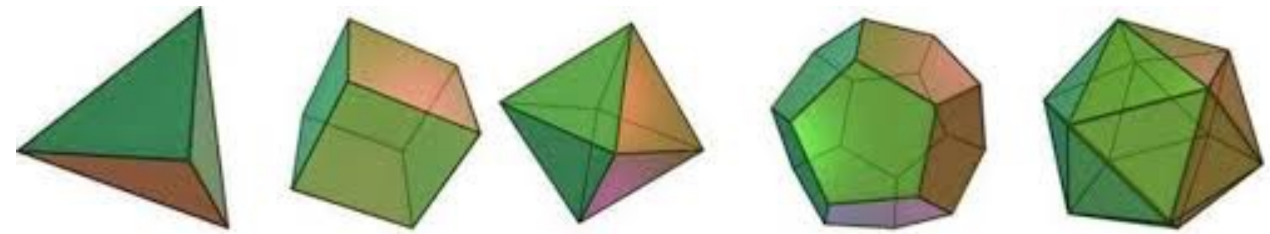


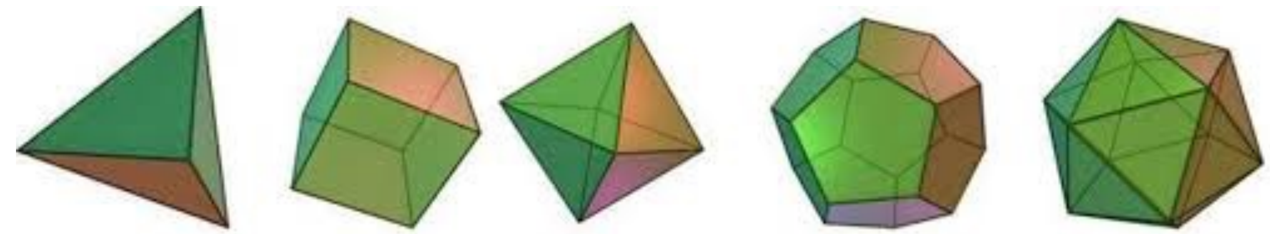
Poinsot (~1810)

Polyhedra



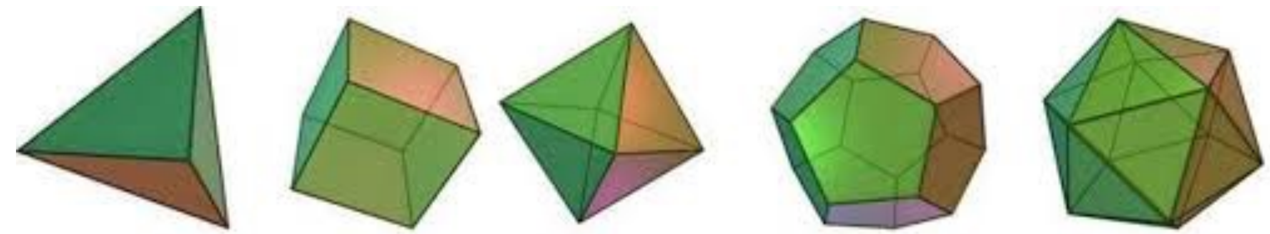
Kepler-Poinsot
polyhedra

Polyhedra



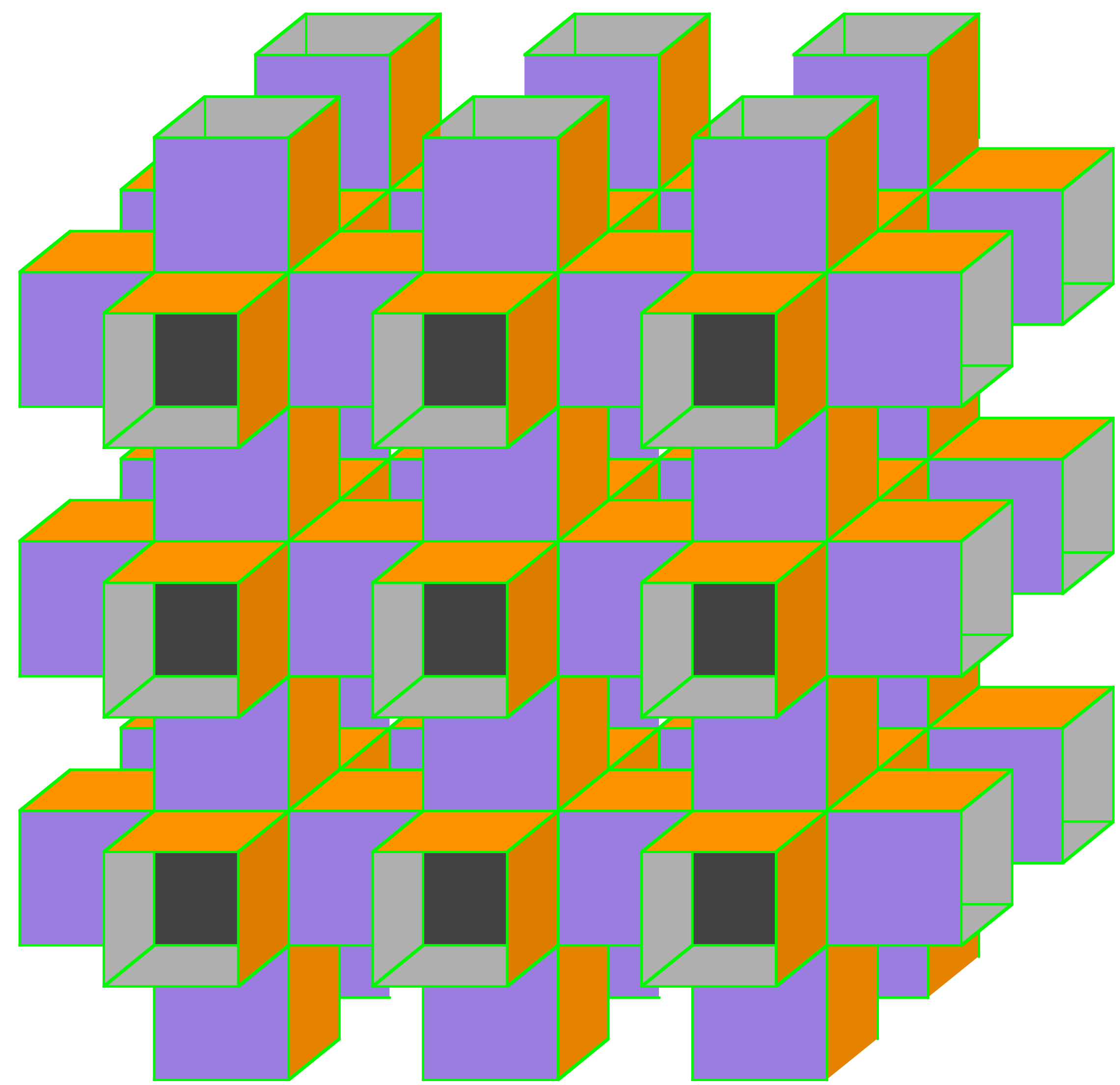
In the 1920's…

Polyhedra



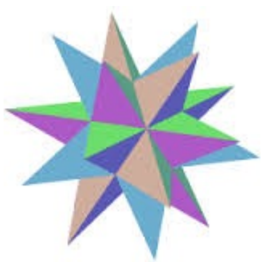
In the 1920's...

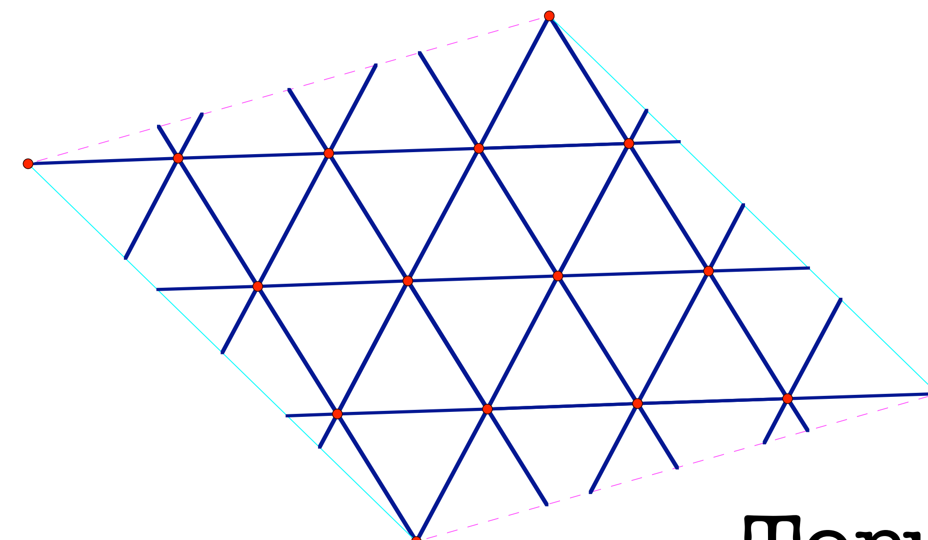
Petrie-Coxeter polyhedra



Polyhedra

In the 1920's...

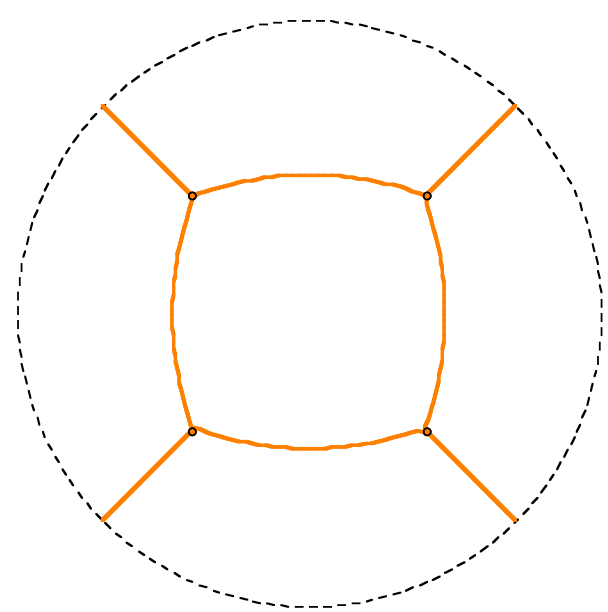


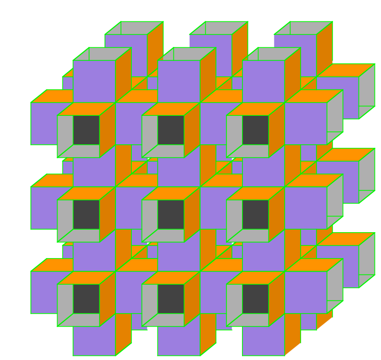


Torus

Brahana:
maps in surfaces
(he was in algebra!)

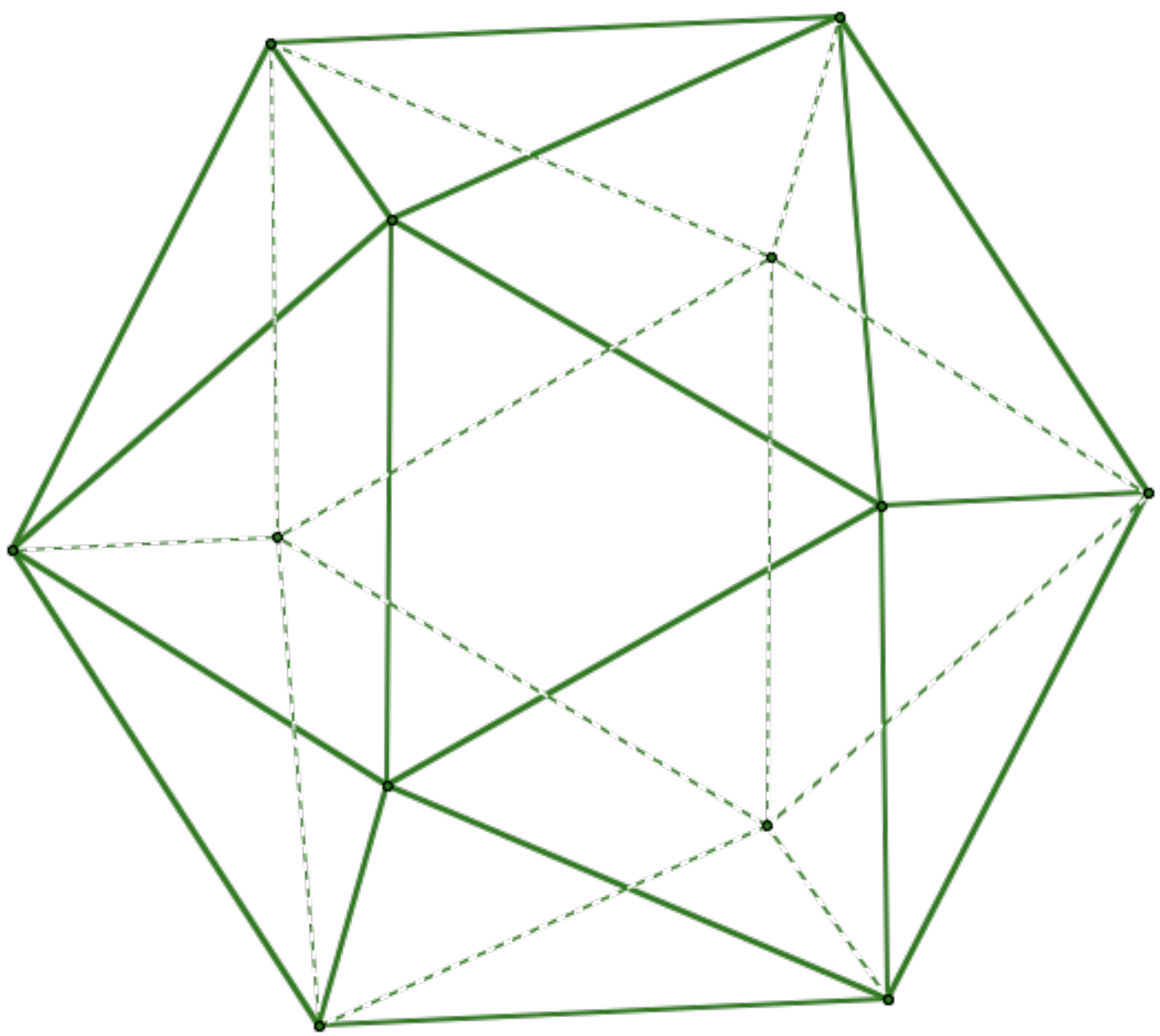
Projective plane

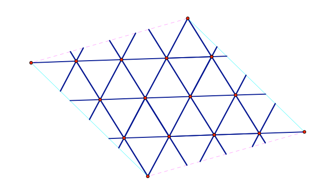


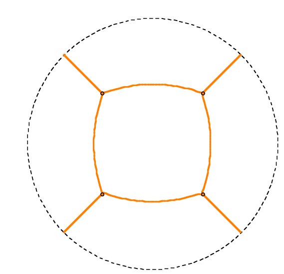


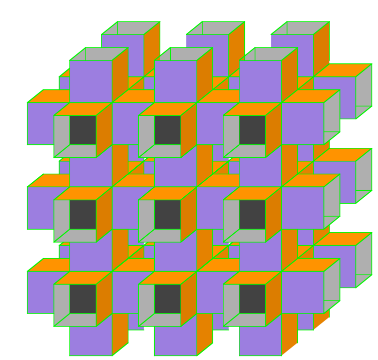
Polyhedra

Coxeter









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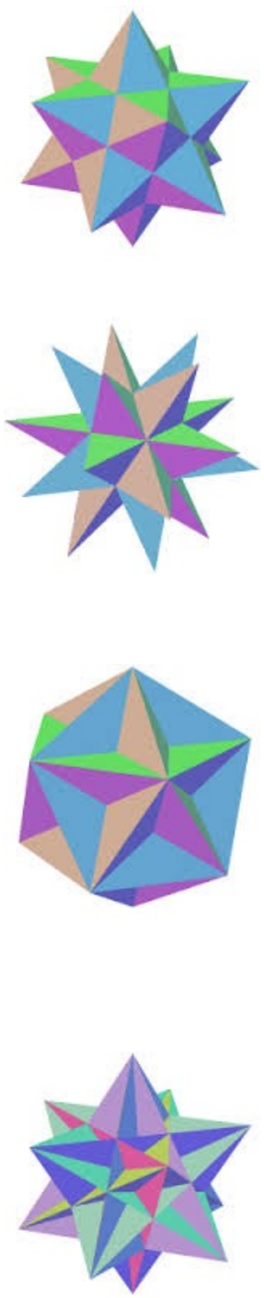

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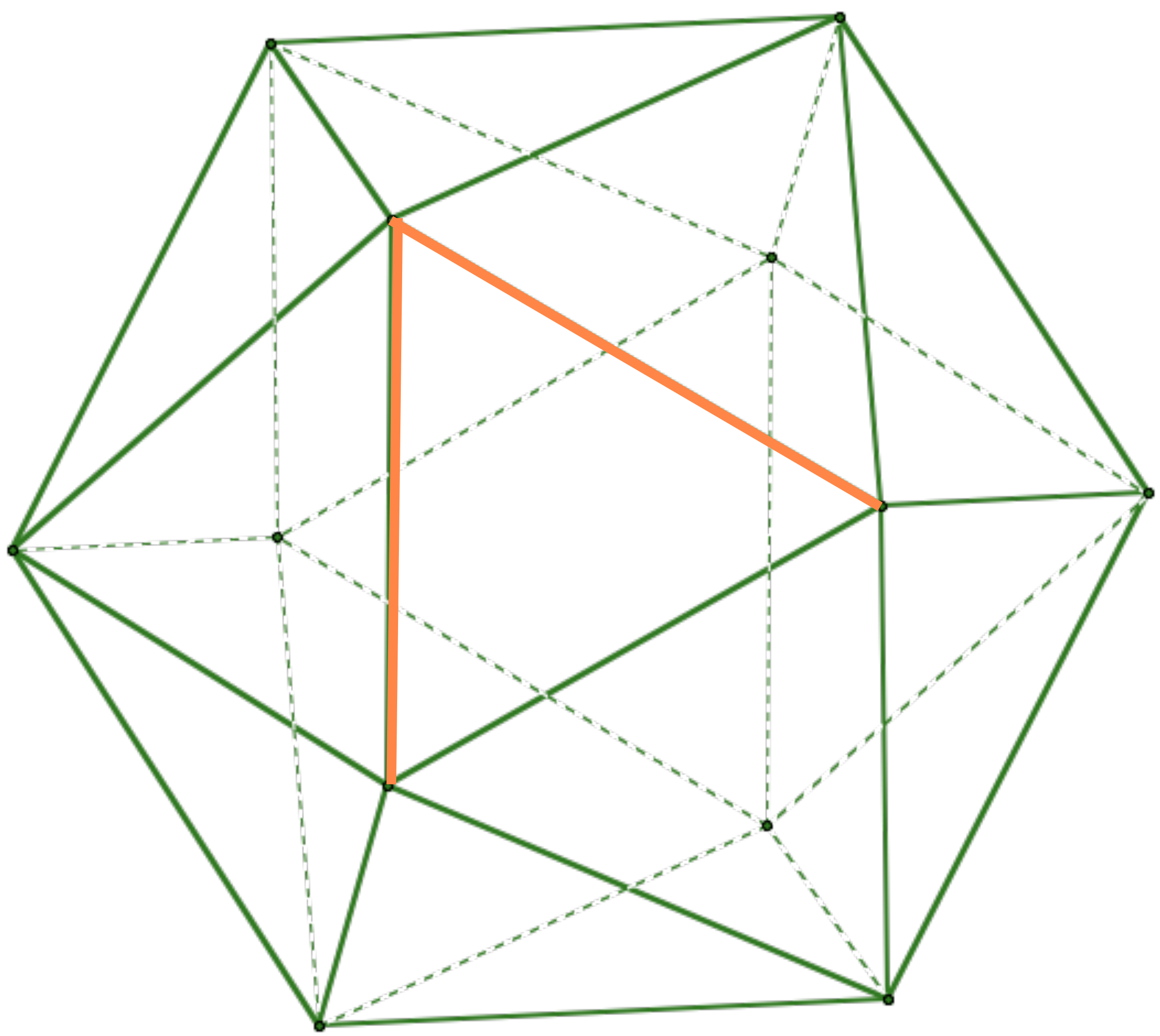

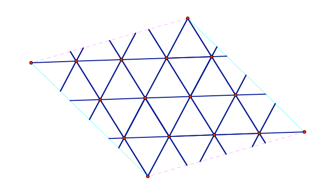

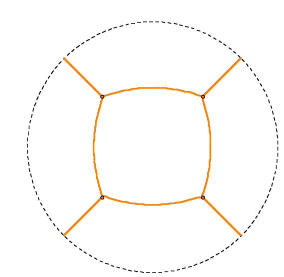

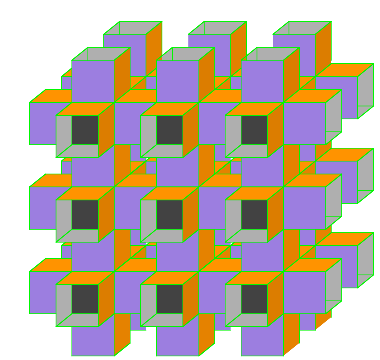

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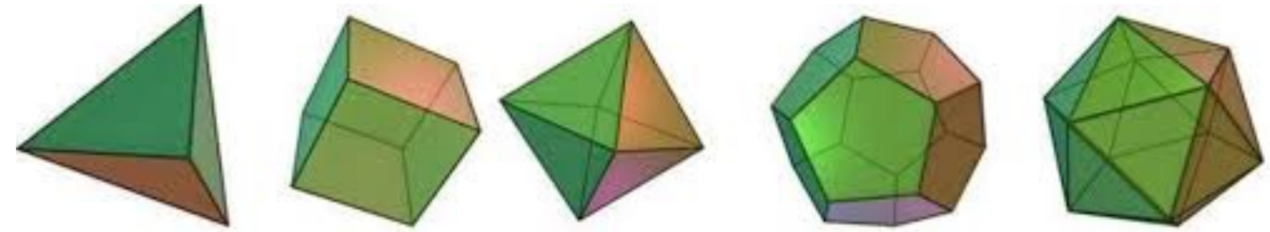

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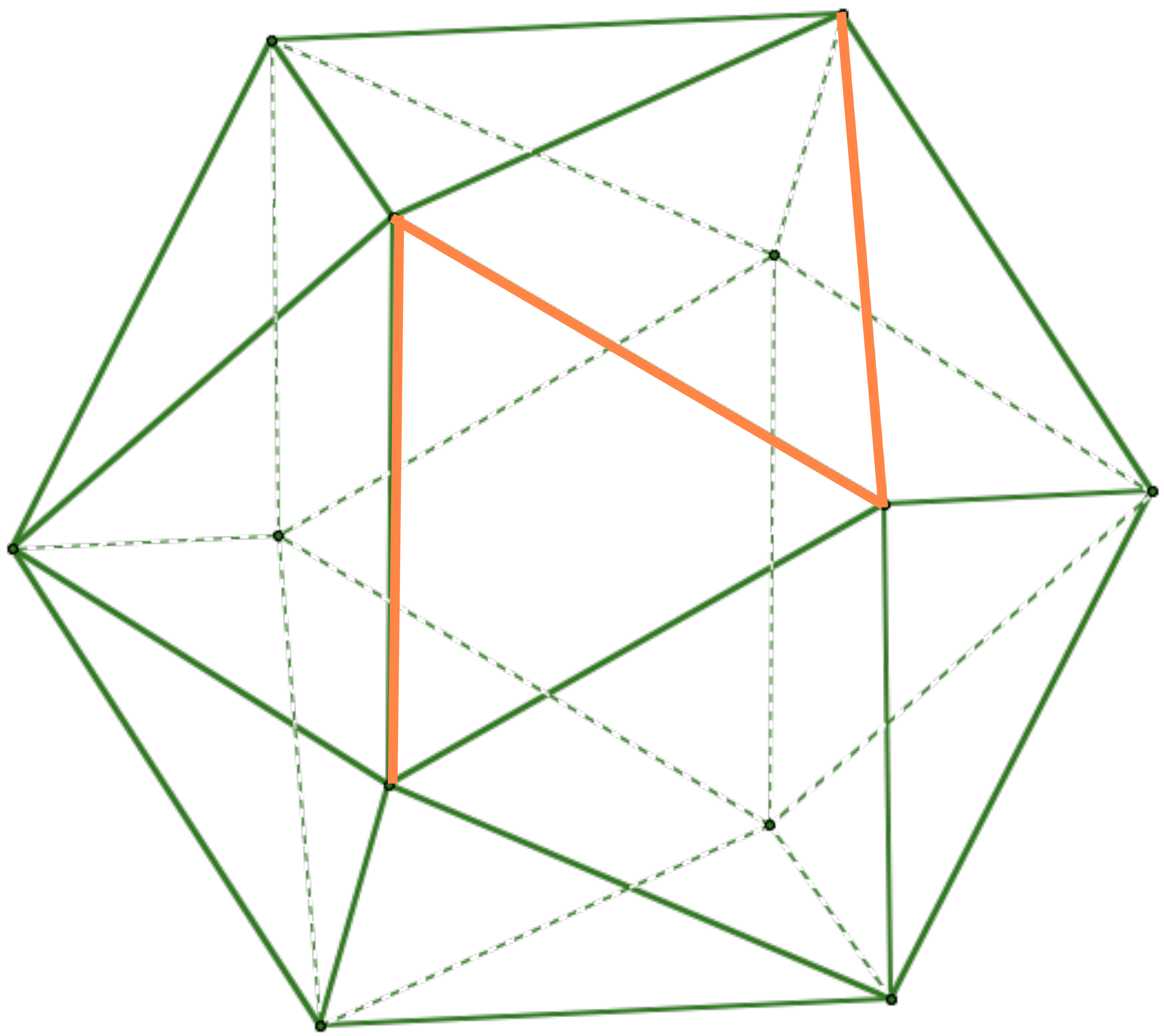

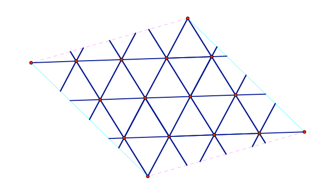

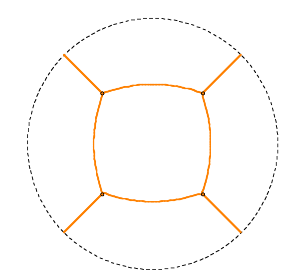

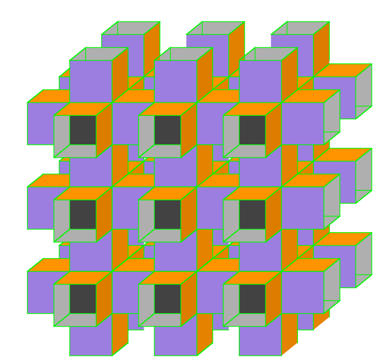

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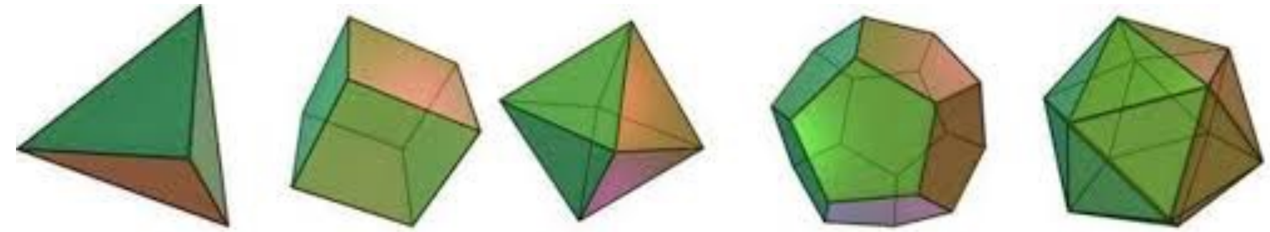

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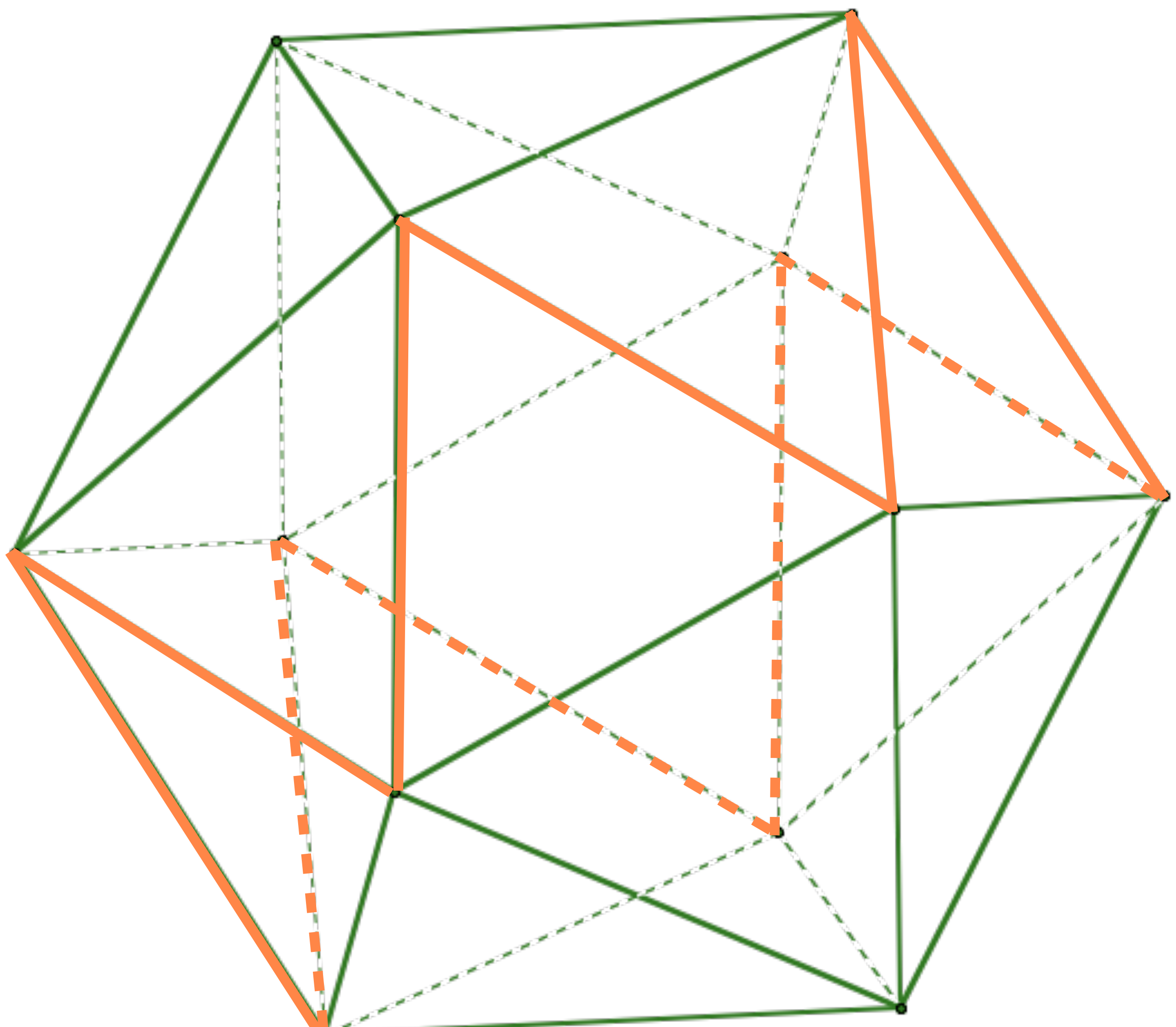

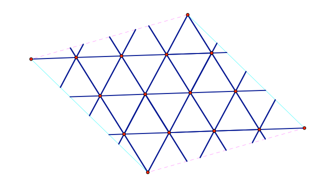

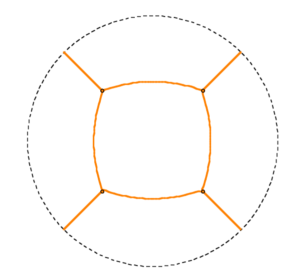

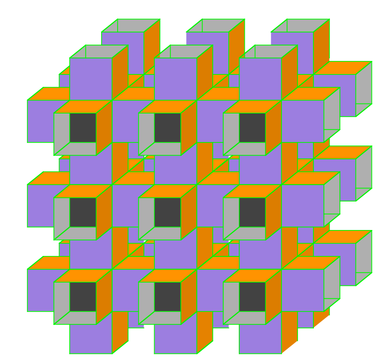

Polyhedra

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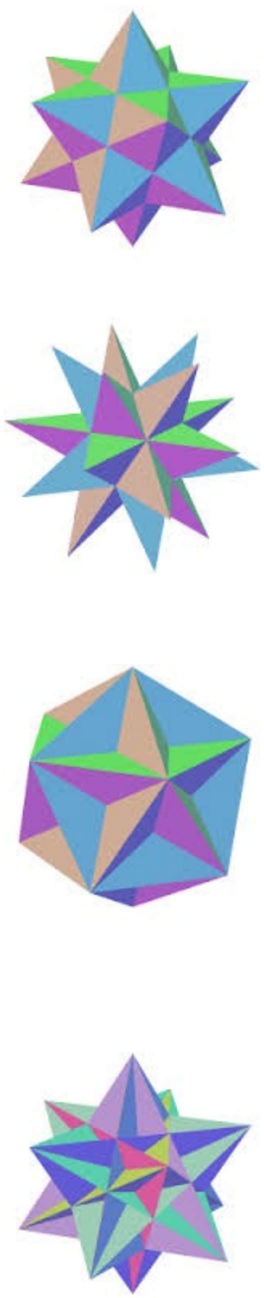

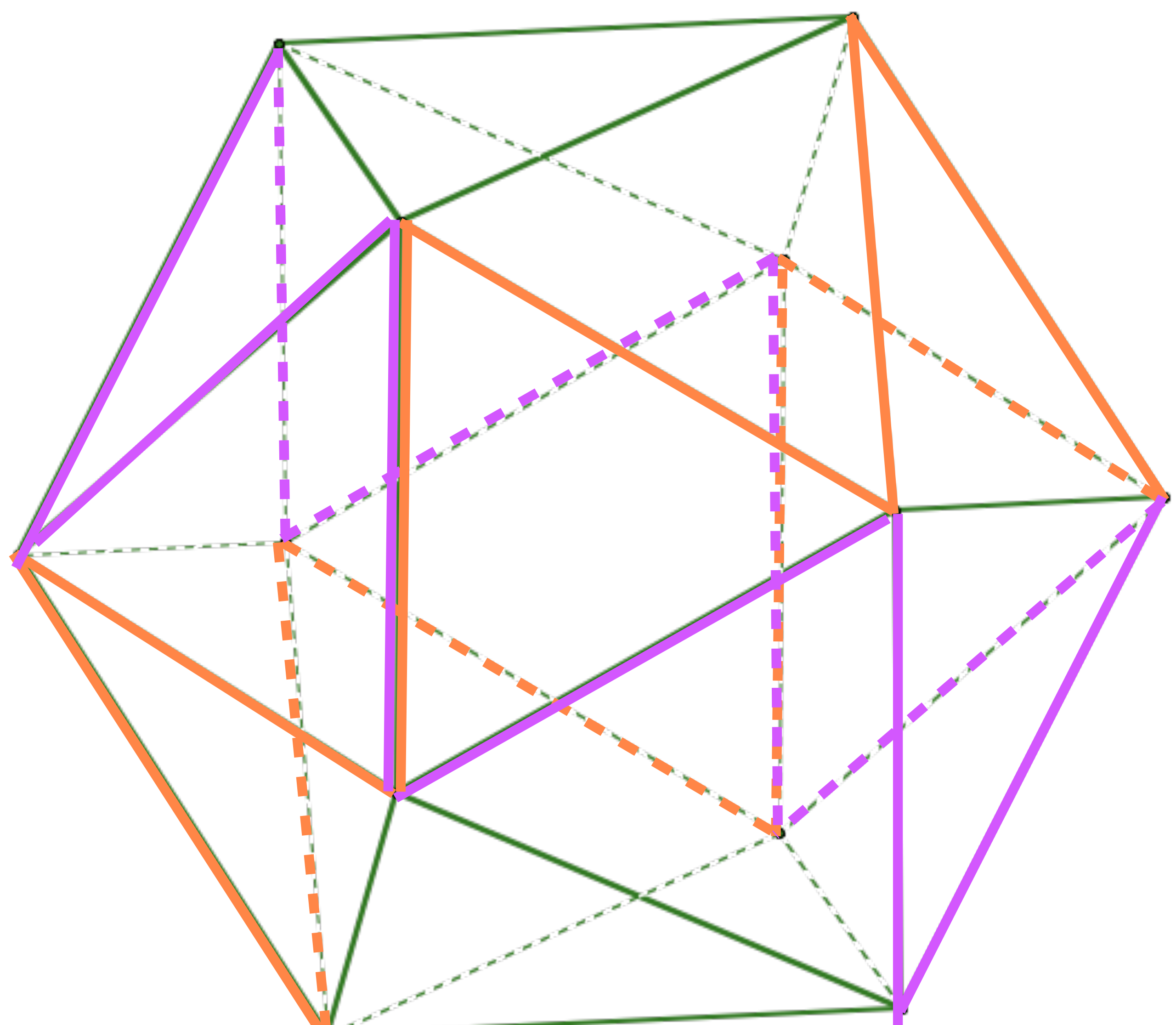

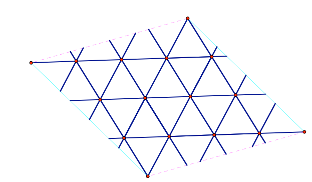

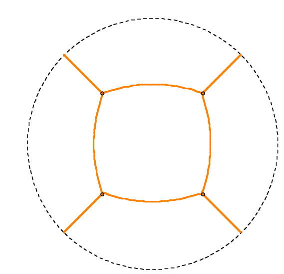

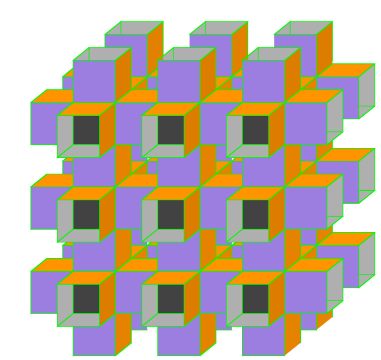

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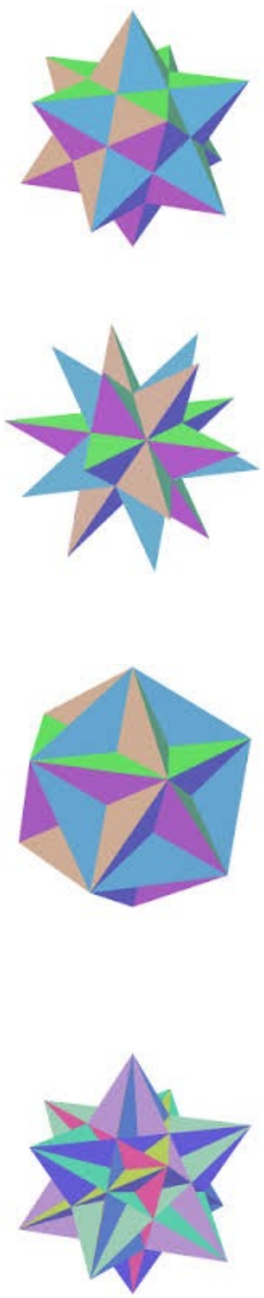

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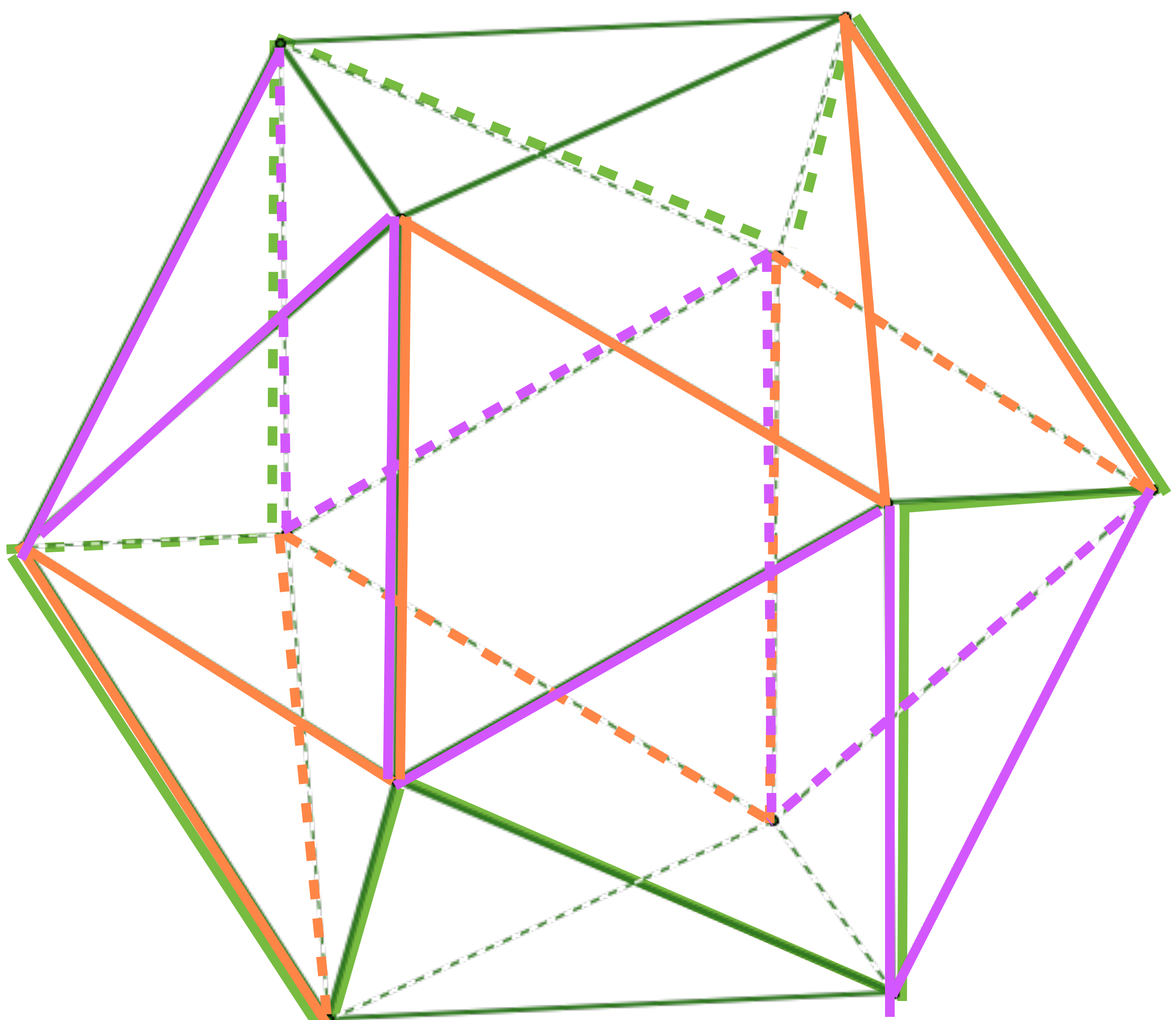

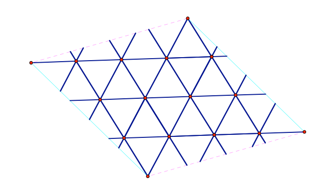

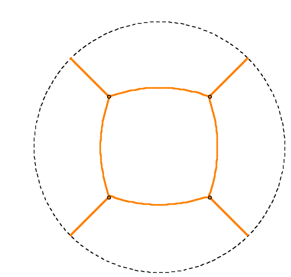

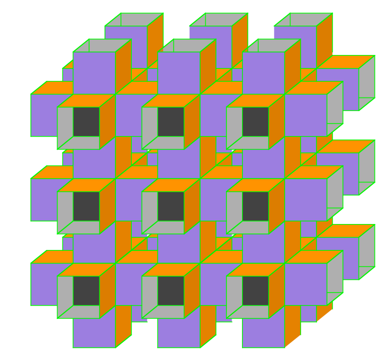

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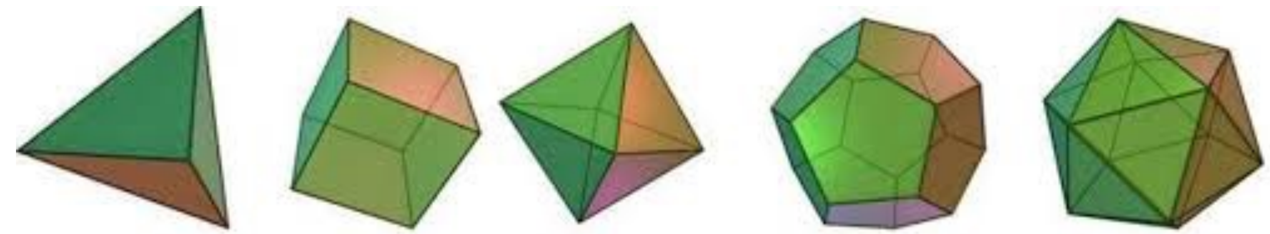

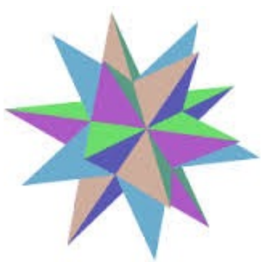

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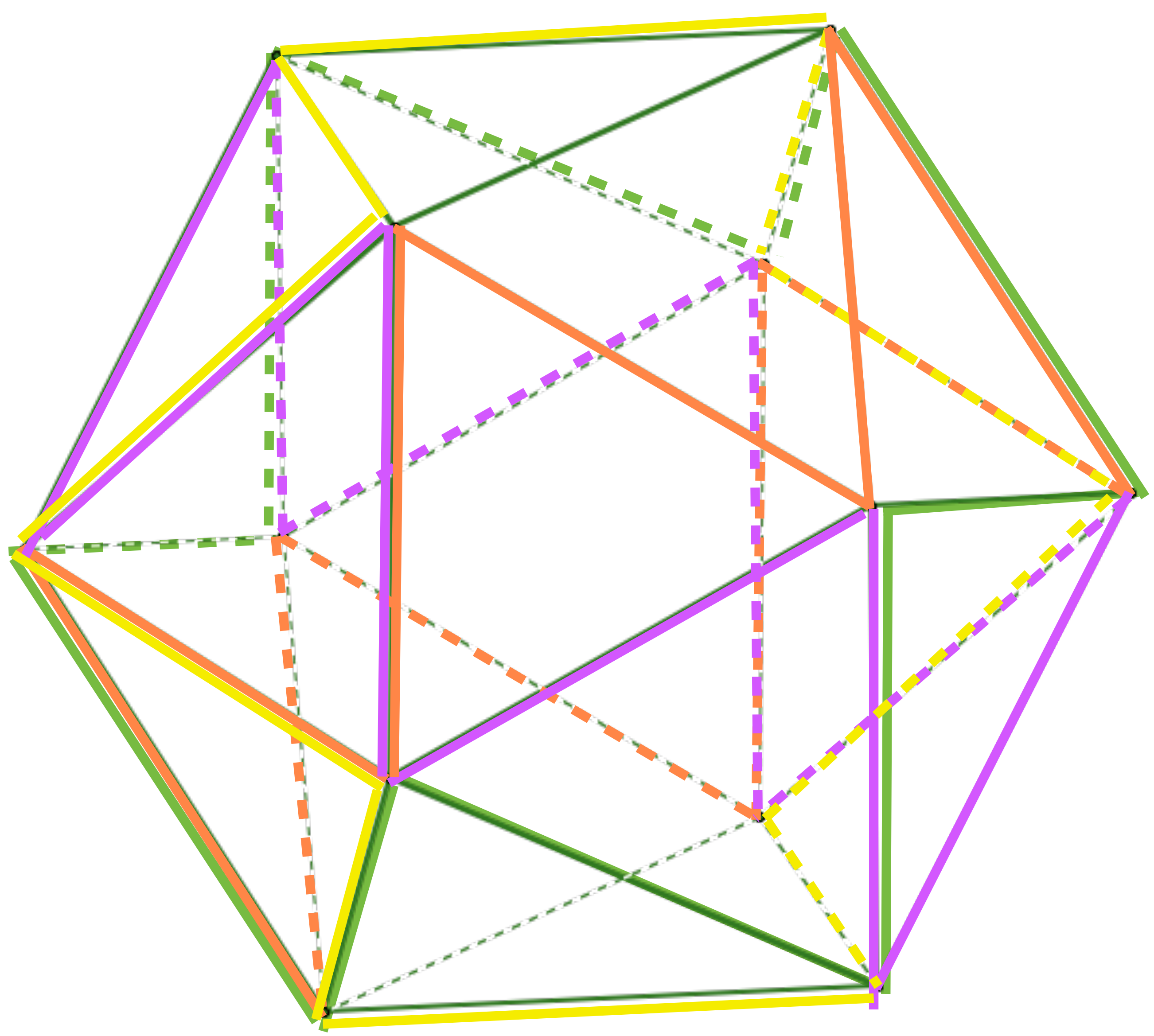

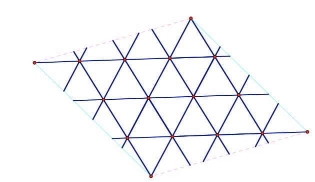

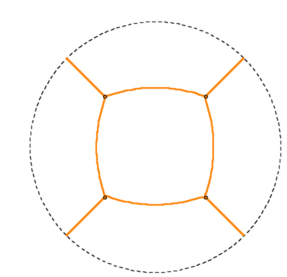

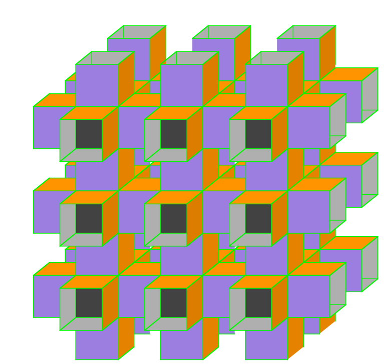

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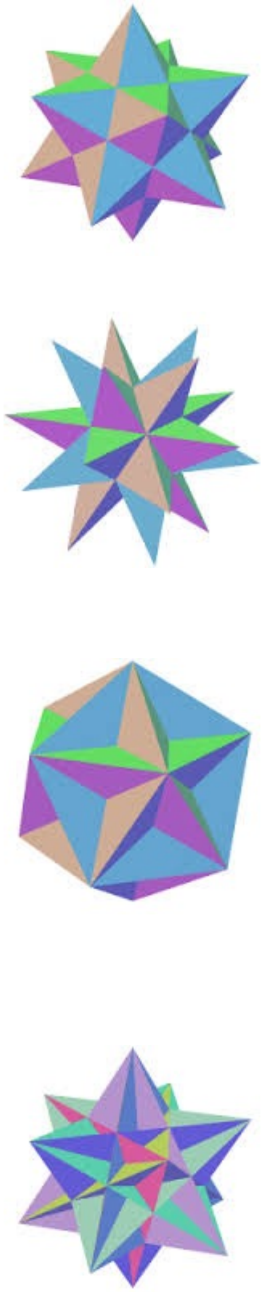

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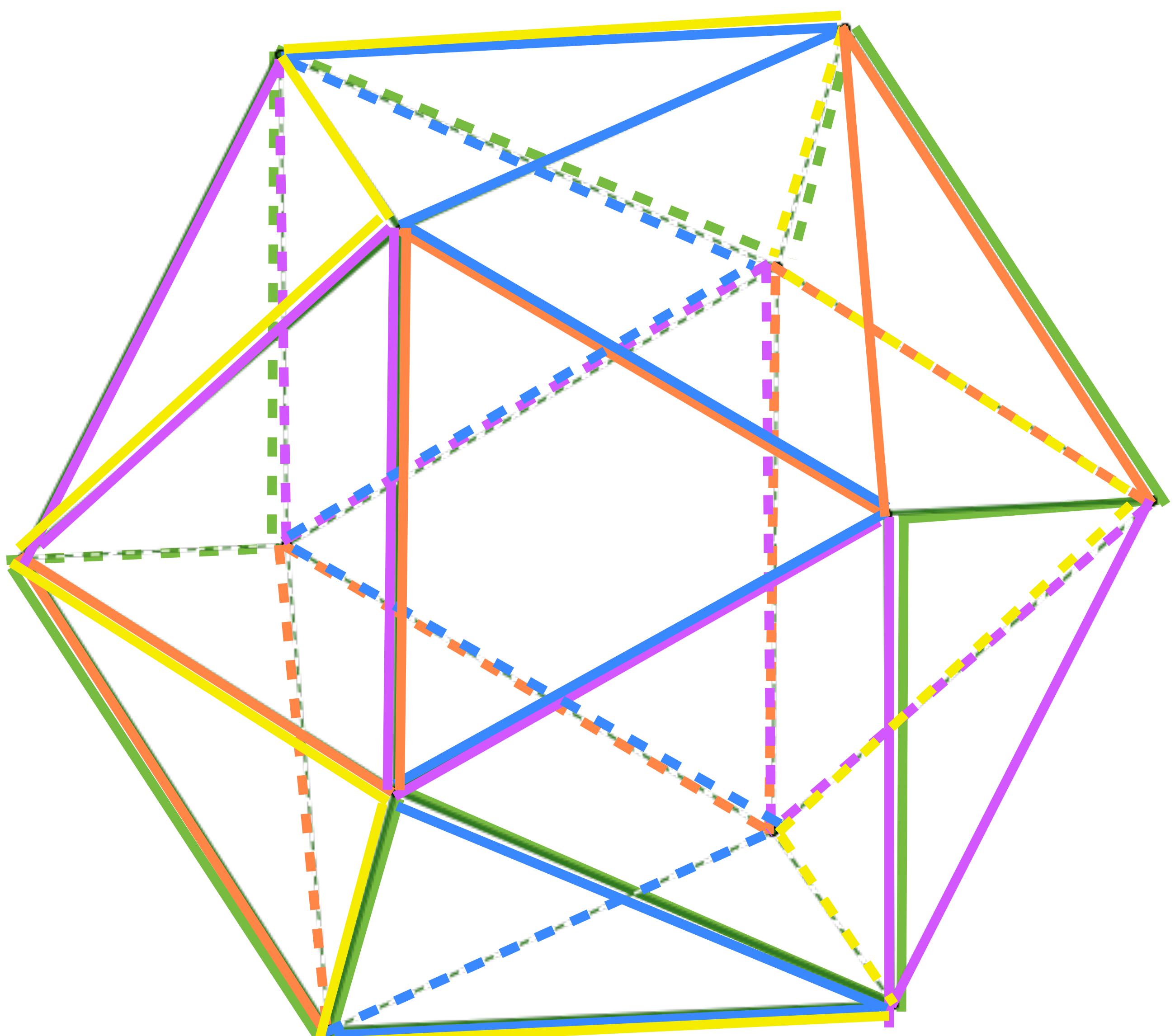

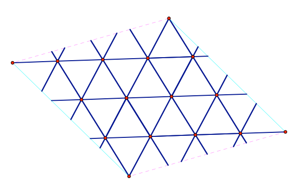

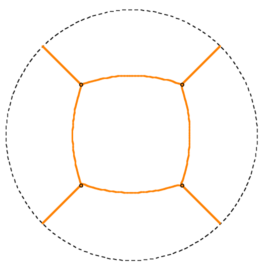

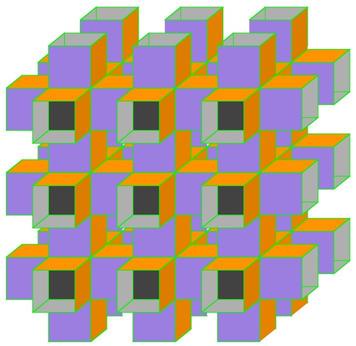

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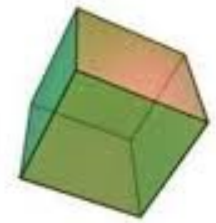
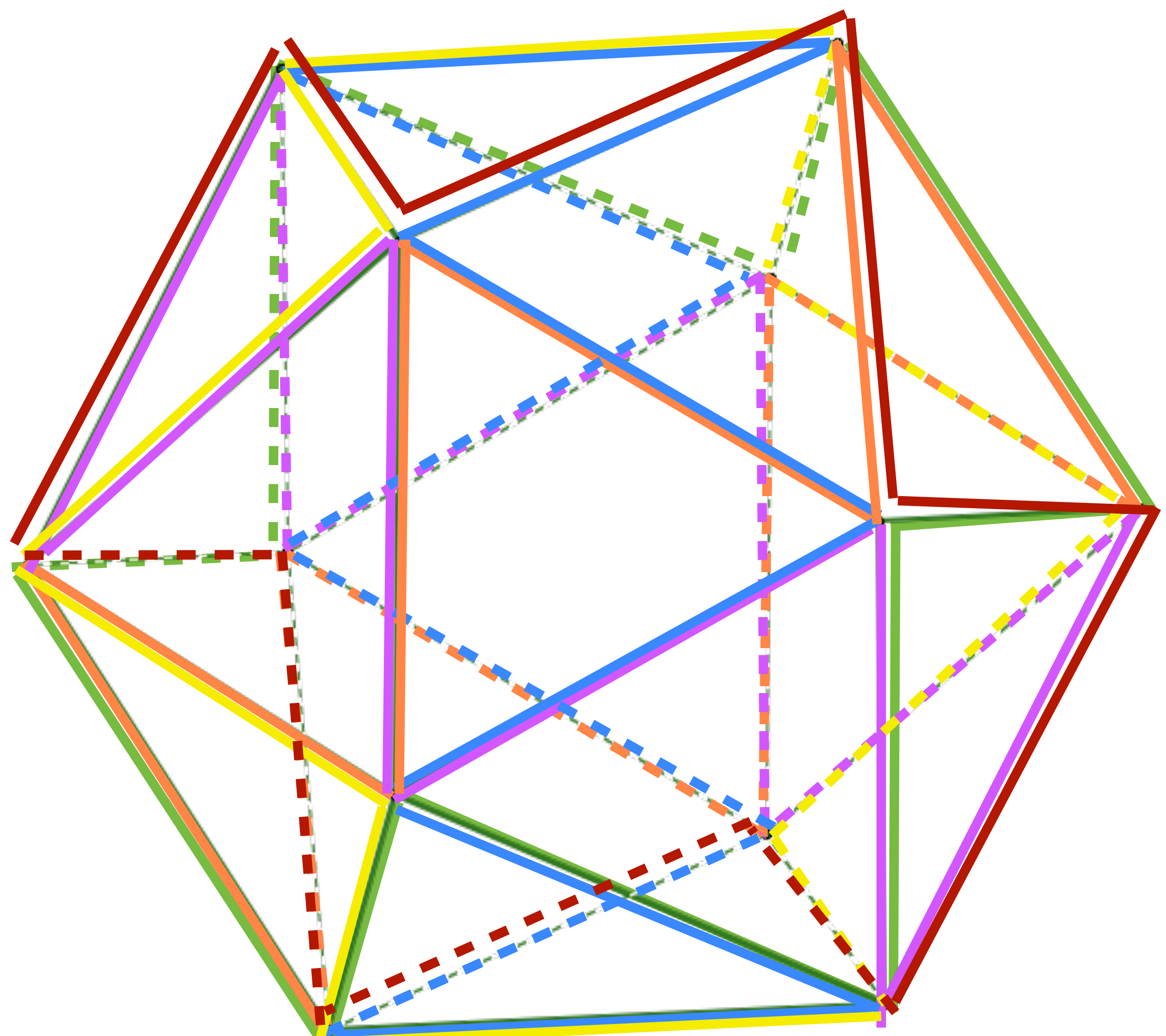
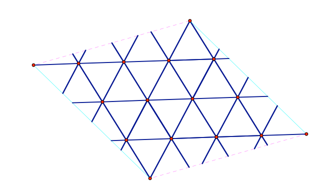
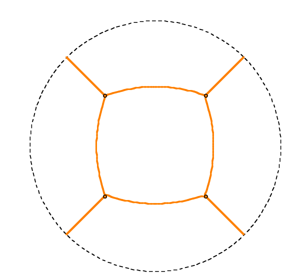
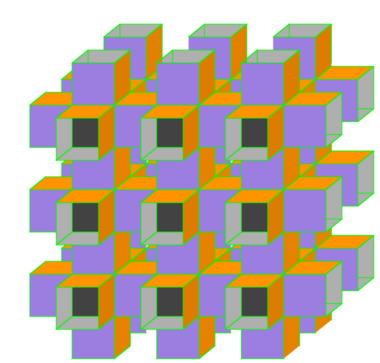






Coxeter



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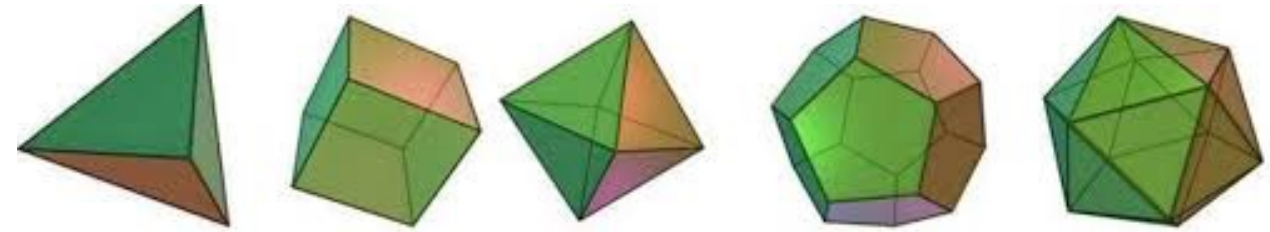
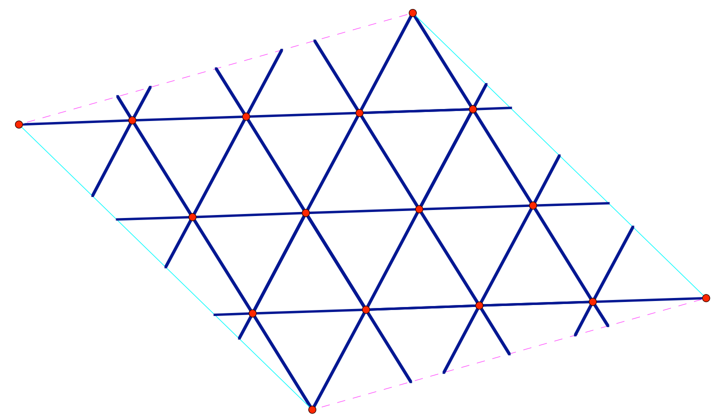
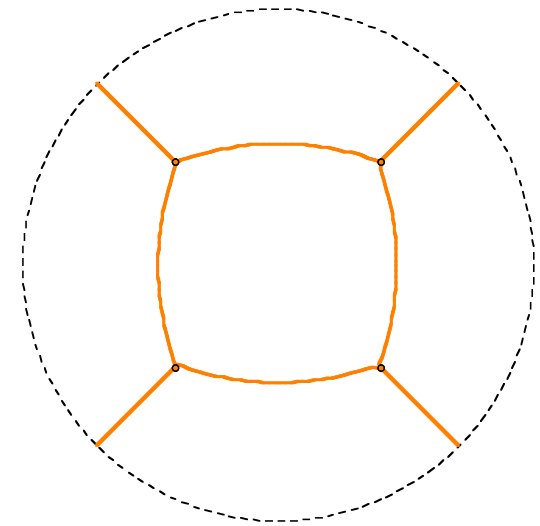
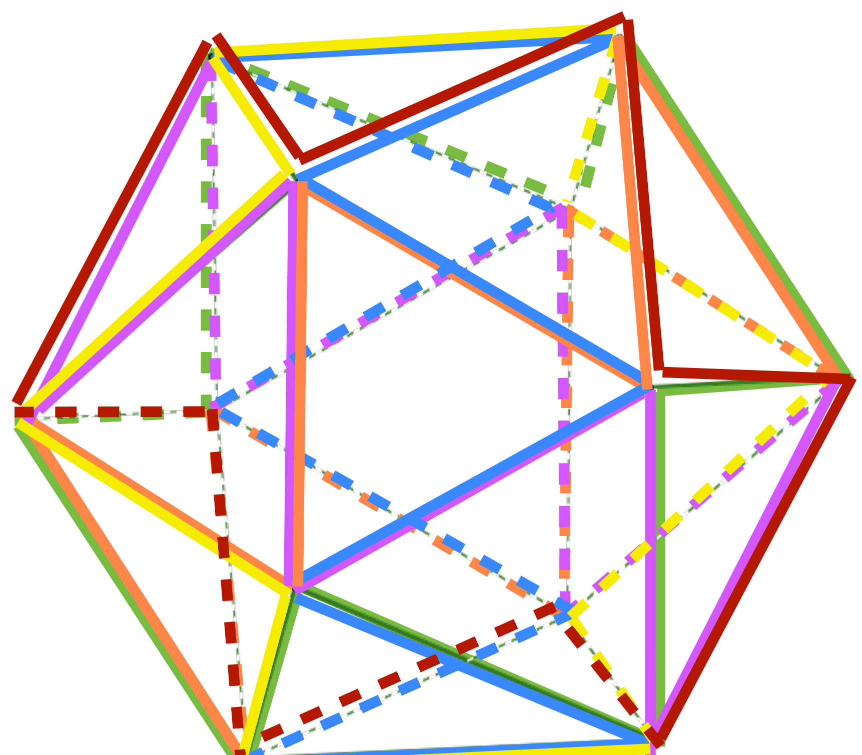
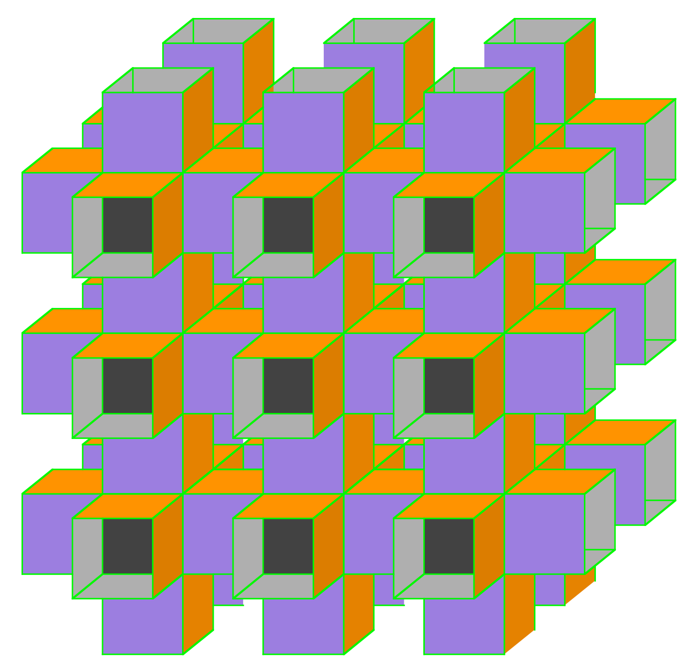




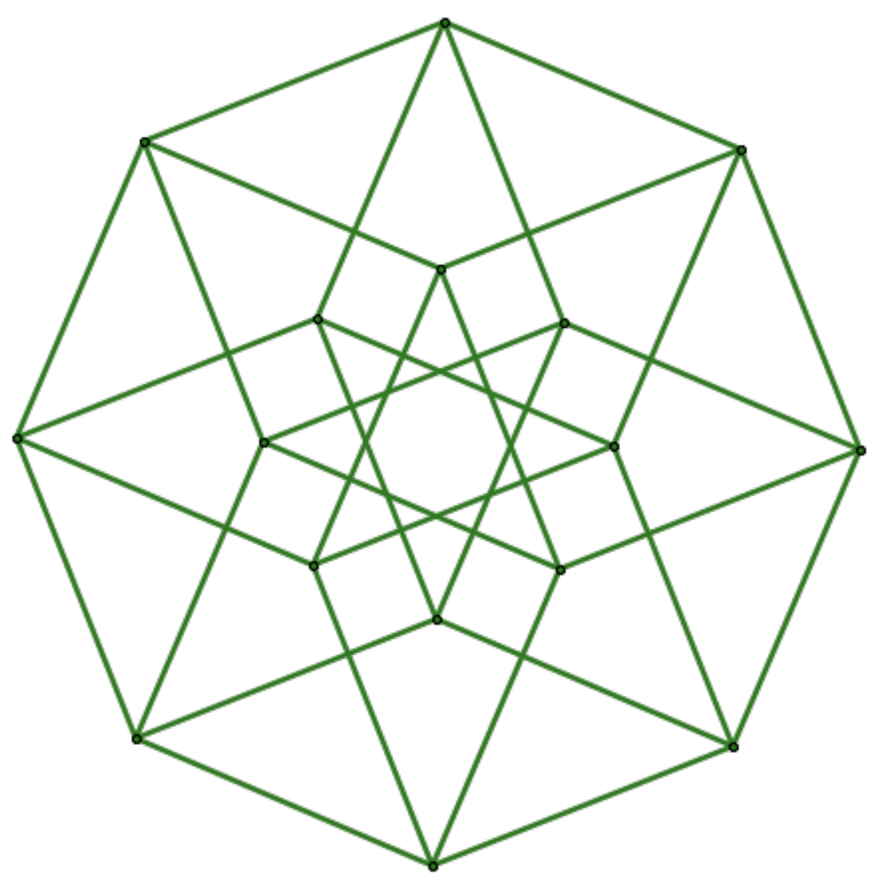






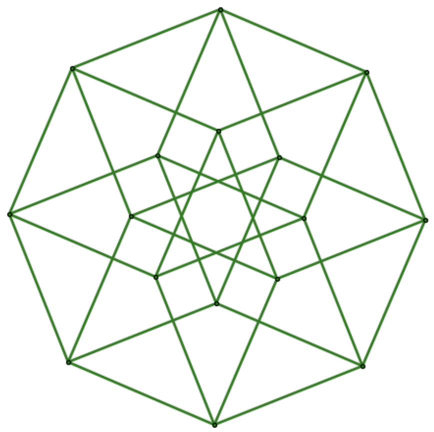
Higher dimensions

Convex polytopes
Ludwig Schläfli (1852)



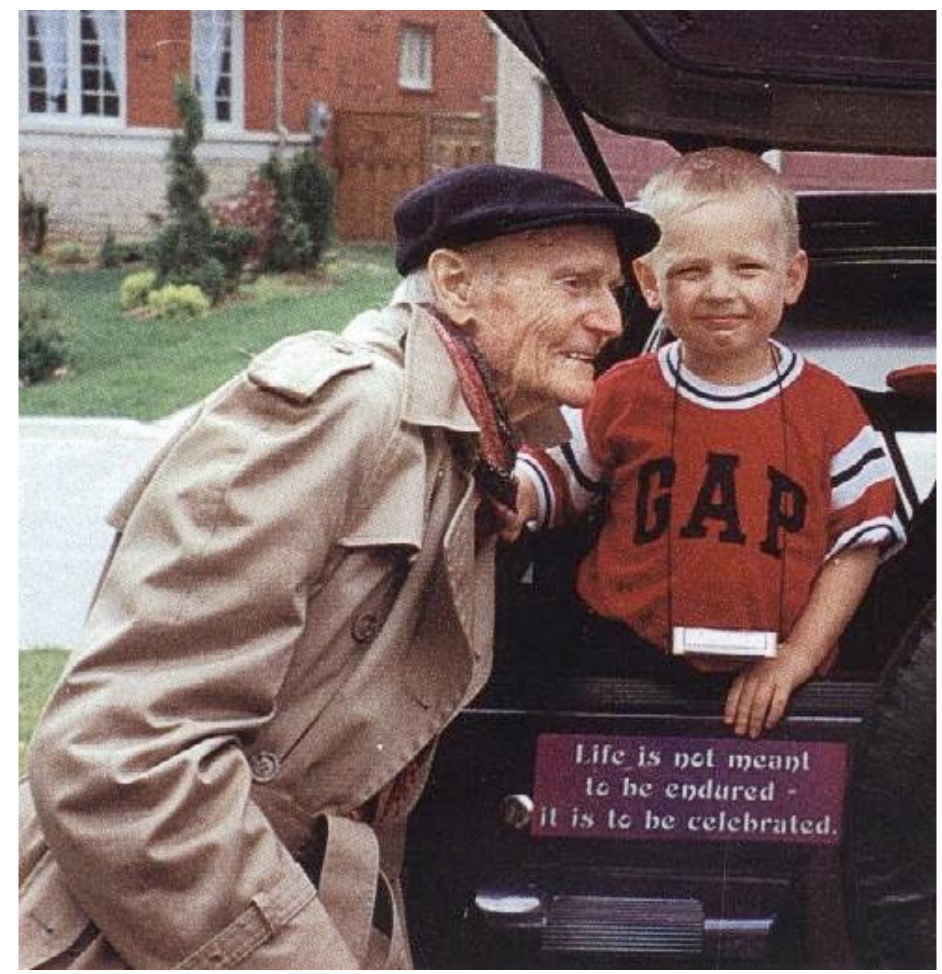
Convex hull of a finite number of points

Higher dimensions

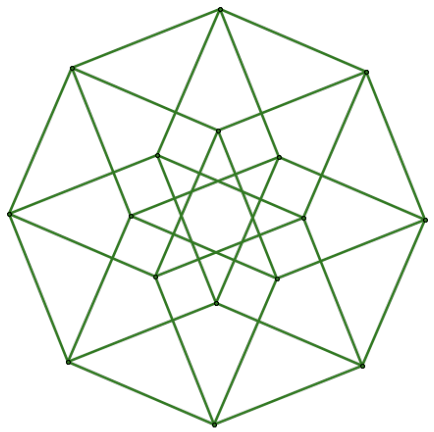


Coxeter
(1907-2003)

Polyhedra and polytopes:



Higher dimensions



Coxeter
(1907-2003)

Polyhedra and polytopes:

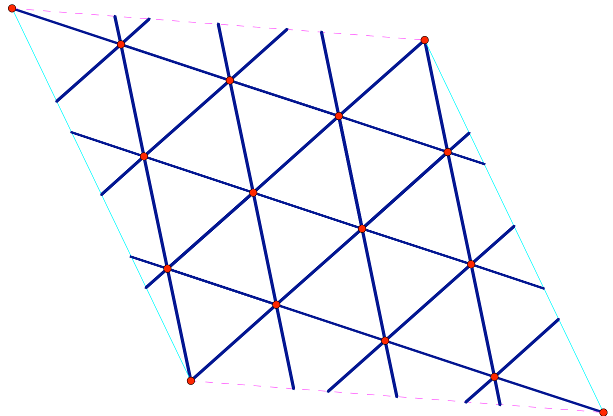
Geometry

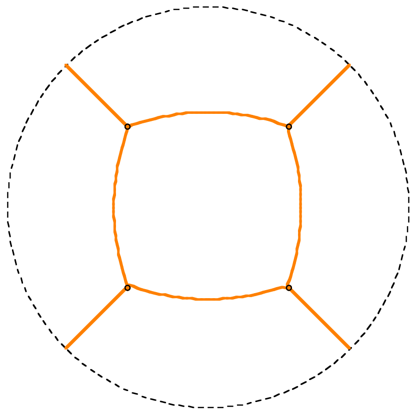
Combinatorics

Group Theory

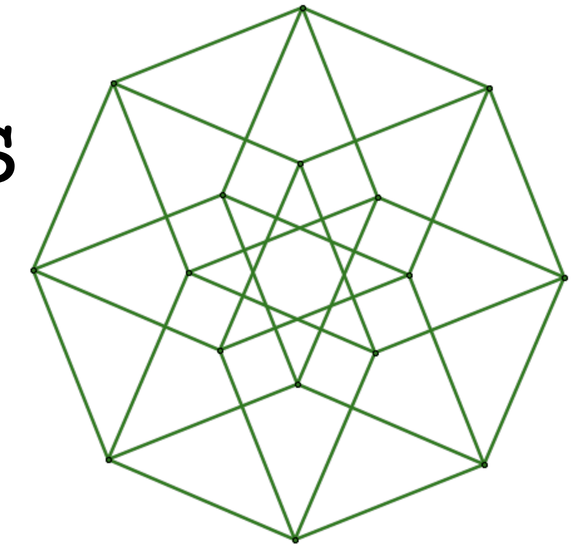
Topology

maps on surfaces





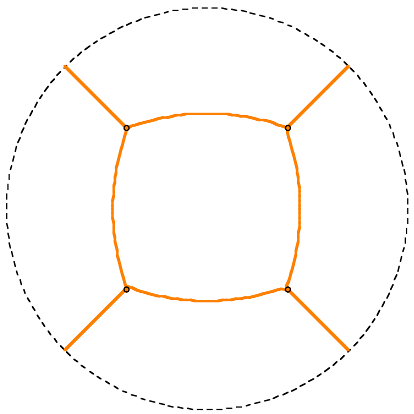
convex polytopes

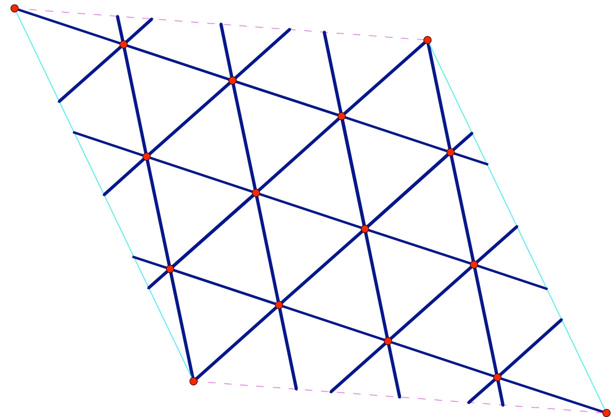


Grünbaum

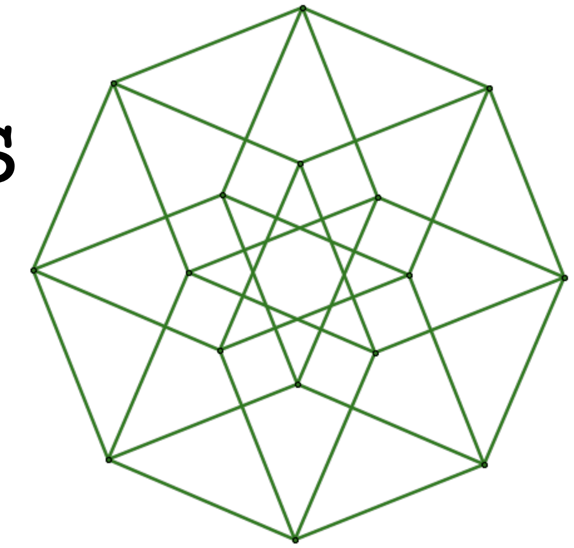
1970´s

maps on
surfaces





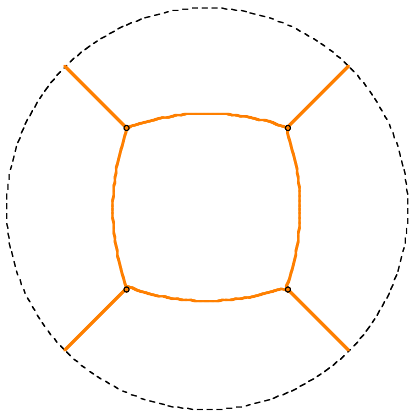
convex
polytopes

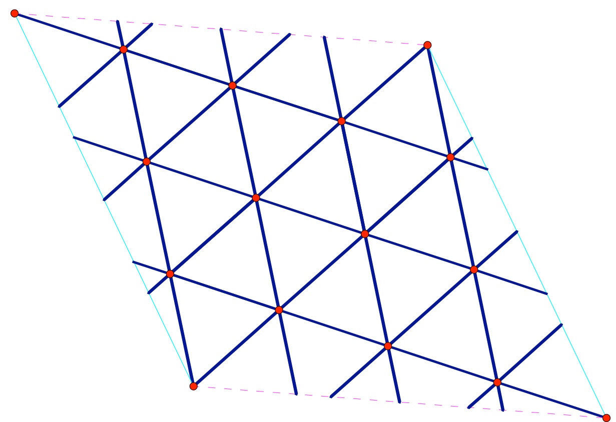


1970´s

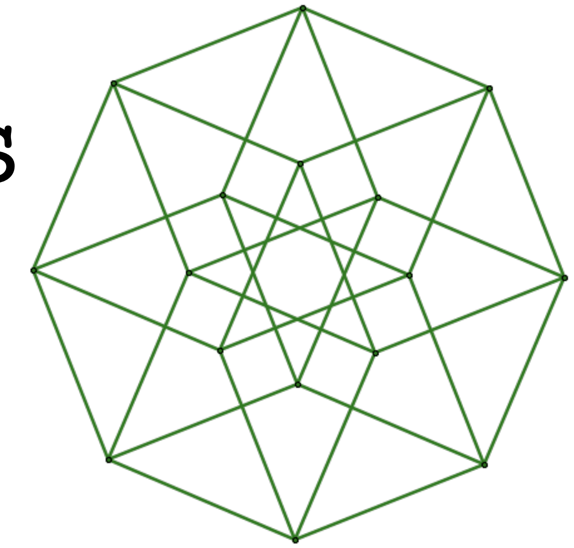
Grünbaum

Proposes to study "polytopes" whose facets and vertex-figures are not spherical maps on surfaces





convex polytopes



1970´s

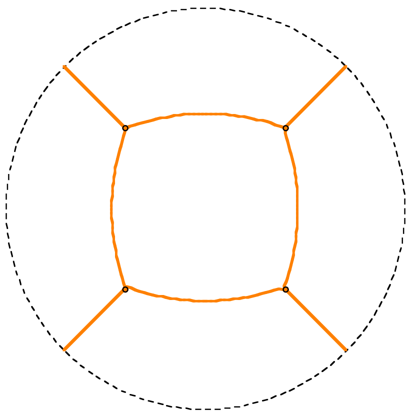
Grünbaum

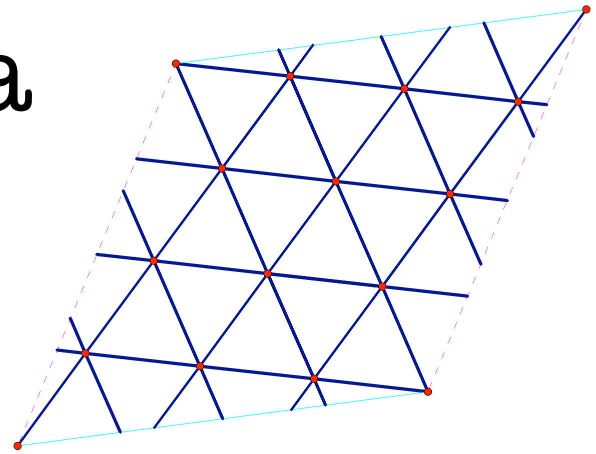
Proposes to study "polytopes" whose facets and vertex-figures are not spherical

Tits

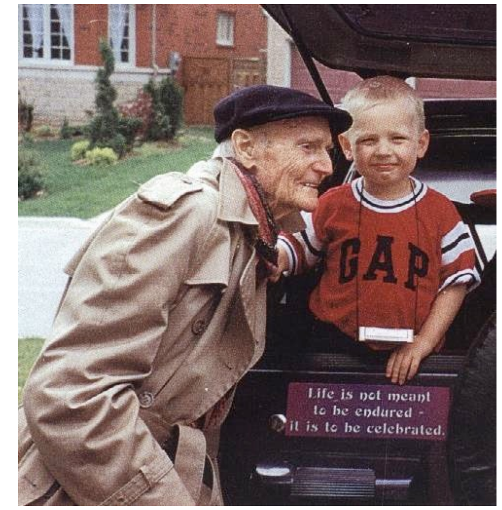
Develops the ideas of incidence geometries

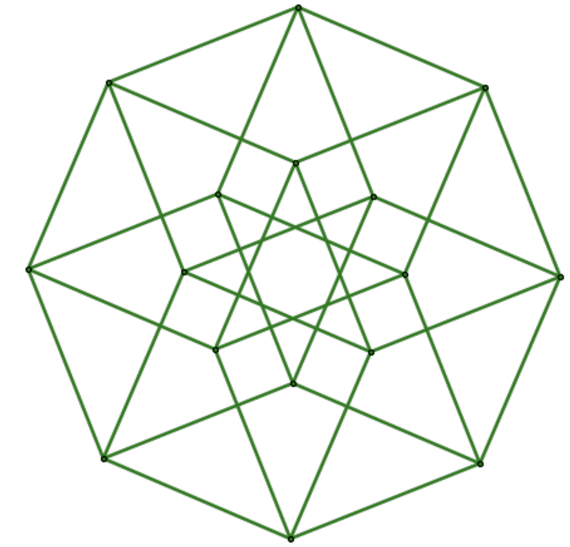
Brahana





Coxeter

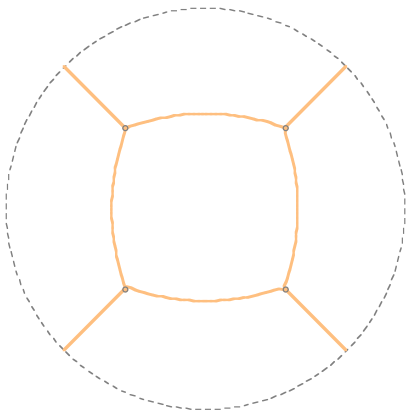


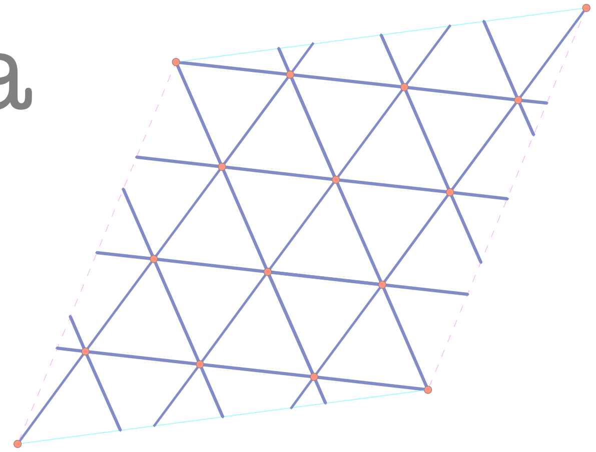


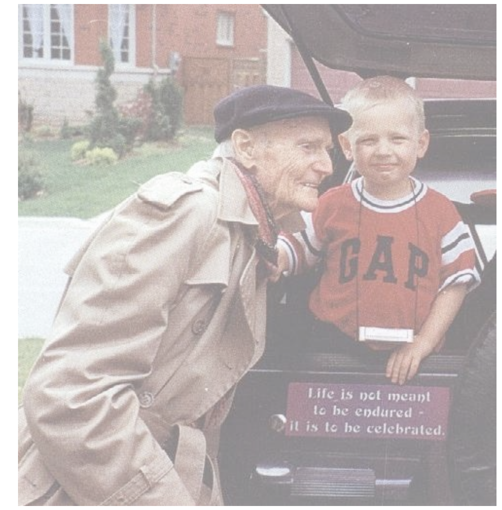
Grünbaum

Tits

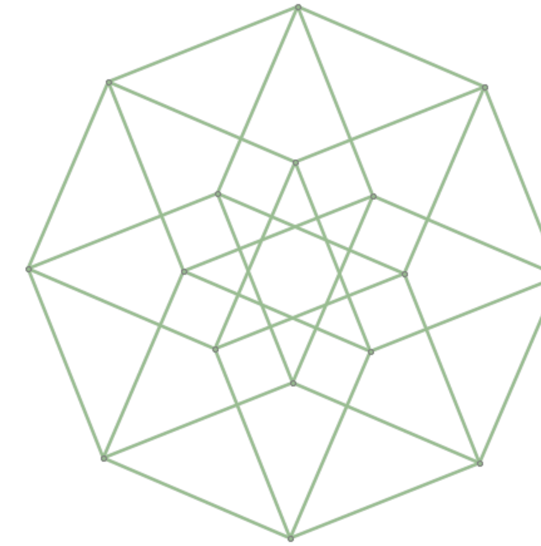
Brahana







Coxeter



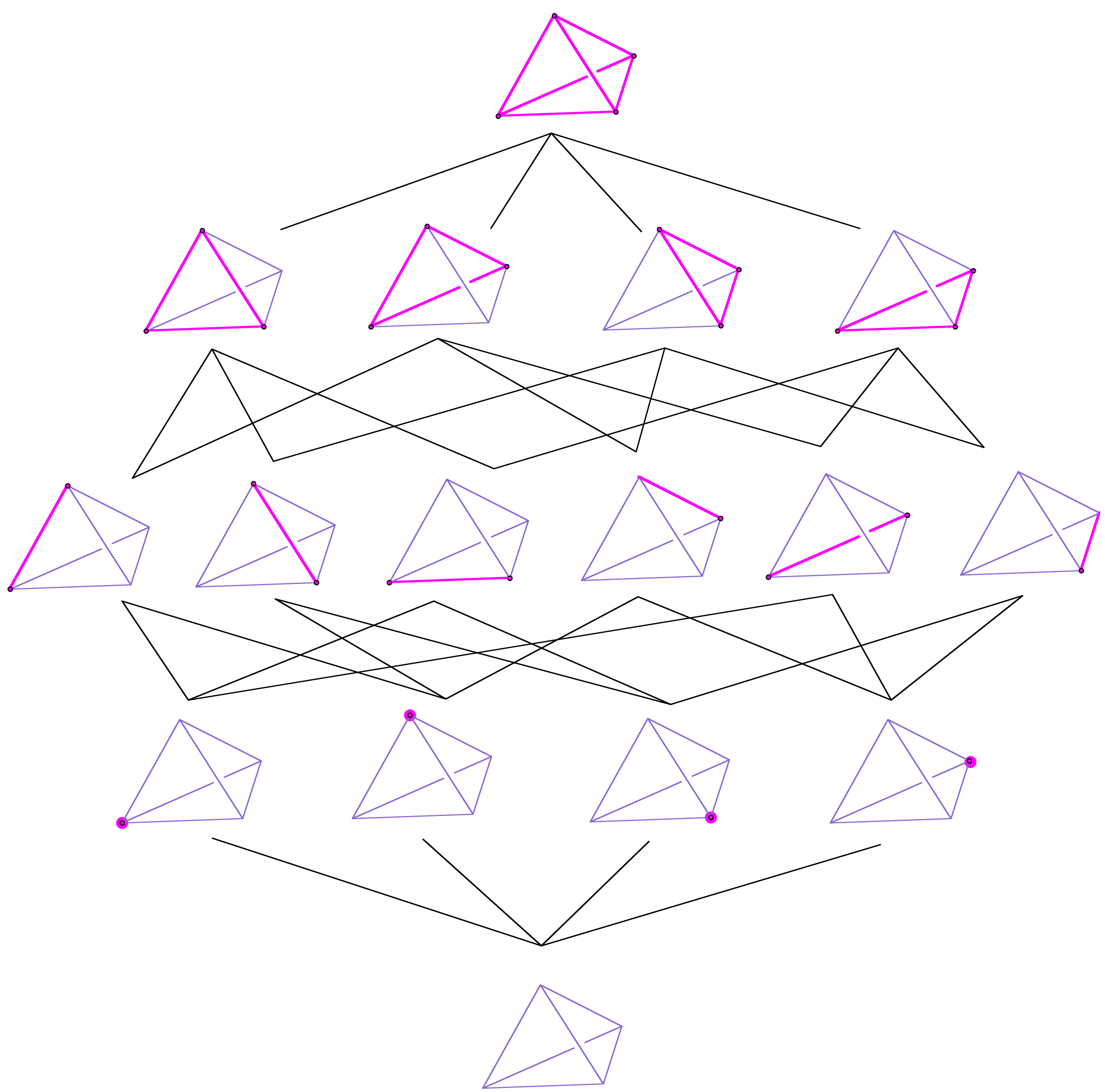
Danzer & Schulte
(early 1980´s)

Grünbaum

Incidence polytopes, now called abstract polytopes

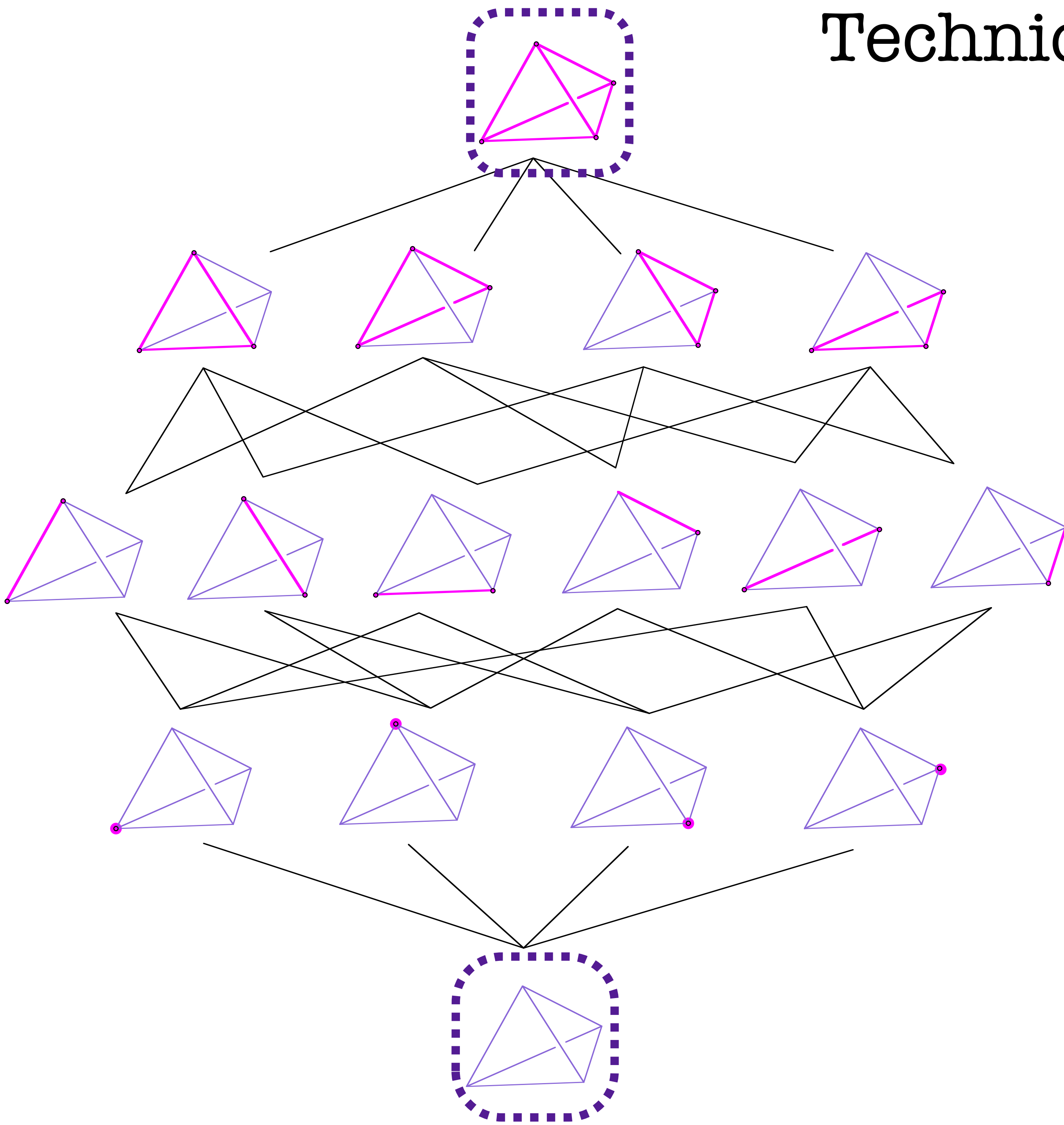
Tits

An abstract n-polytope is a partially ordered set endowed with a rank function to {-1,0,...,n} (dimension, in the convex case)



An abstract n-polytope is a partially ordered set endowed with a rank function to {-1,0,…,n} (dimension, in the convex case)

Technical: Min and max element



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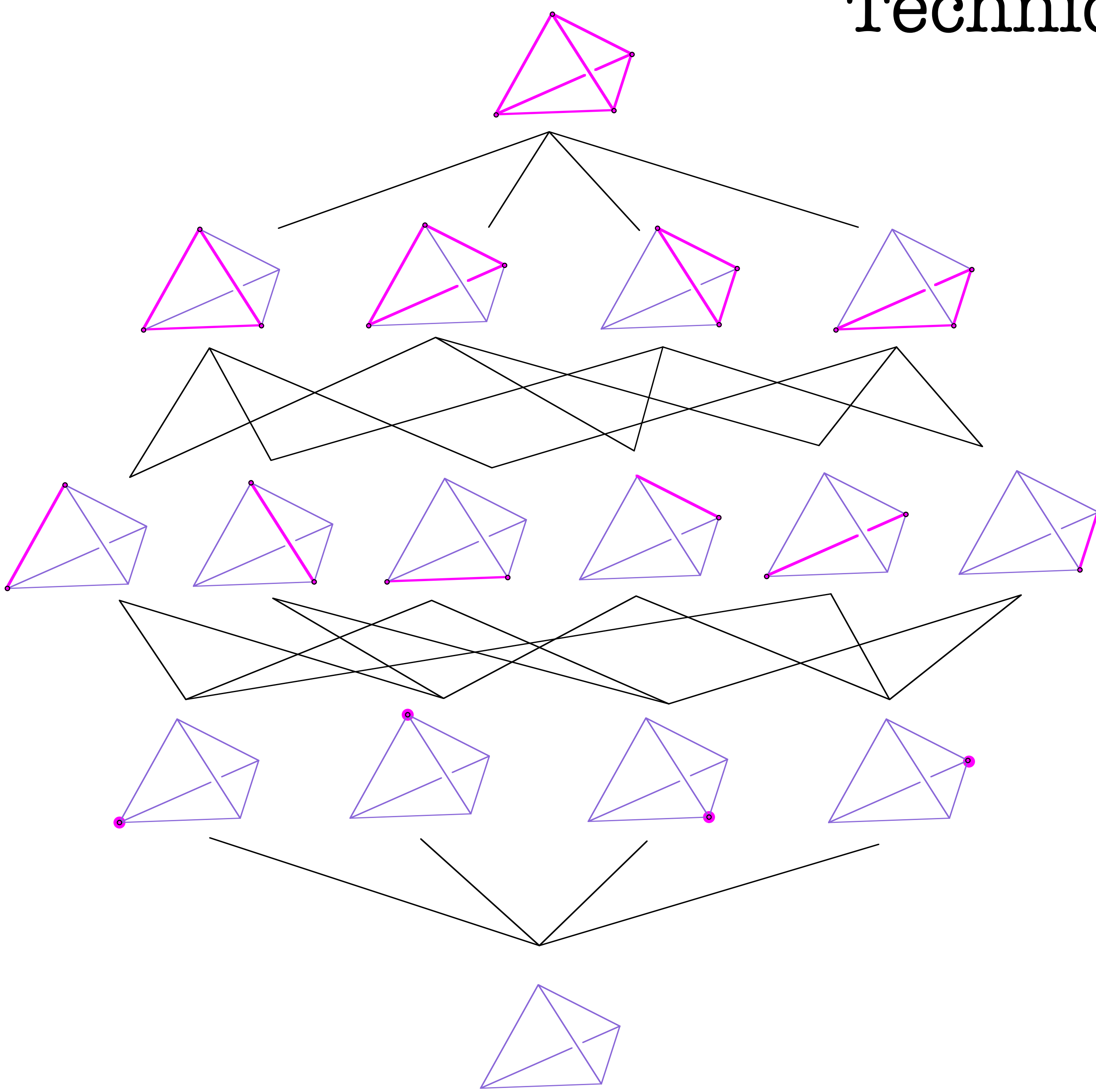

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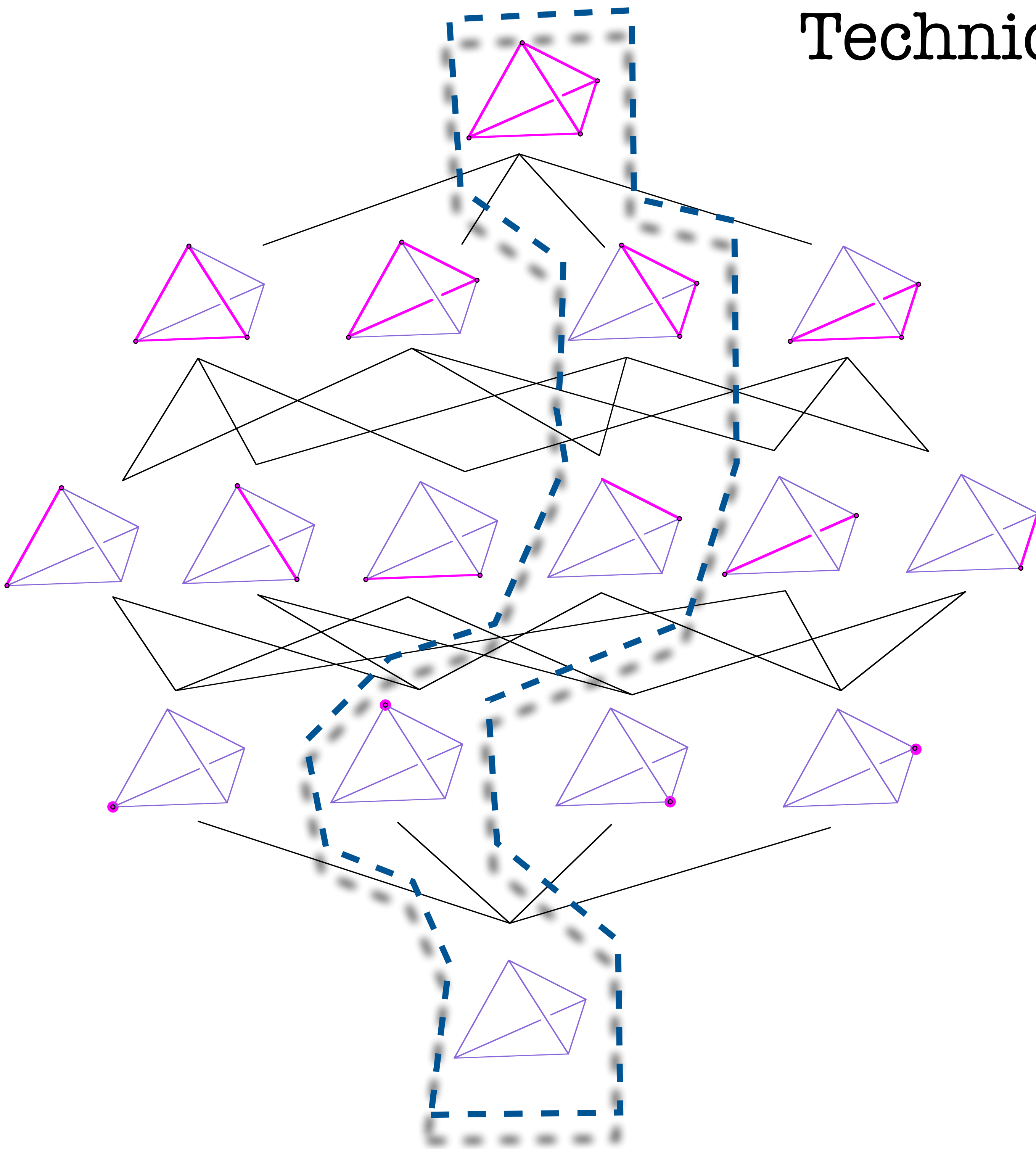

Technical: Min and max element

Flags: n+2 elements

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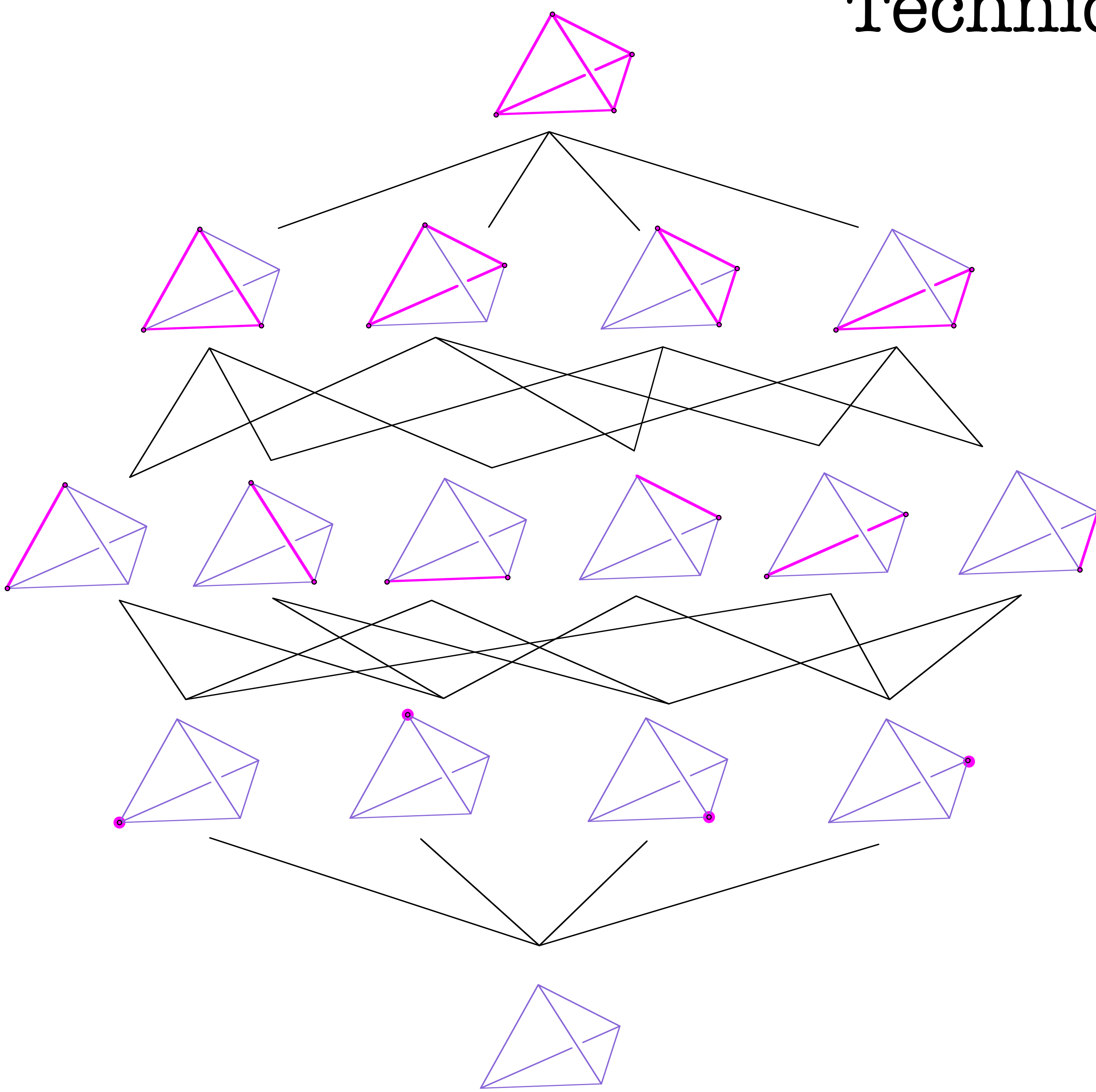

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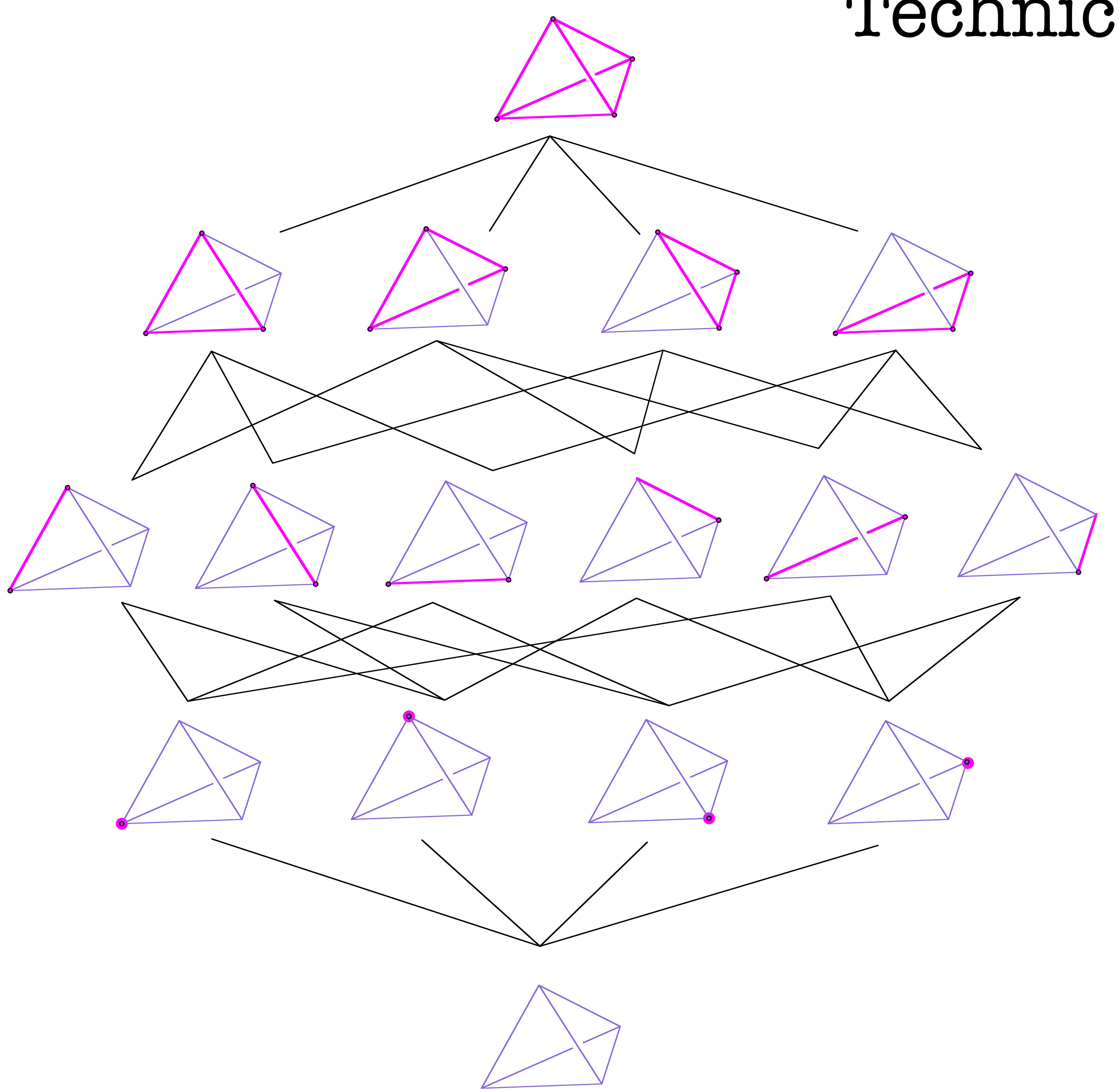
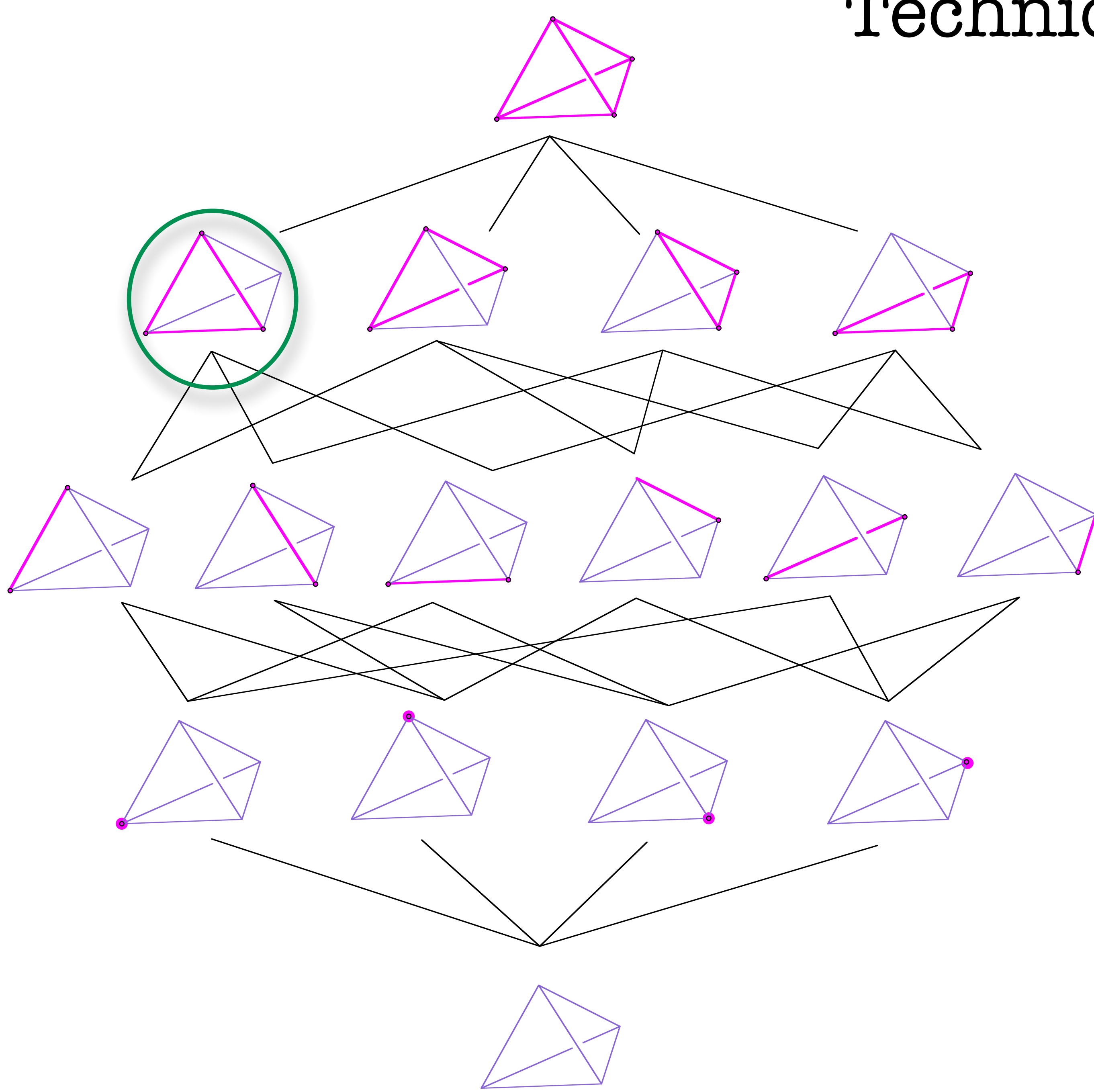


Flags: n+2 elements

Strongly connected

An abstract n-polytope is a partially ordered set endowed with a rank function to {-1,0,...,n} (dimension, in the convex case)



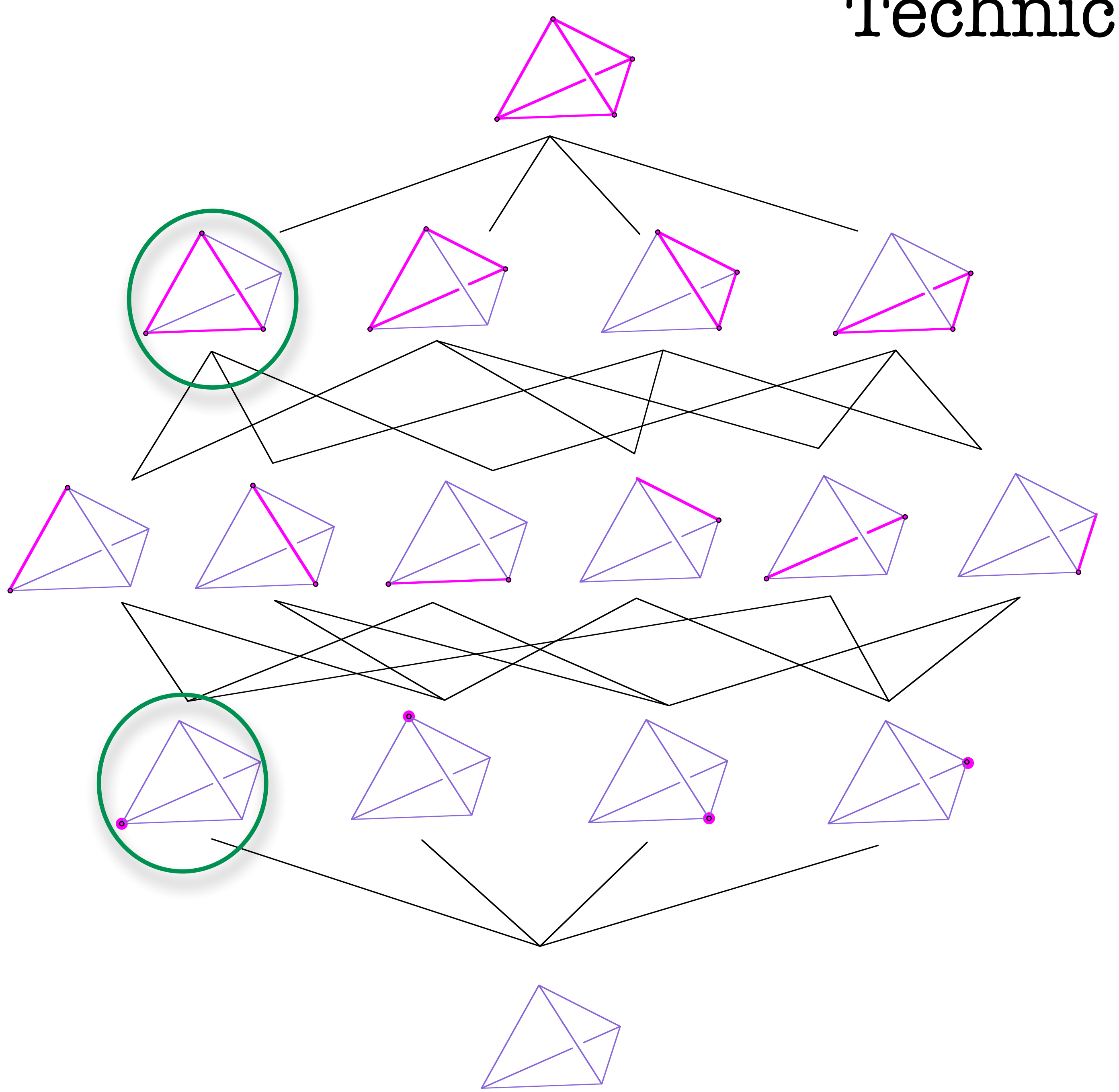
Technical: Min and max element

Flags: n+2 elements

Strongly connected

Diamond

An abstract n-polytope is a partially ordered set endowed with a rank function to {-1,0,...,n} (dimension, in the convex case)



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Flags: n+2 elements

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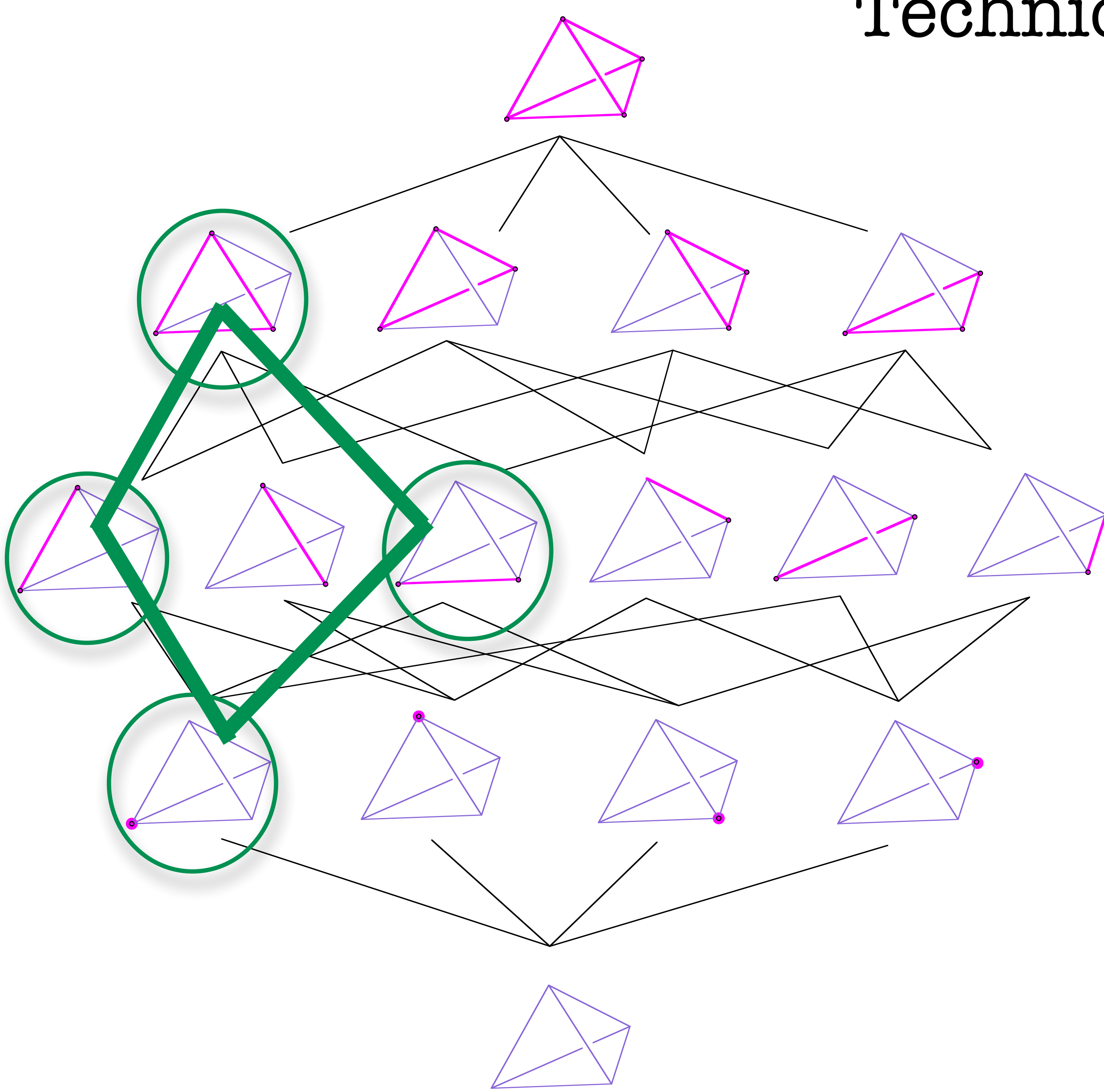
An abstract n-polytope is a partially ordered set endowed with a rank function to {-1,0,...,n} (dimension, in the convex case)



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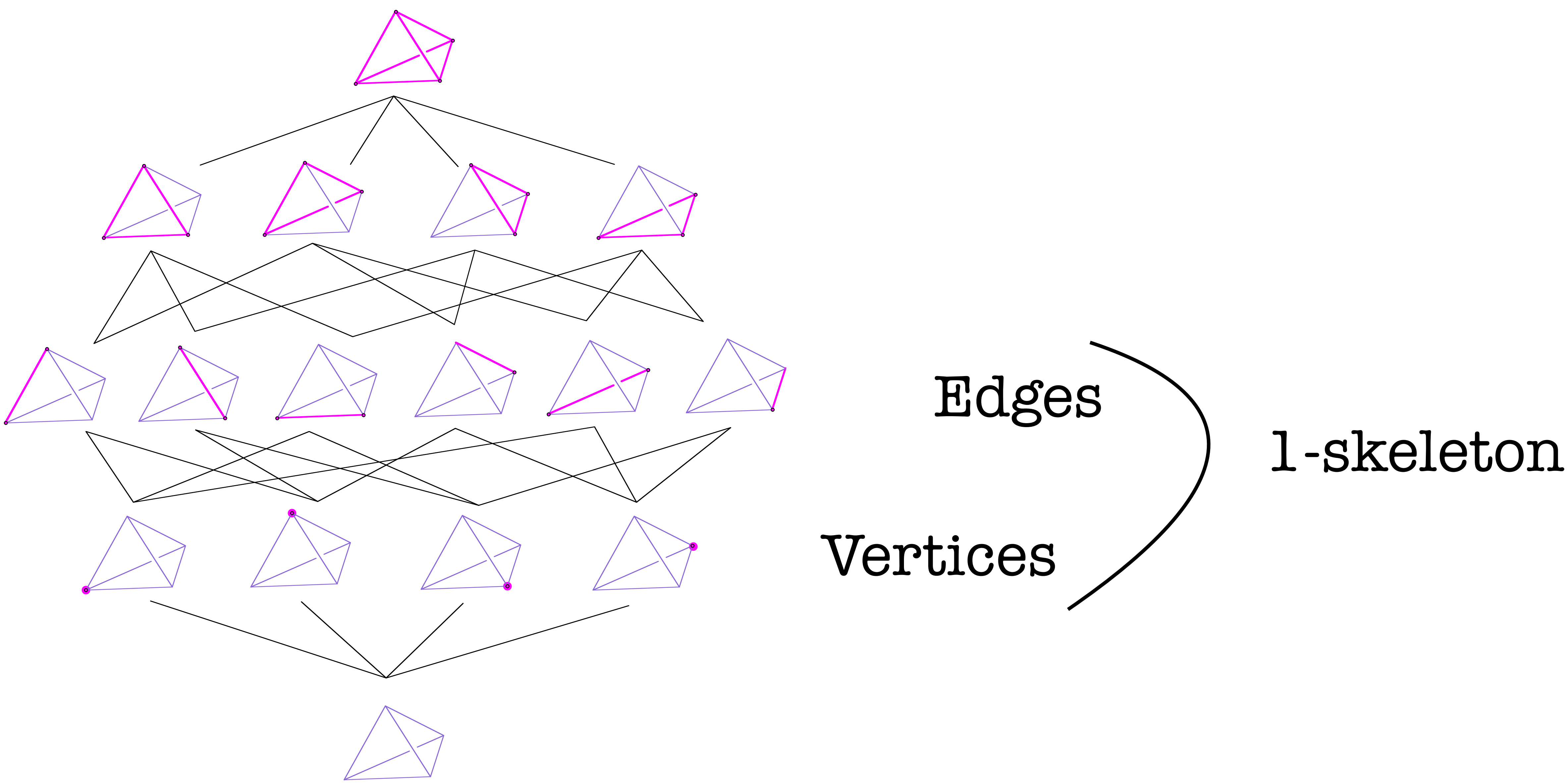
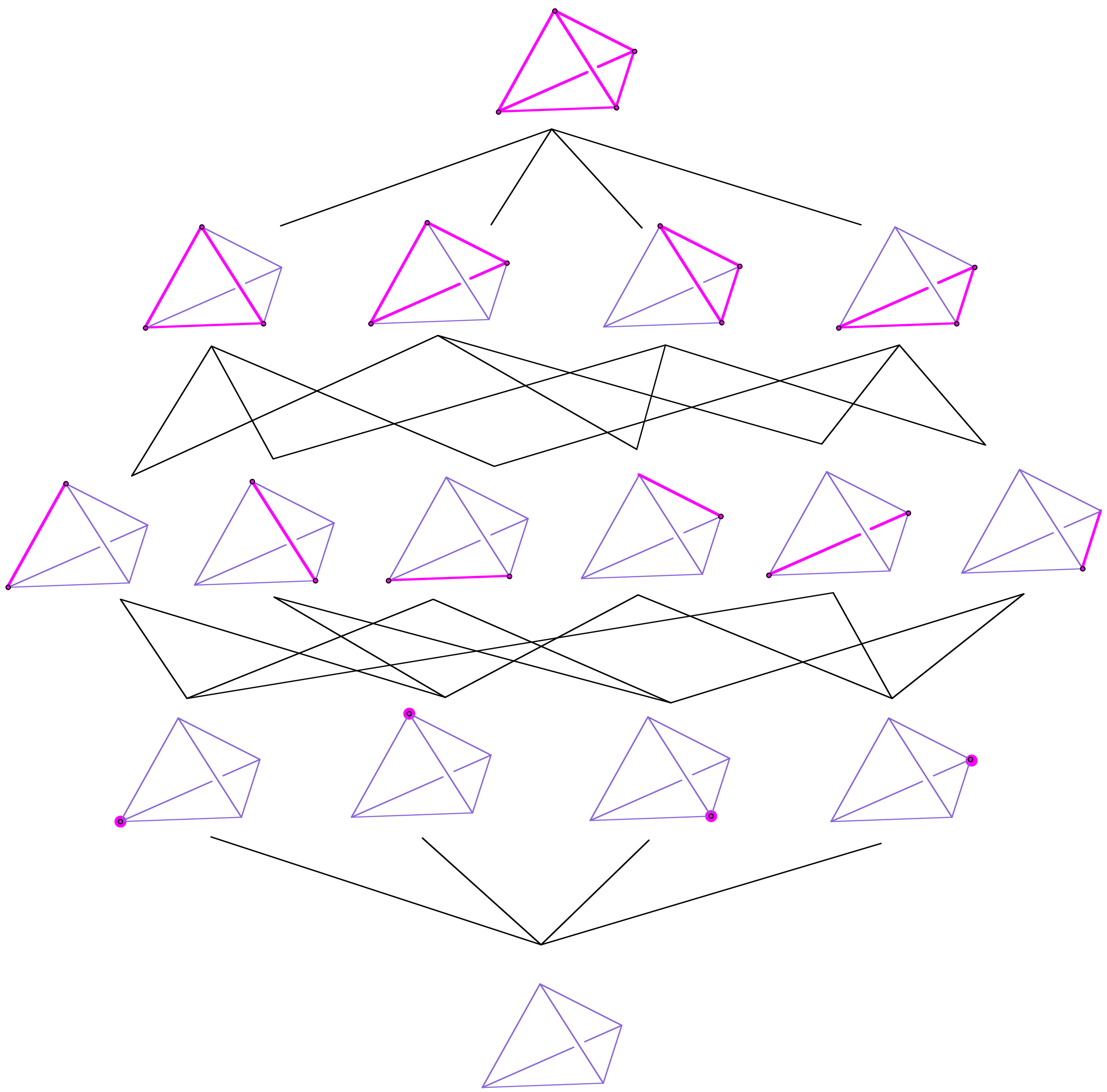
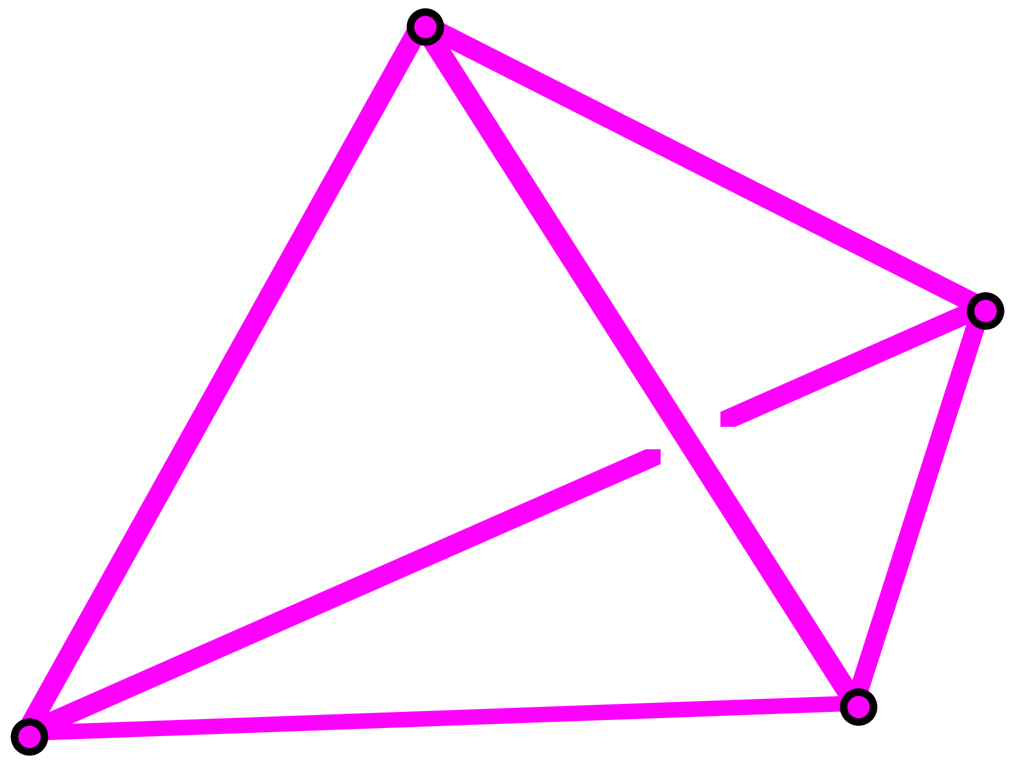
Diamond

Abstract polytope P



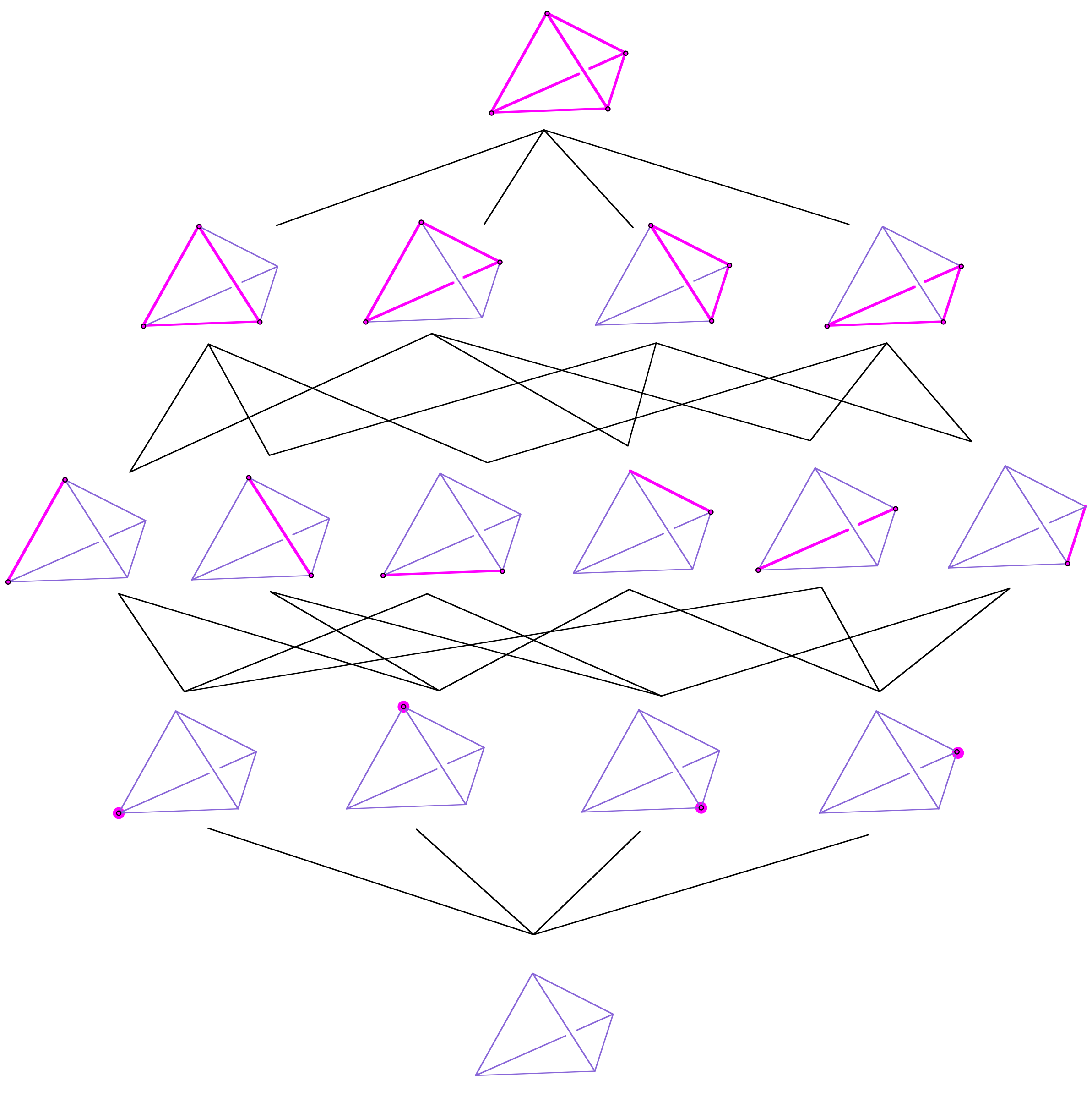
Abstract polytope P

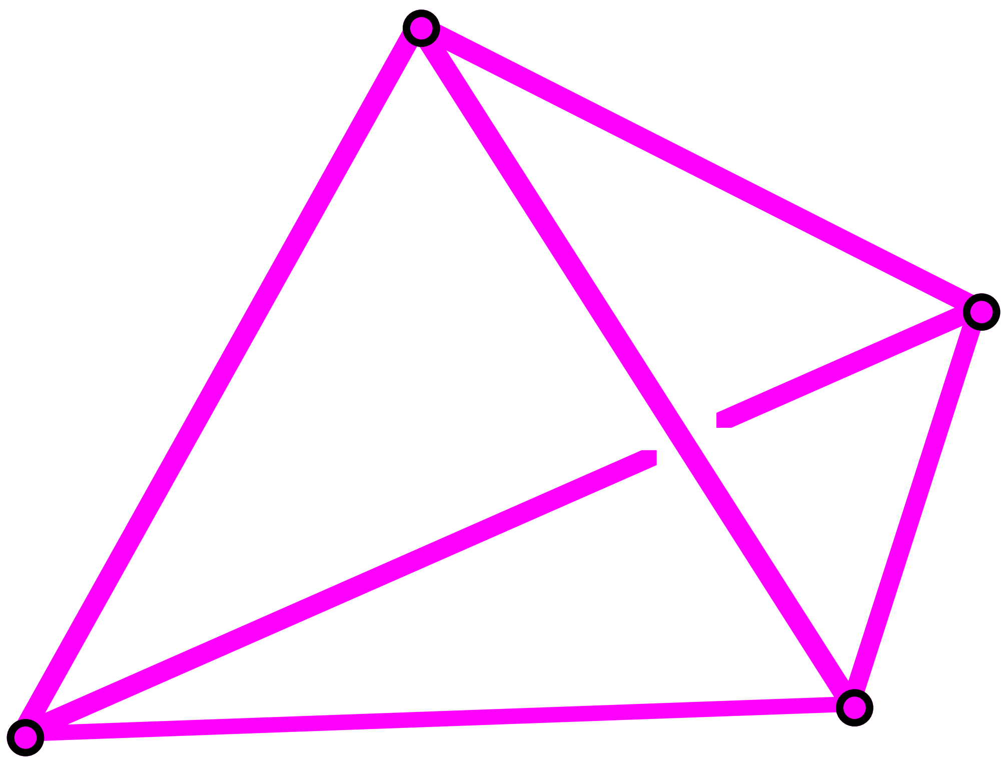




Abstract polytope P

is NOT determined by its 1-skeleton





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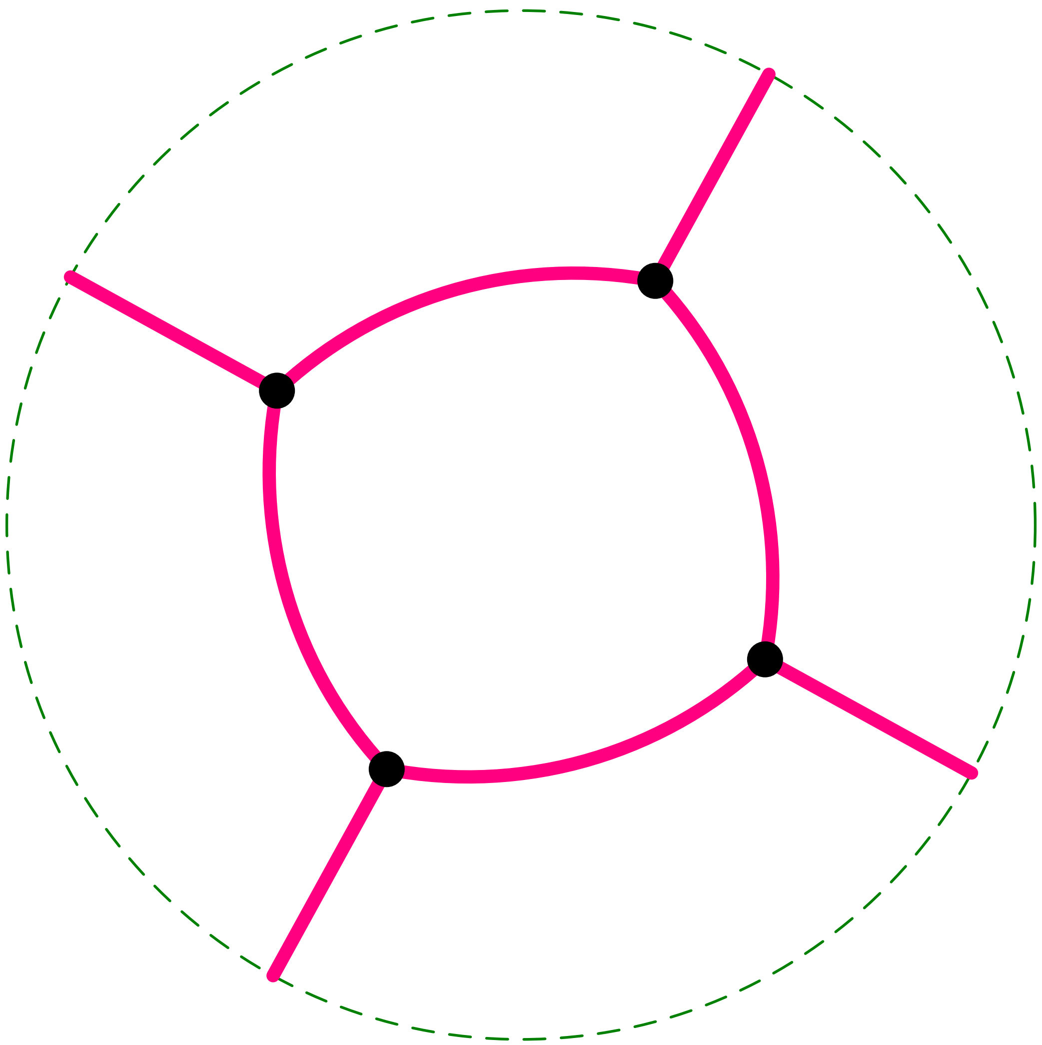
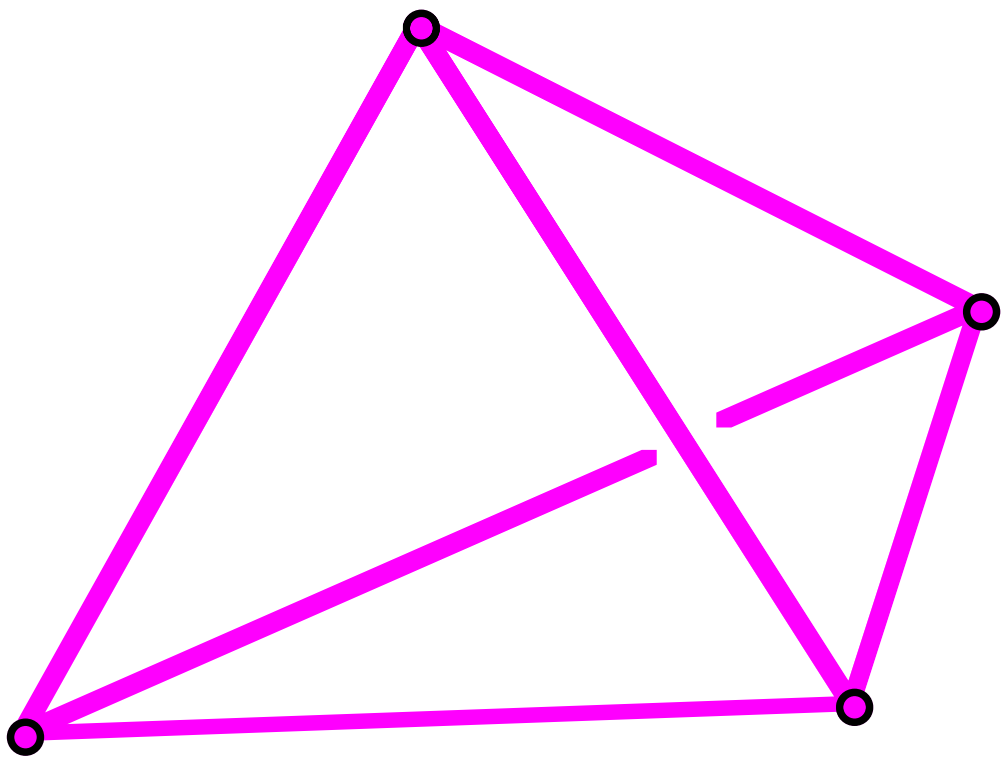
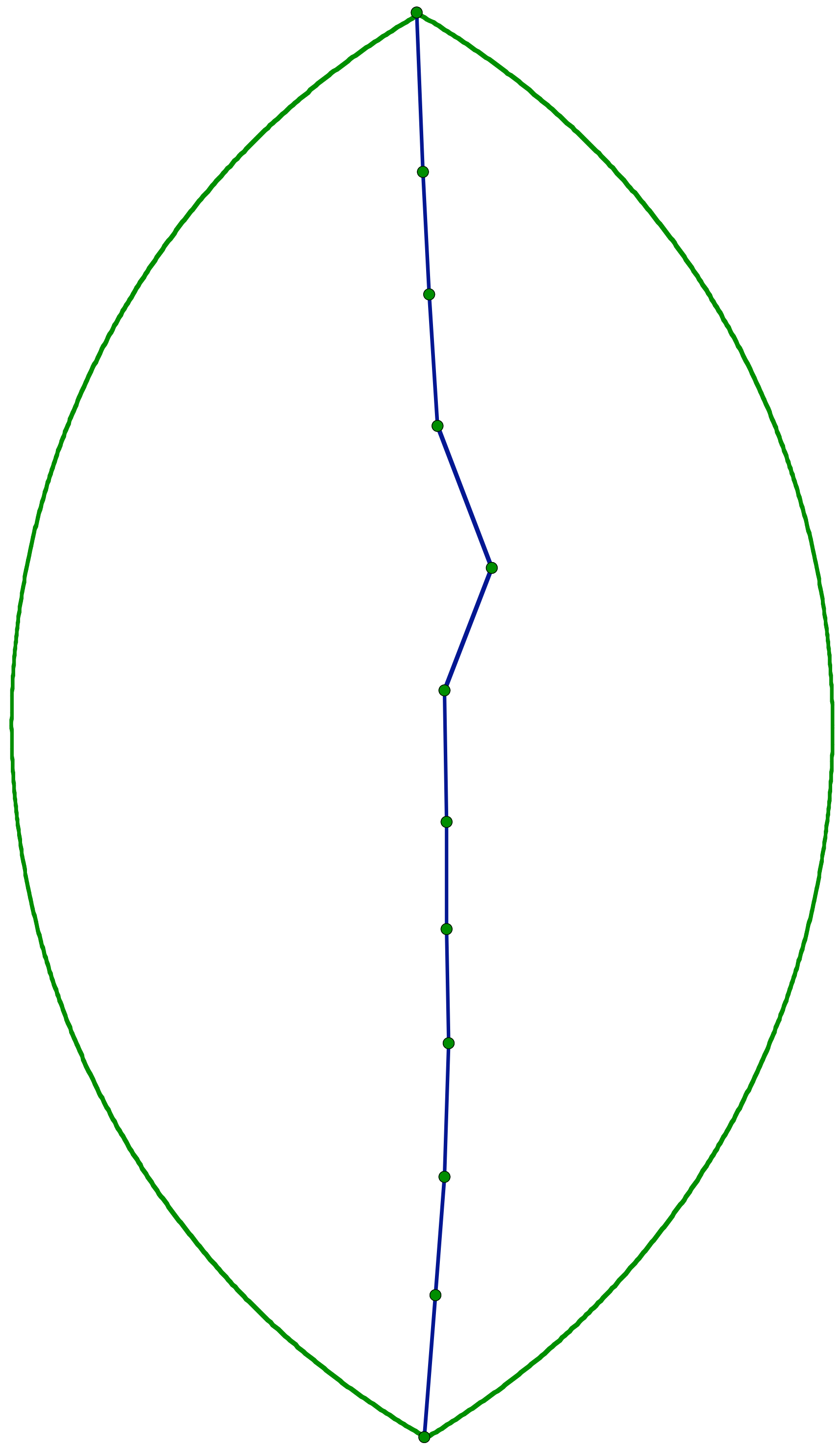




Given an abstract polytope P

Two flags are adjacent if they differ in exactly on one face.



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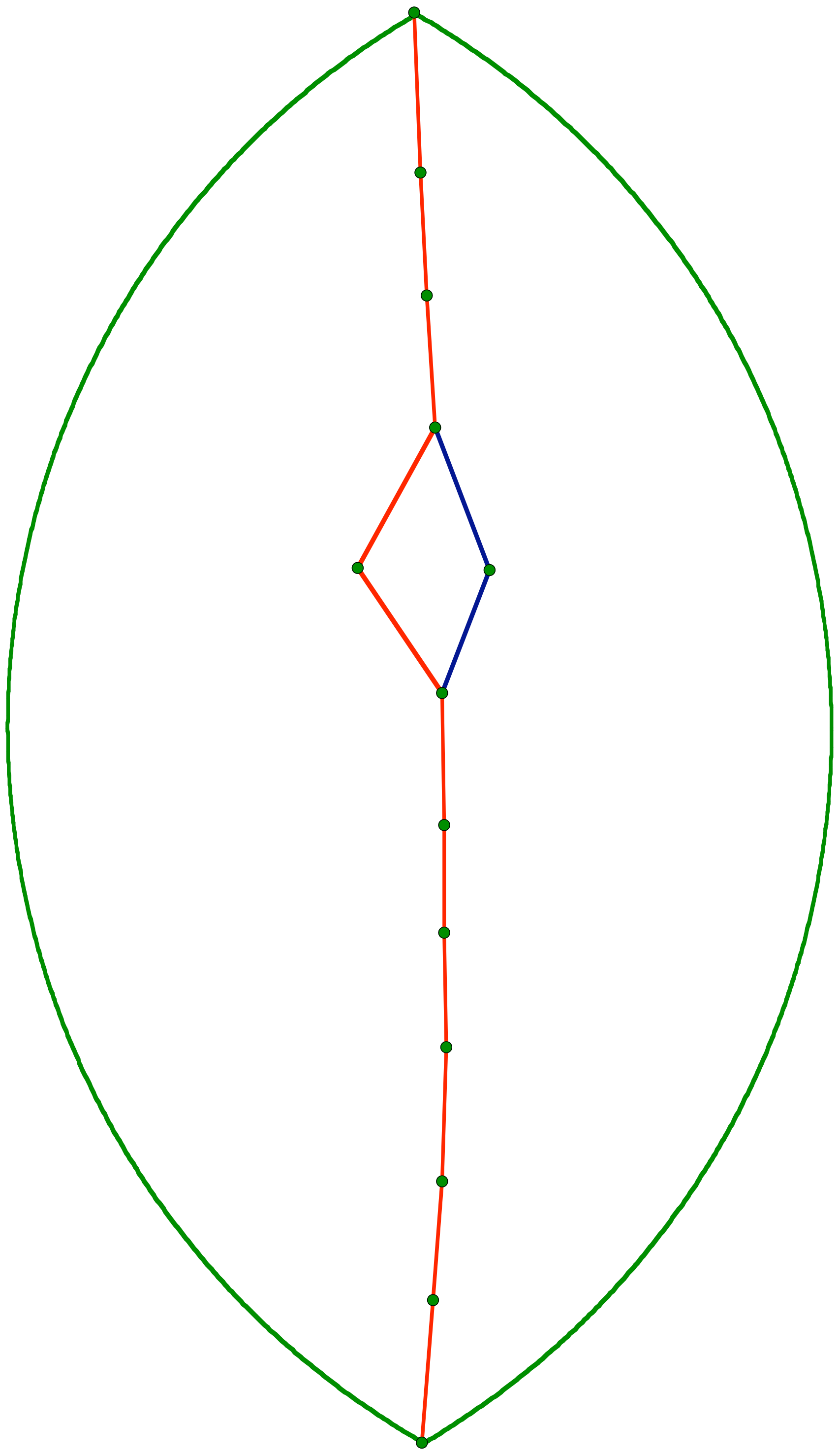


Given an abstract polytope P

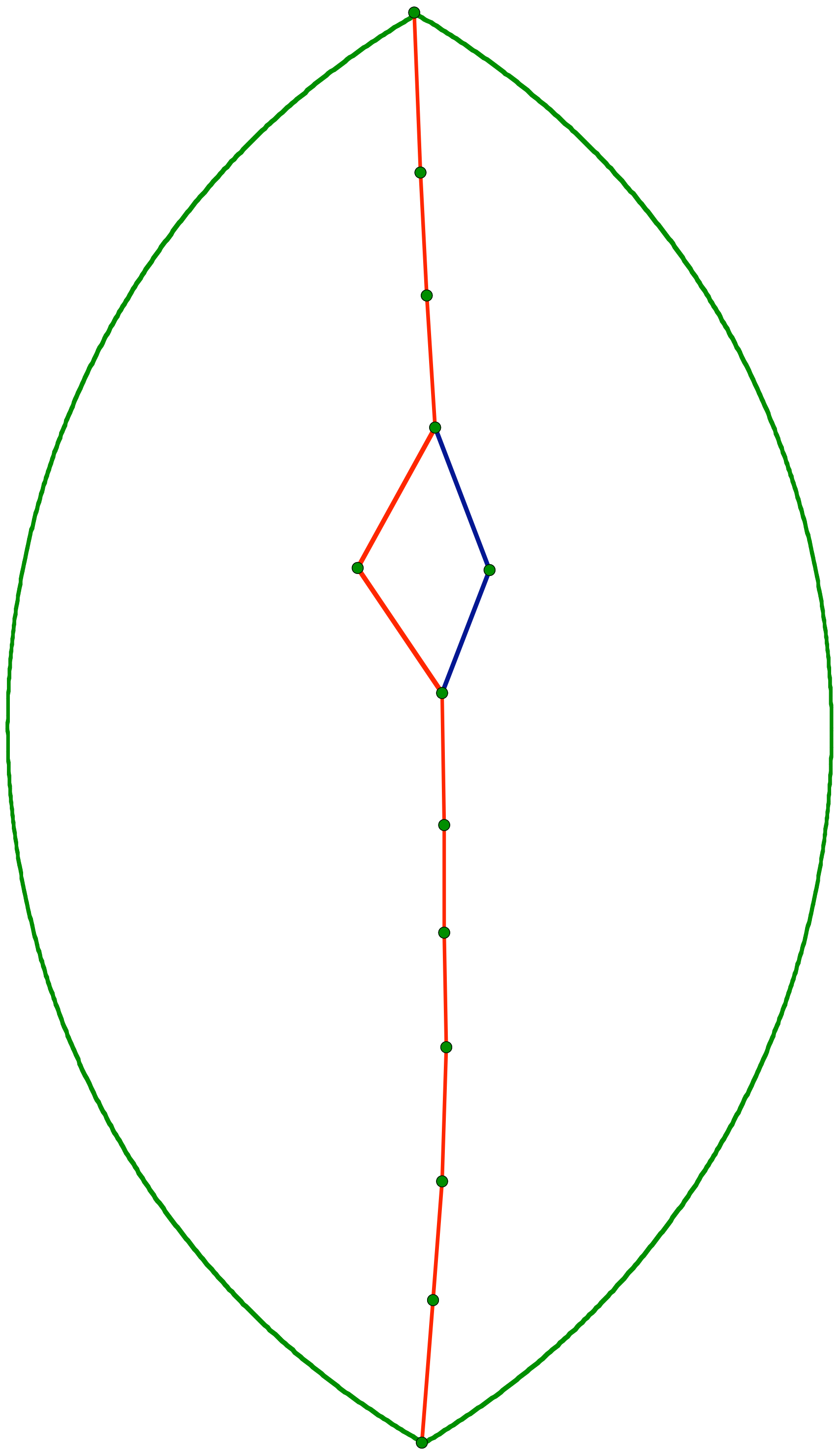
Two flags are adjacent if they differ in exactly on one face.

Φ

Flag

Φ^i

Its (unique!) i-adjacent



Symmetries

An automorphism of a polytope P is a order preserving bijection of P.

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An automorphism of a polytope P is a bijection of the set of flags of P that preserves the incidences.

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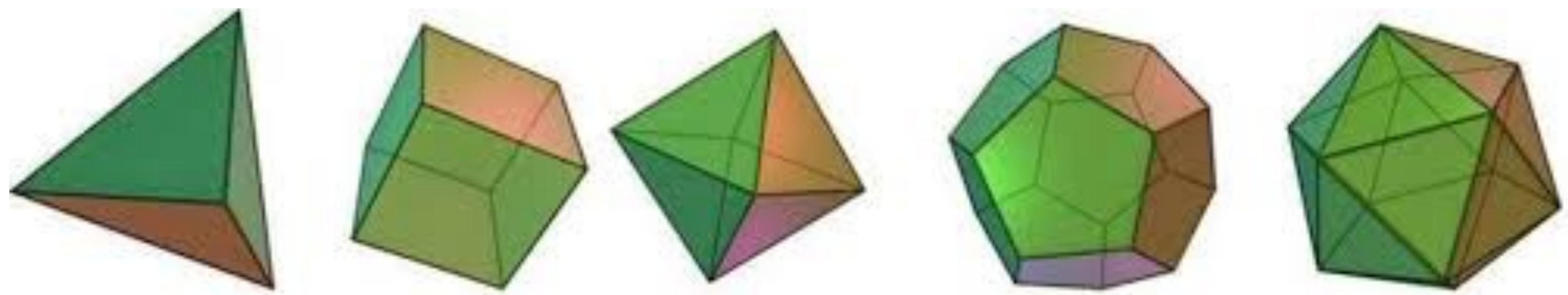
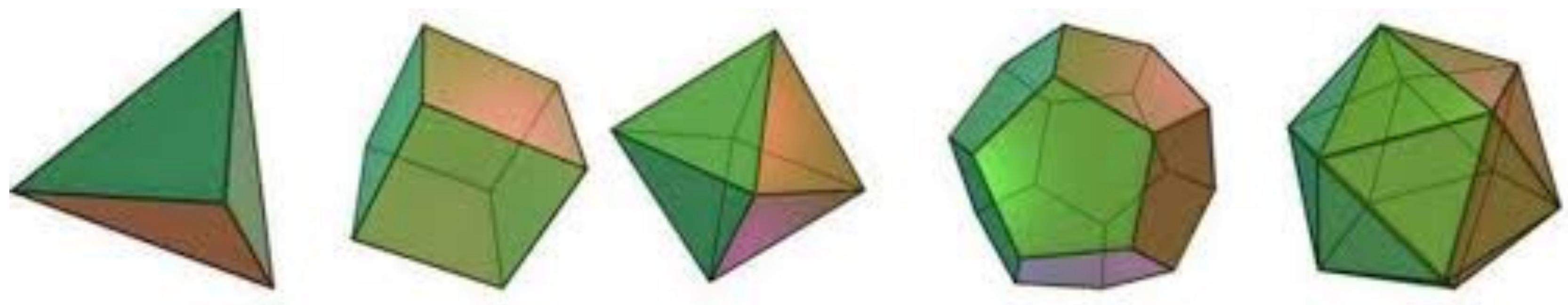
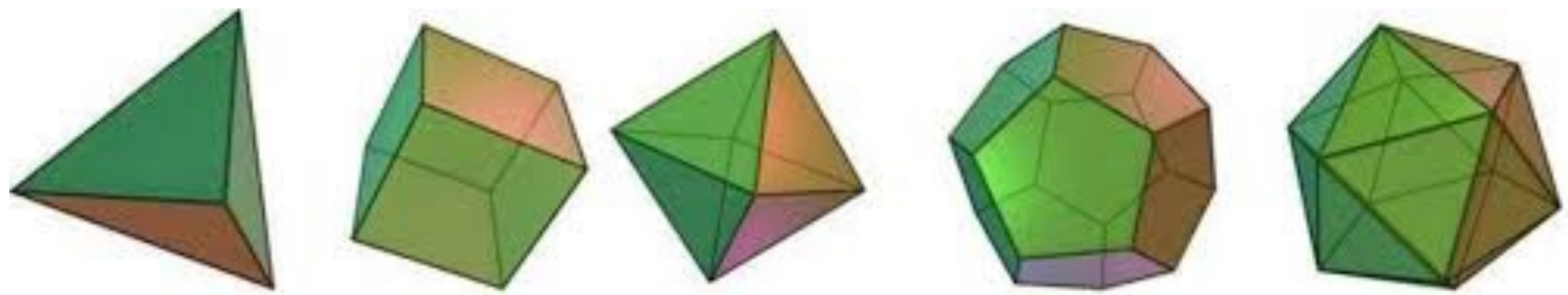
An automorphism of a polytope P is a bijection of the set of flags of P that preserves the incidences.

We often study automorphisms through their action on the flags of the polytope.





A polytope is regular if its automorphism group acts transitively on the flags.



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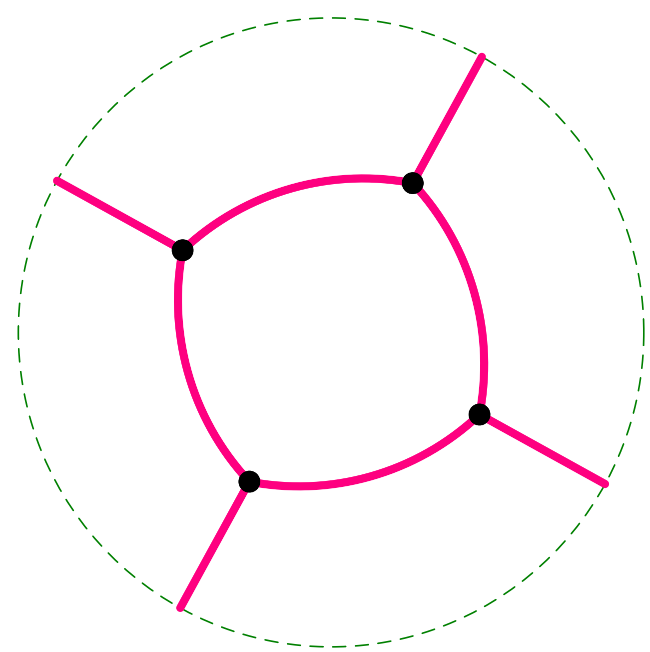

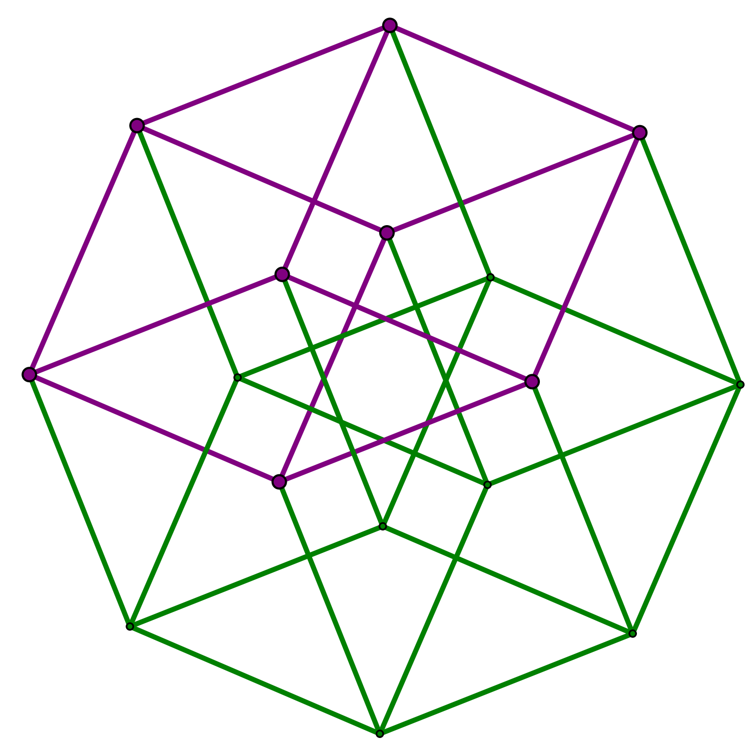

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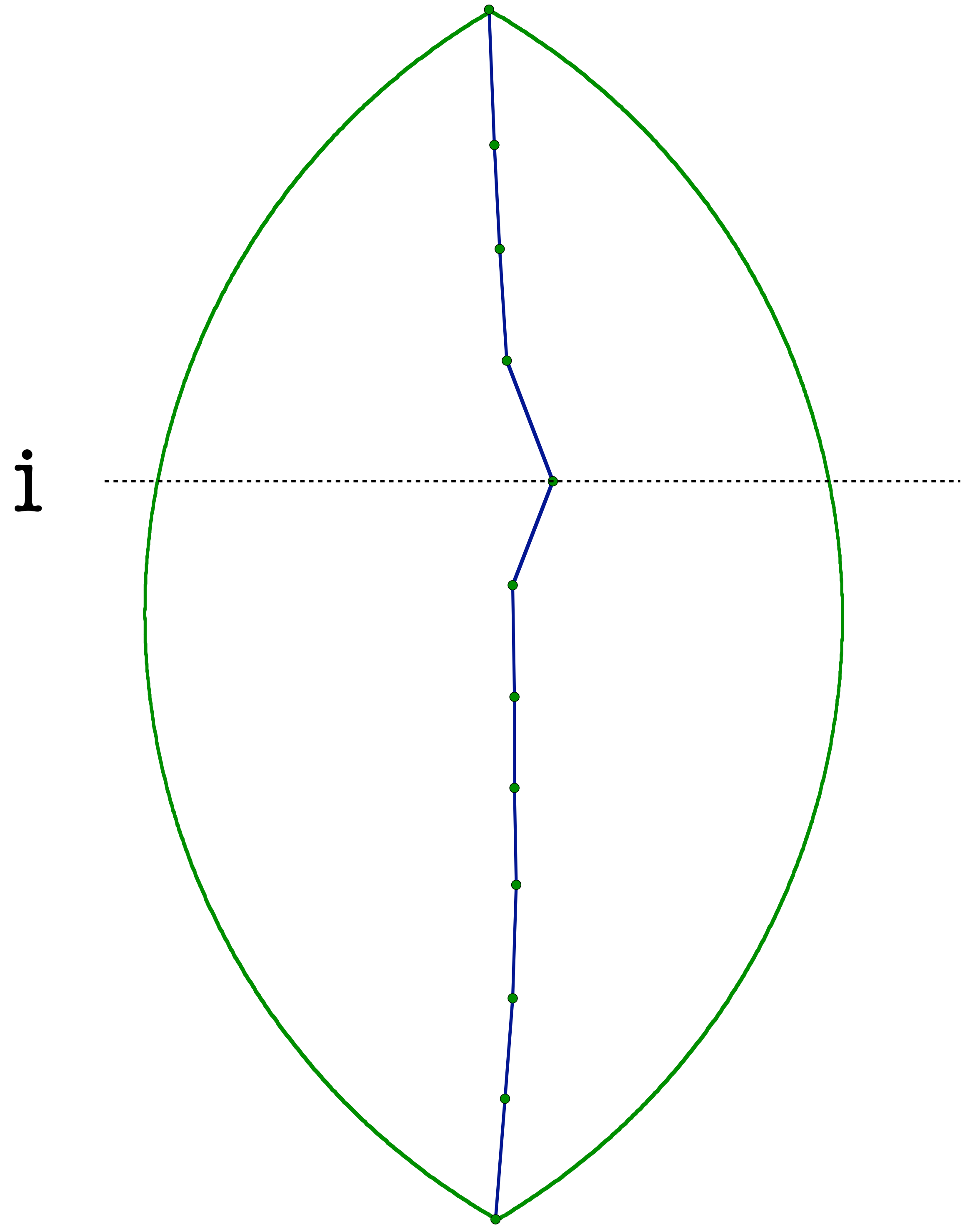

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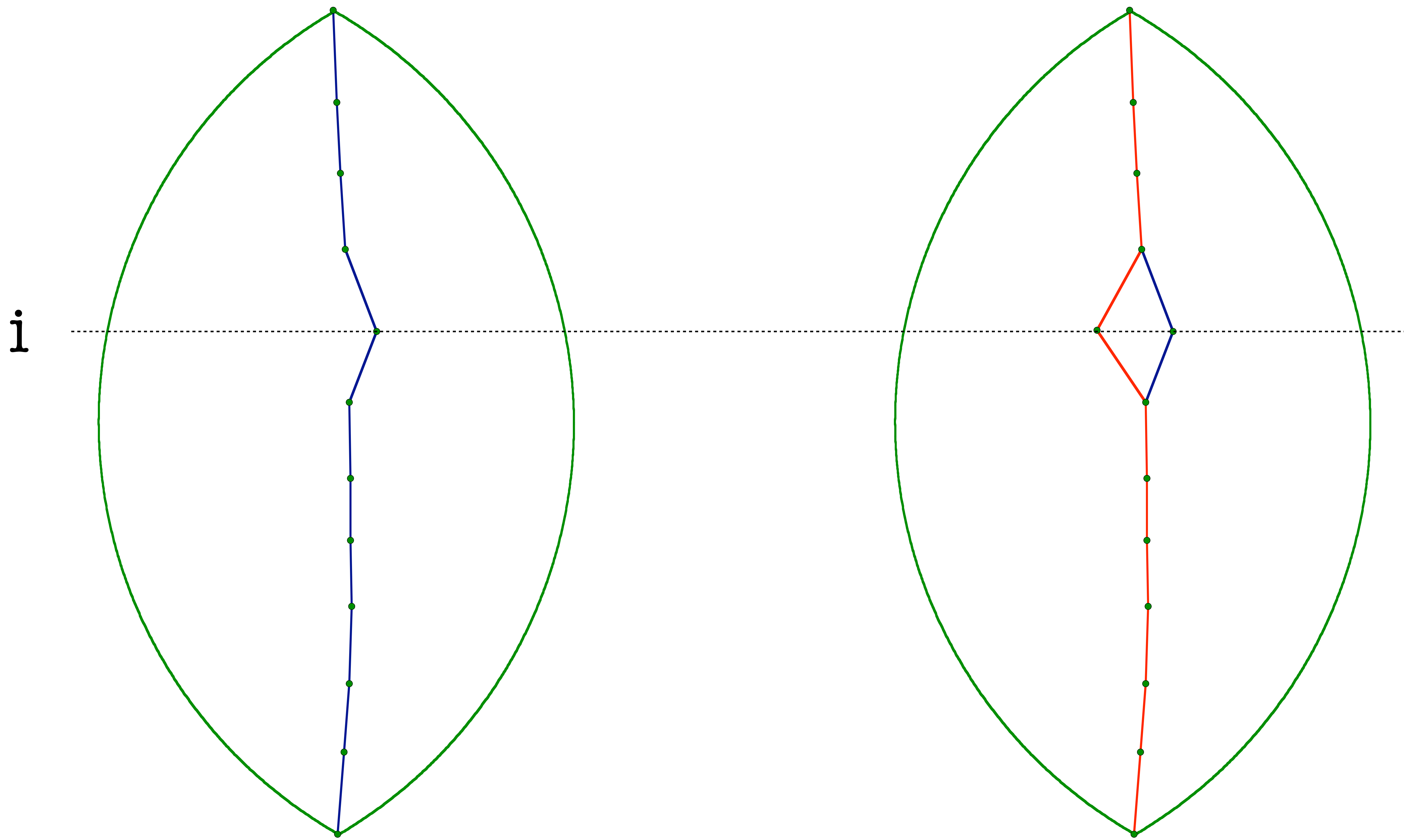

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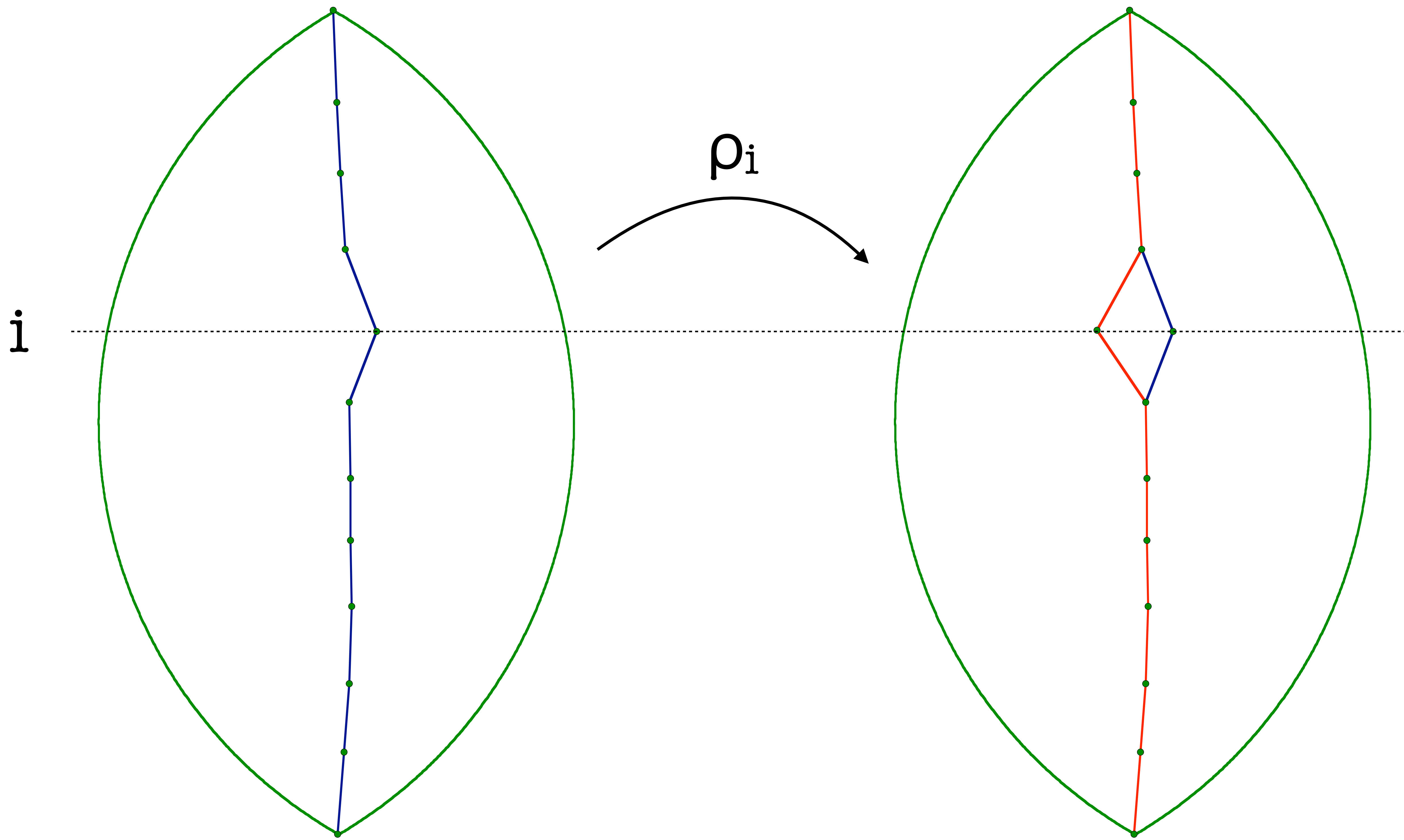
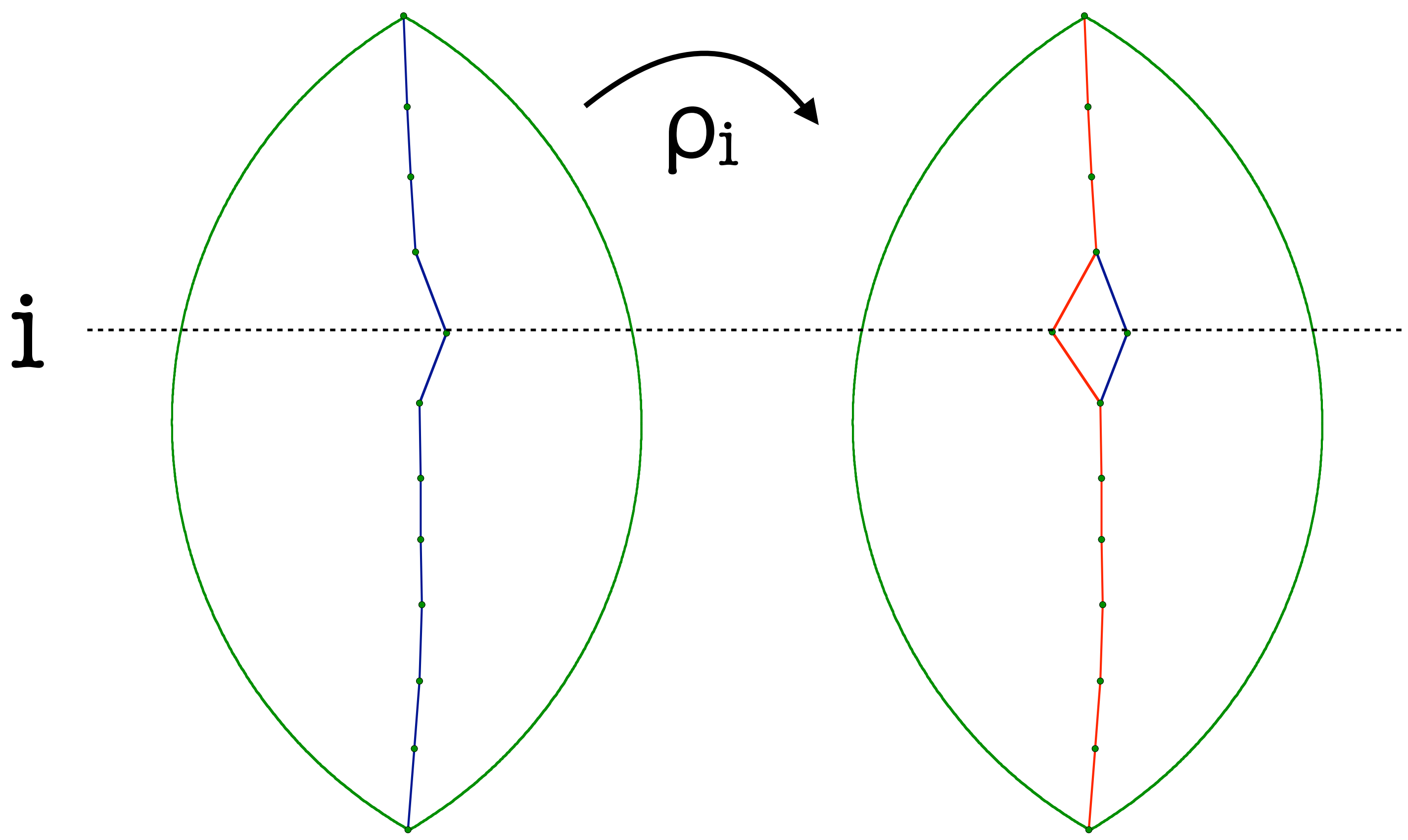
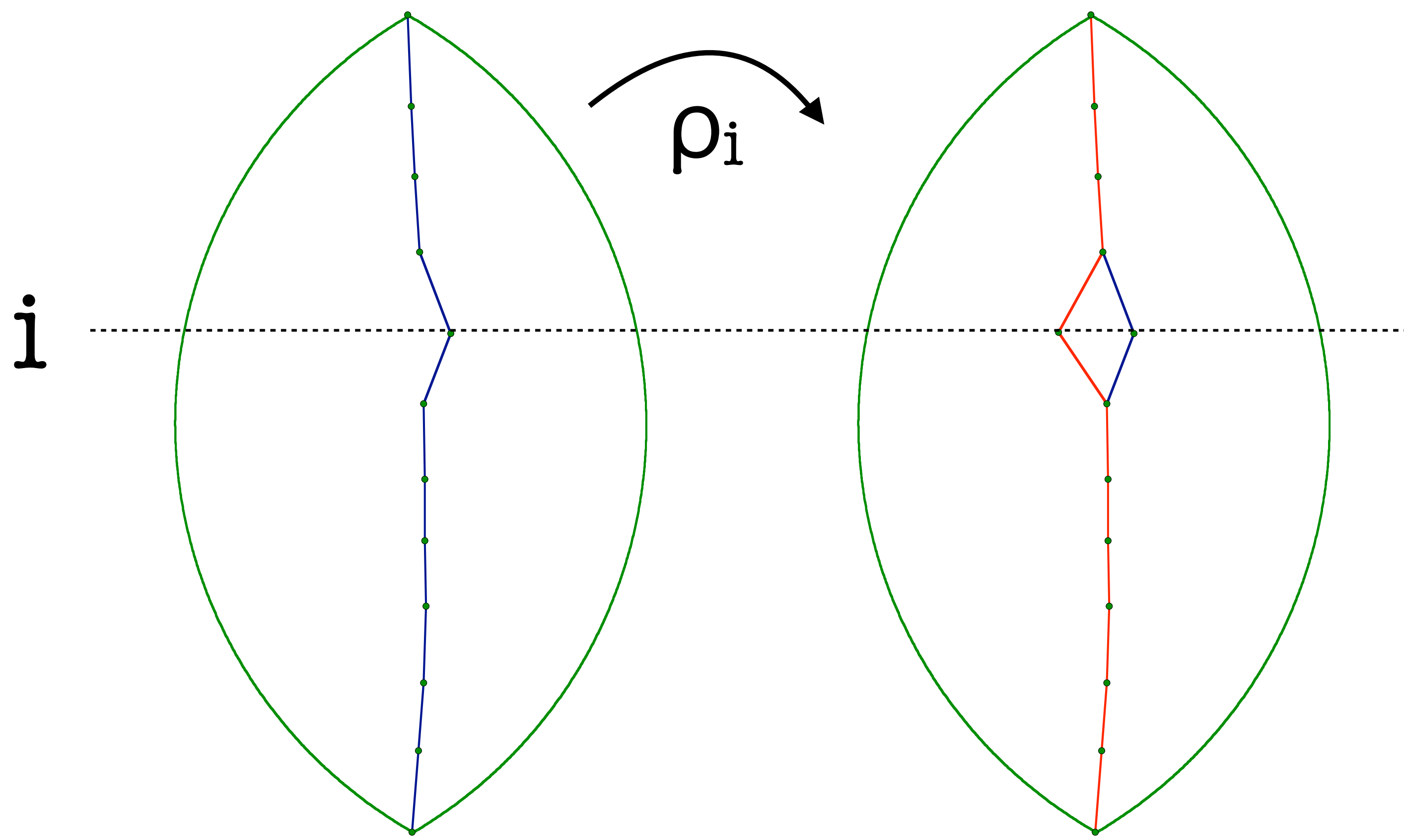


If a polytope is regular, fixing a base flag Φ, there exist automorphisms ρ_i, for each i, such that

$$\Phi\rho_i = \Phi^i$$





If a polytope is such that, fixing a base flag Φ, there exist automorphisms ρ_i, for each i, satisfying

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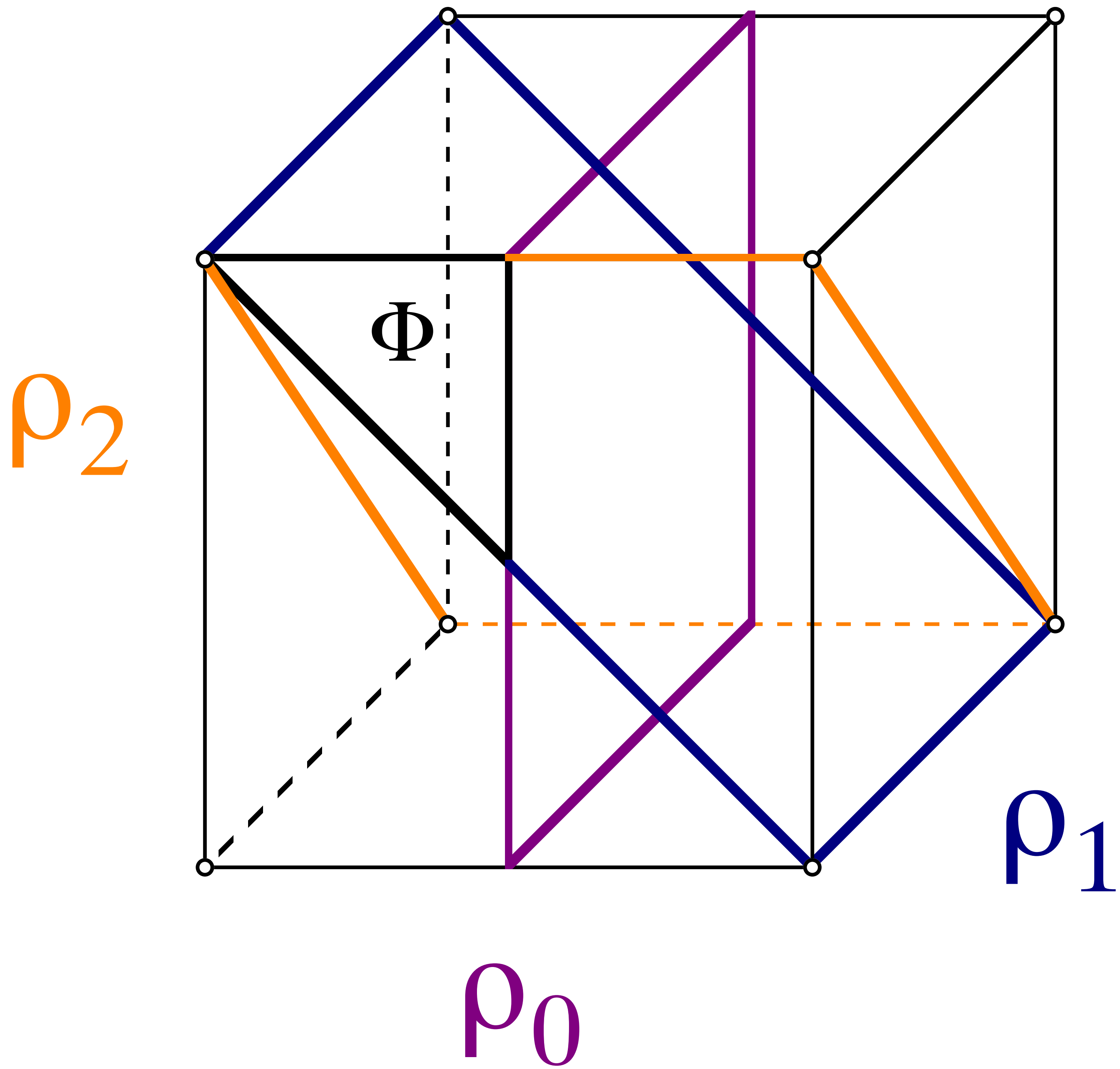
then the polytope is regular.



Φ
ρ2
ρ1
ρ0

Chirality in polytopes

A polytope is chiral if its automorphism group has two orbits on flags with adjacent flags in different orbits.

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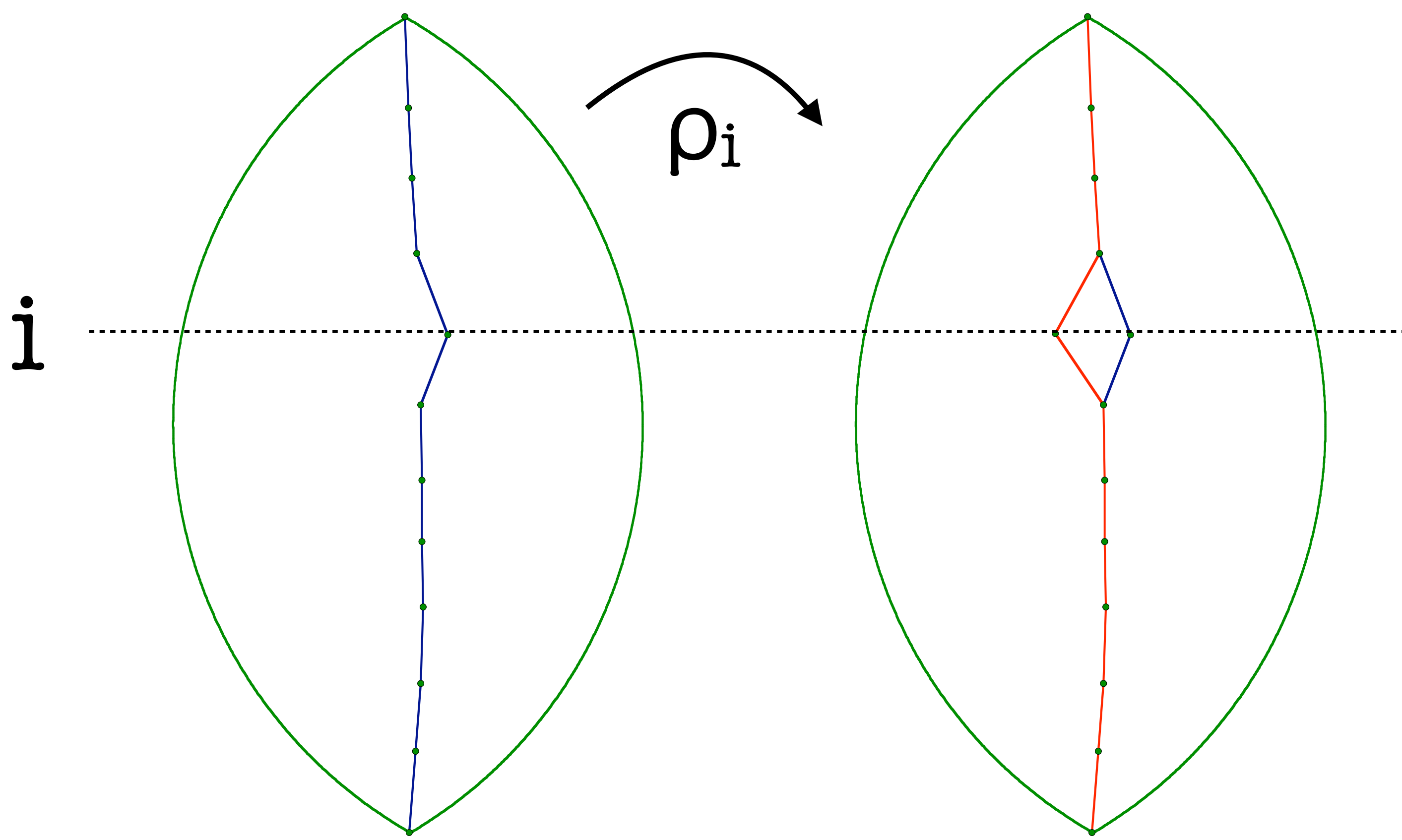
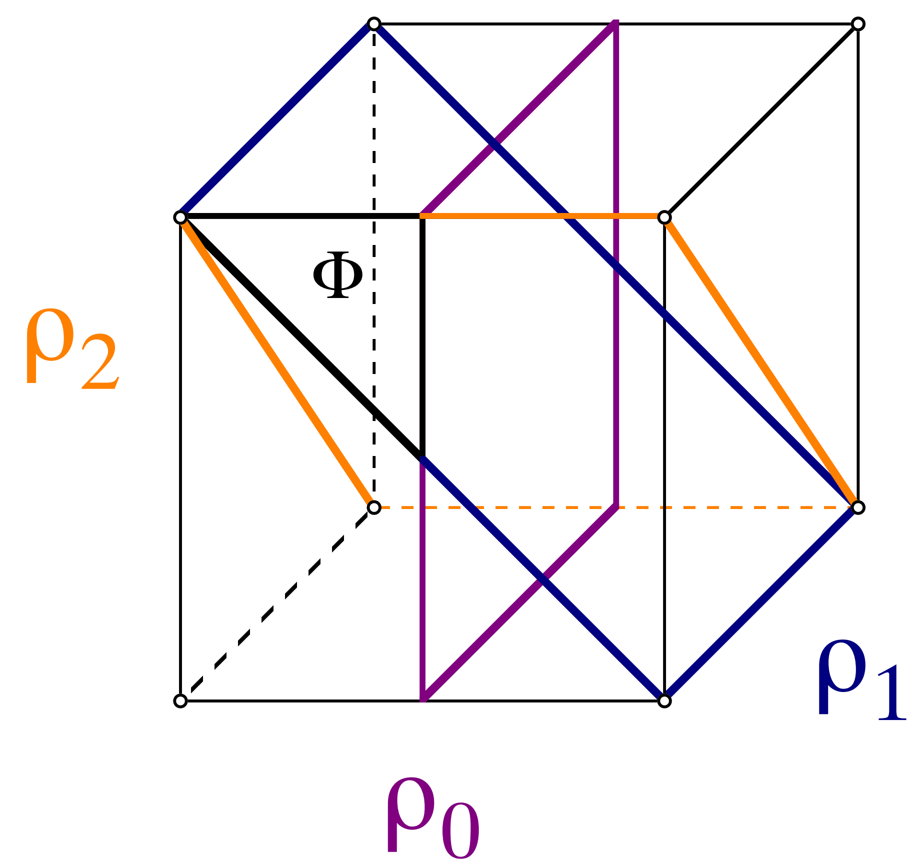
Regular case:





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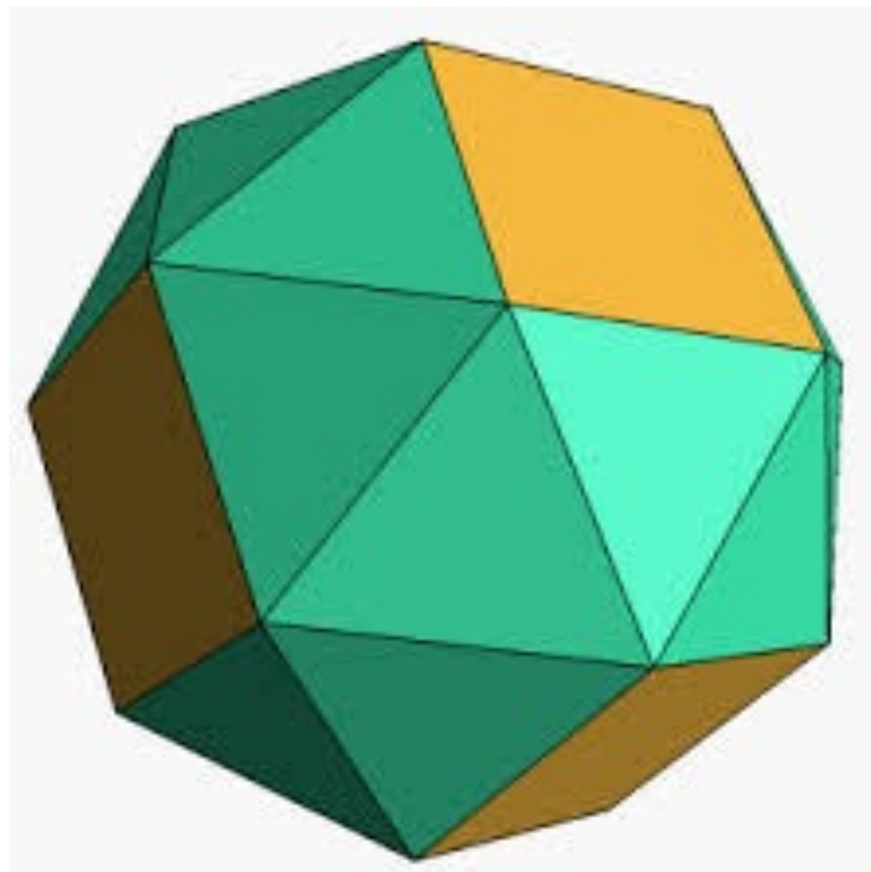
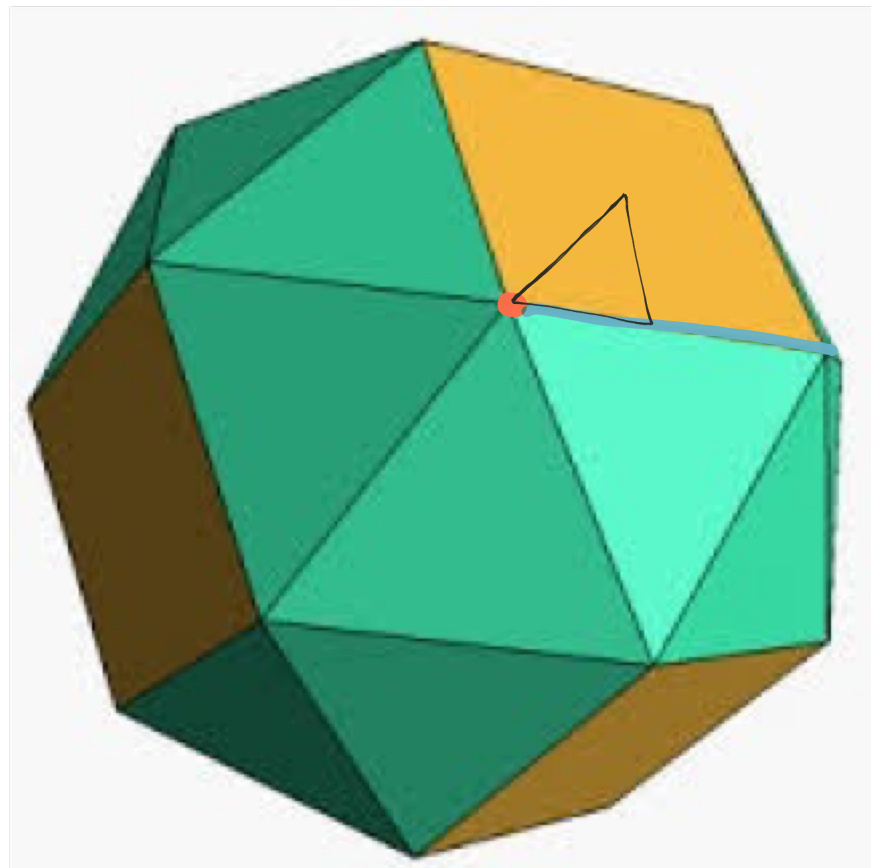


Snob cube

A polytope is chiral if its automorphism group has two orbits on flags with adjacent flags in different orbits.



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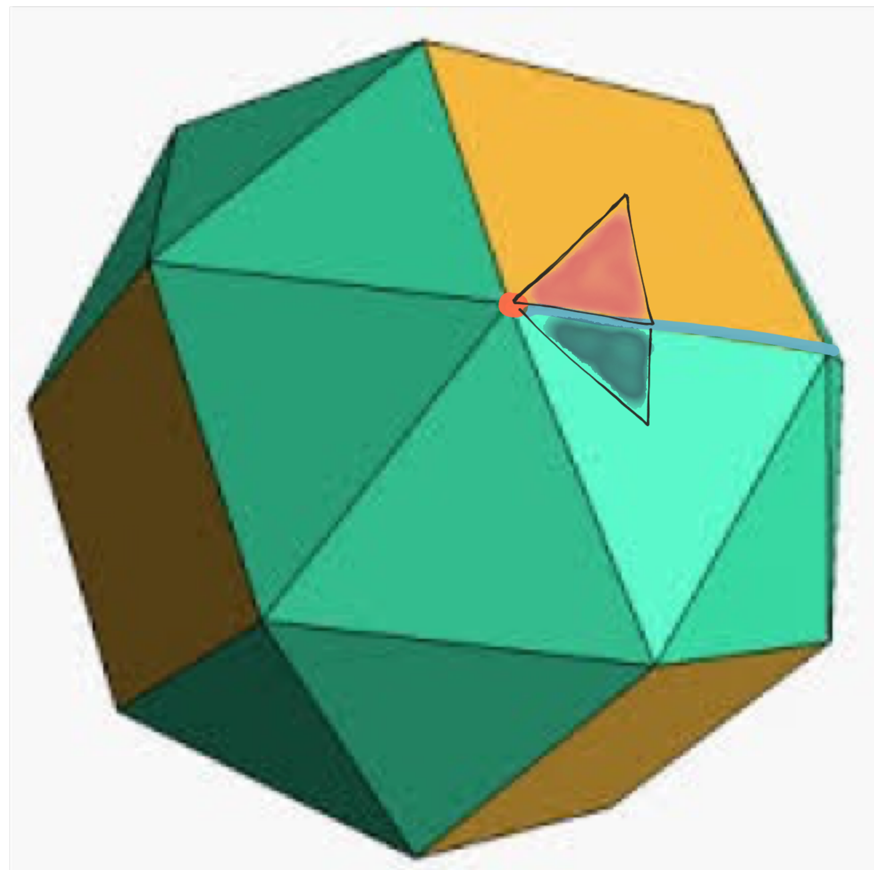

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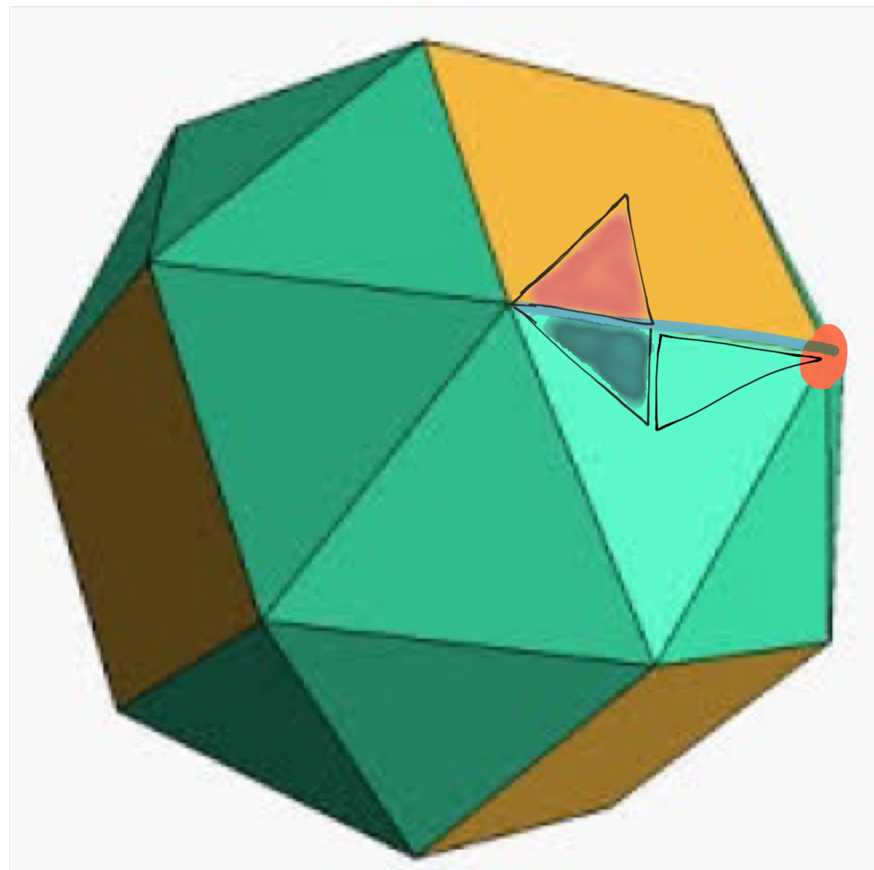

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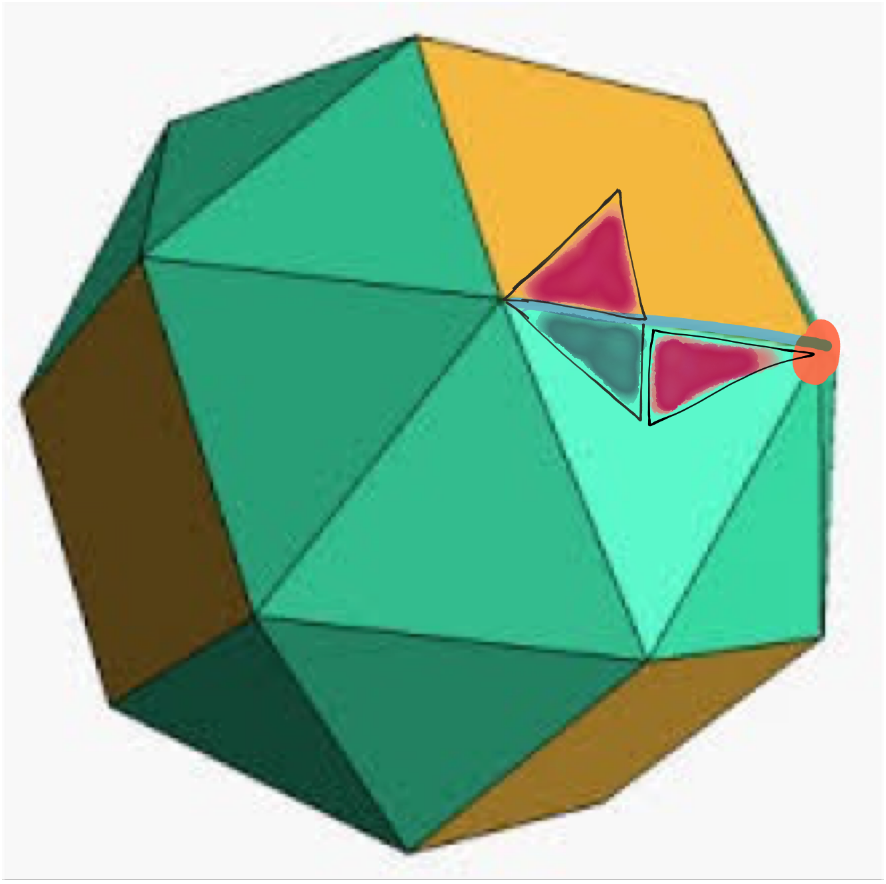

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A polytope is chiral if its automorphism group has two orbits on flags with adjacent flags in different orbits.

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All rank 2 polytopes are regular (easy to see)

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There are no finite chiral polytopes in Euclidian 3-space (Schulte)

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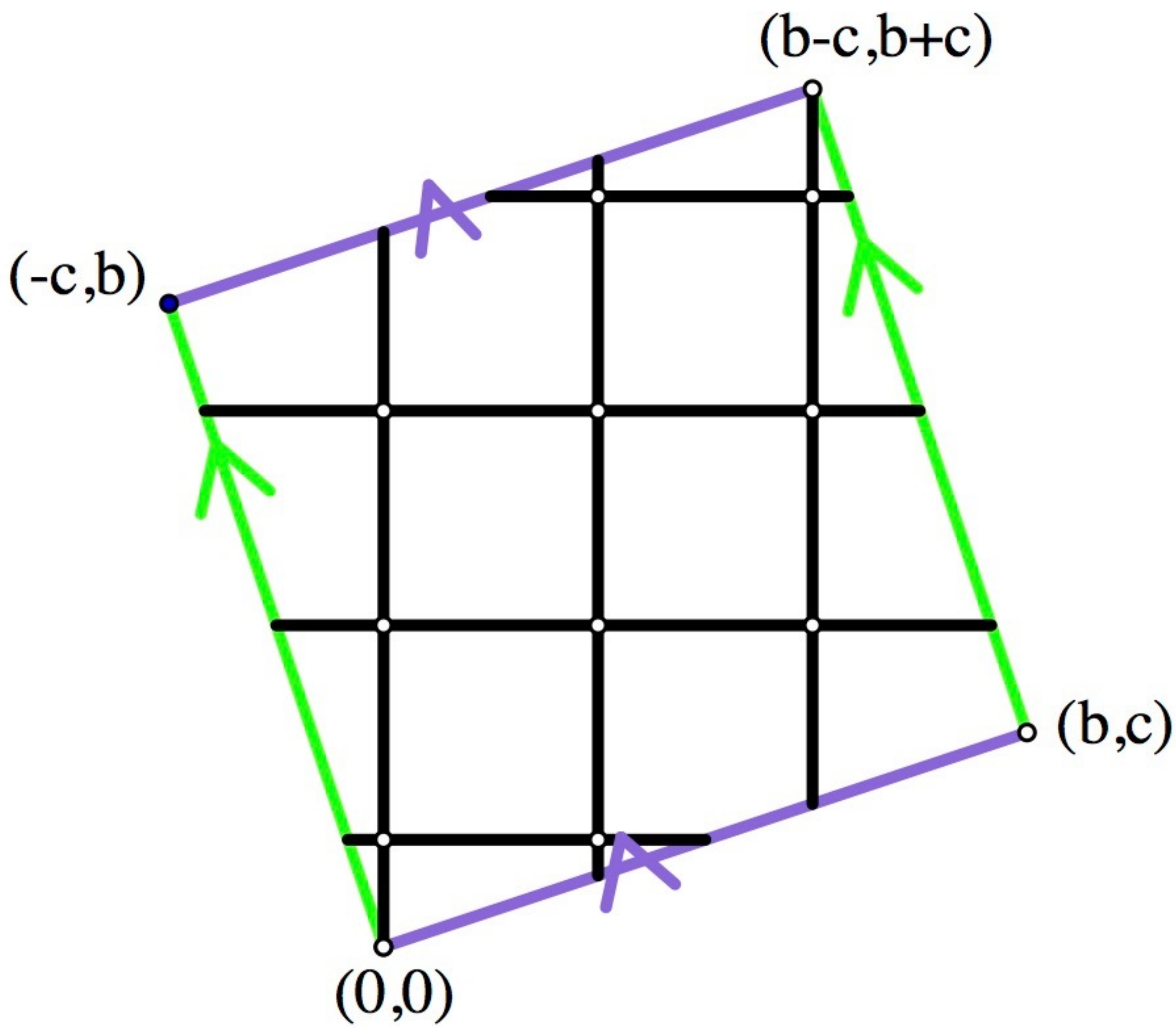
All rank 2 polytopes are regular (easy to see)

There are no finite chiral polytopes in Euclidian 3-space (Schulte)

There are no convex chiral polytopes (McMullen)

A polytope is chiral if its automorphism group has two orbits on flags with adjacent flags in different orbits



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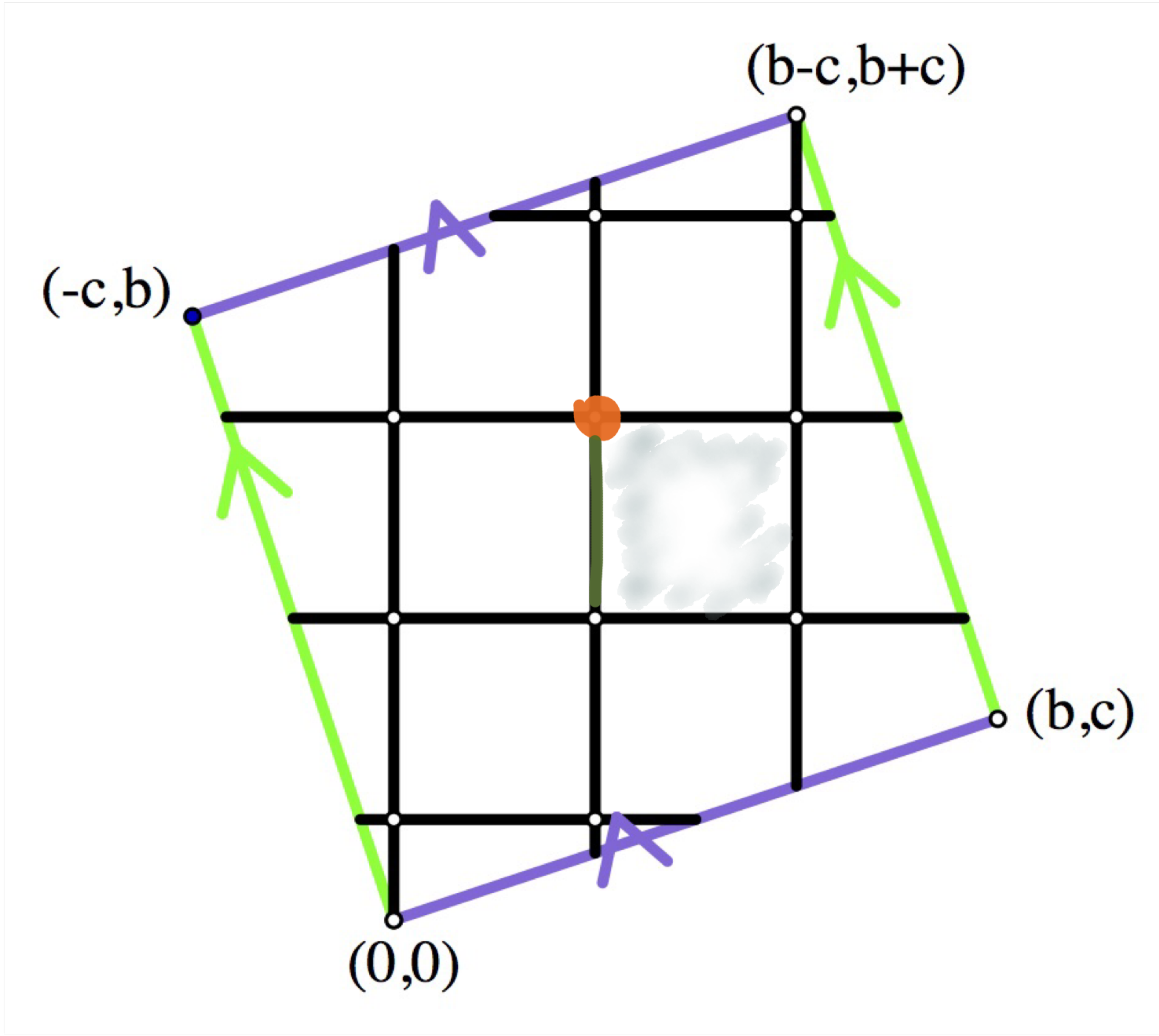

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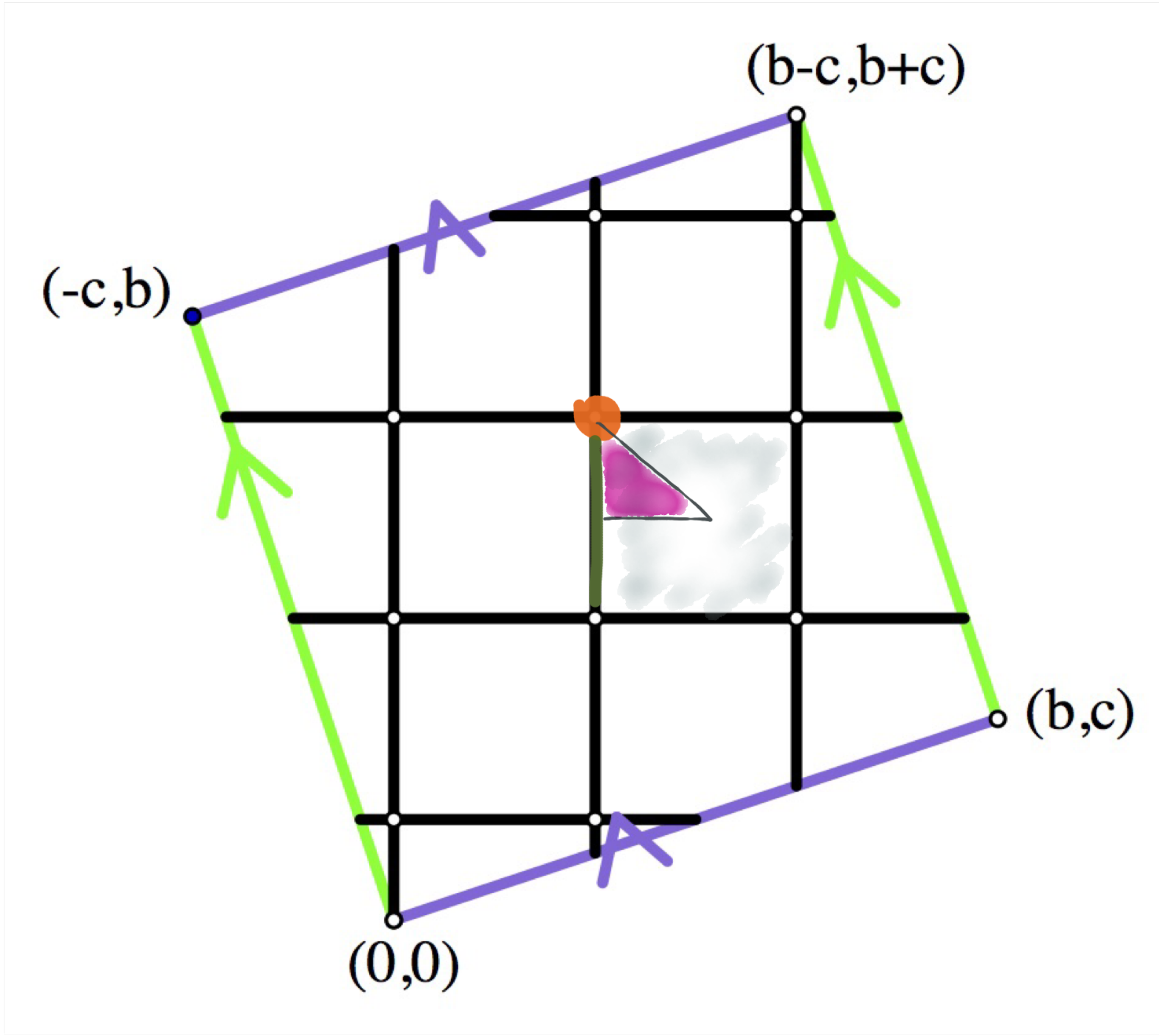

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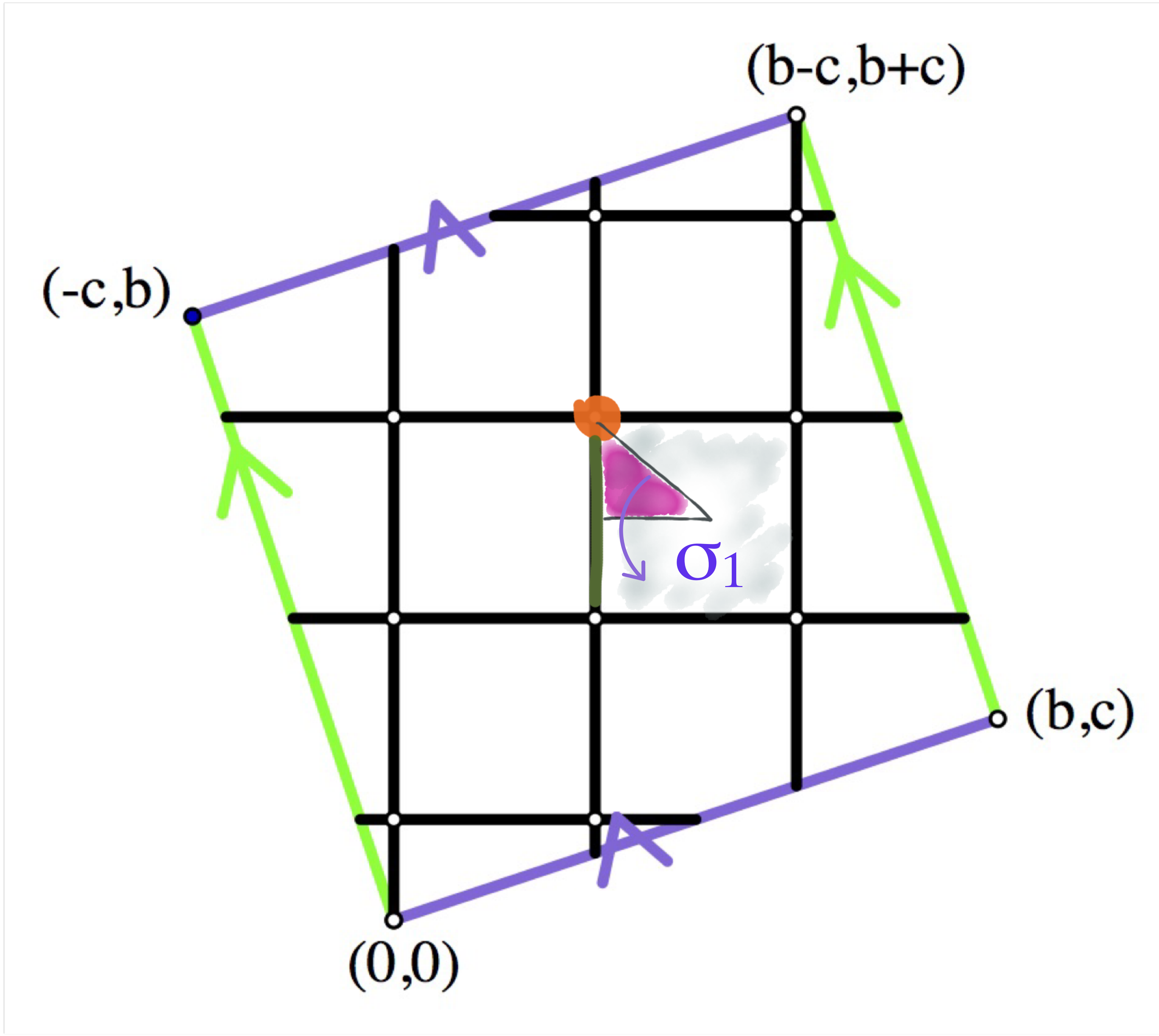

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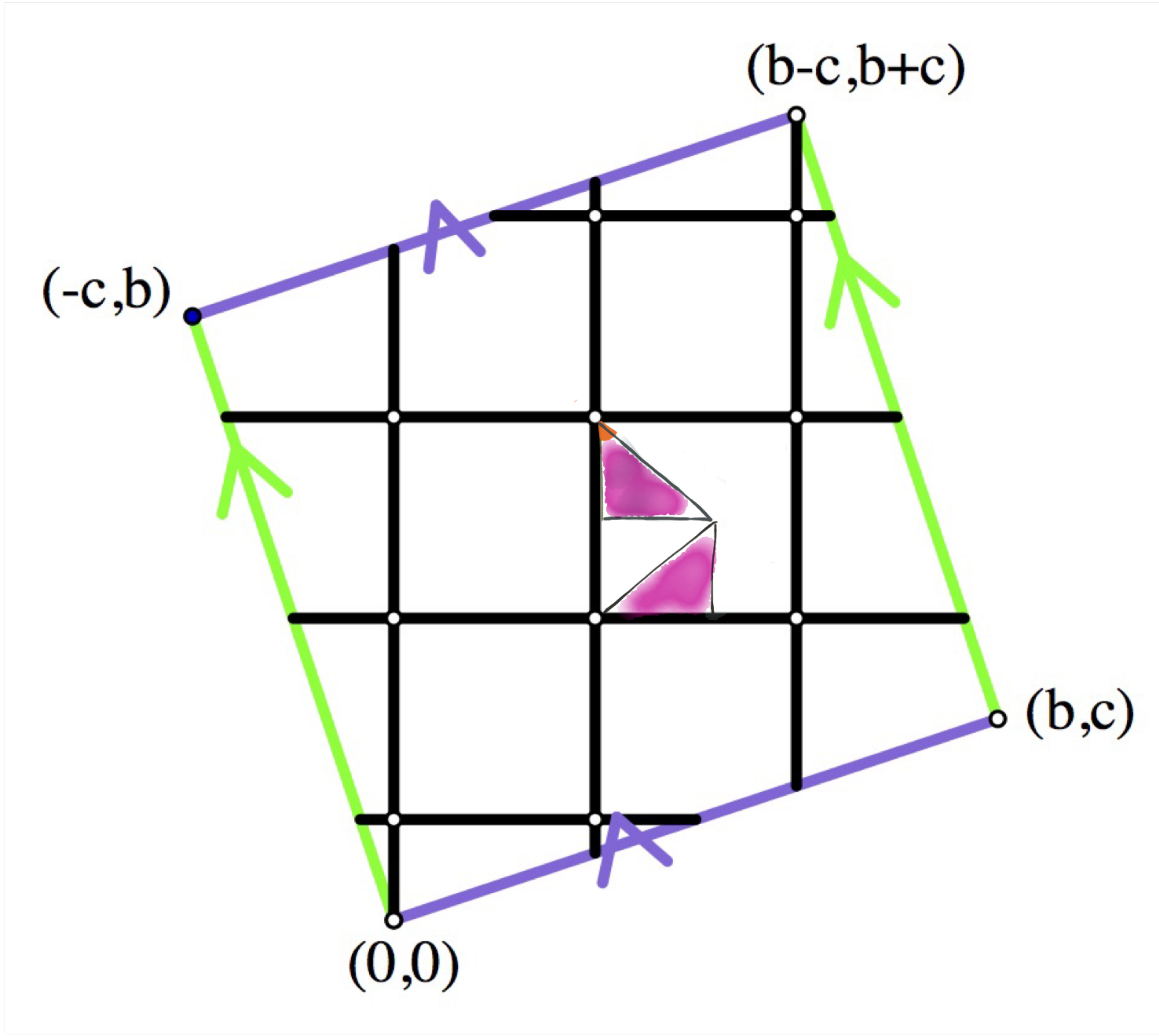

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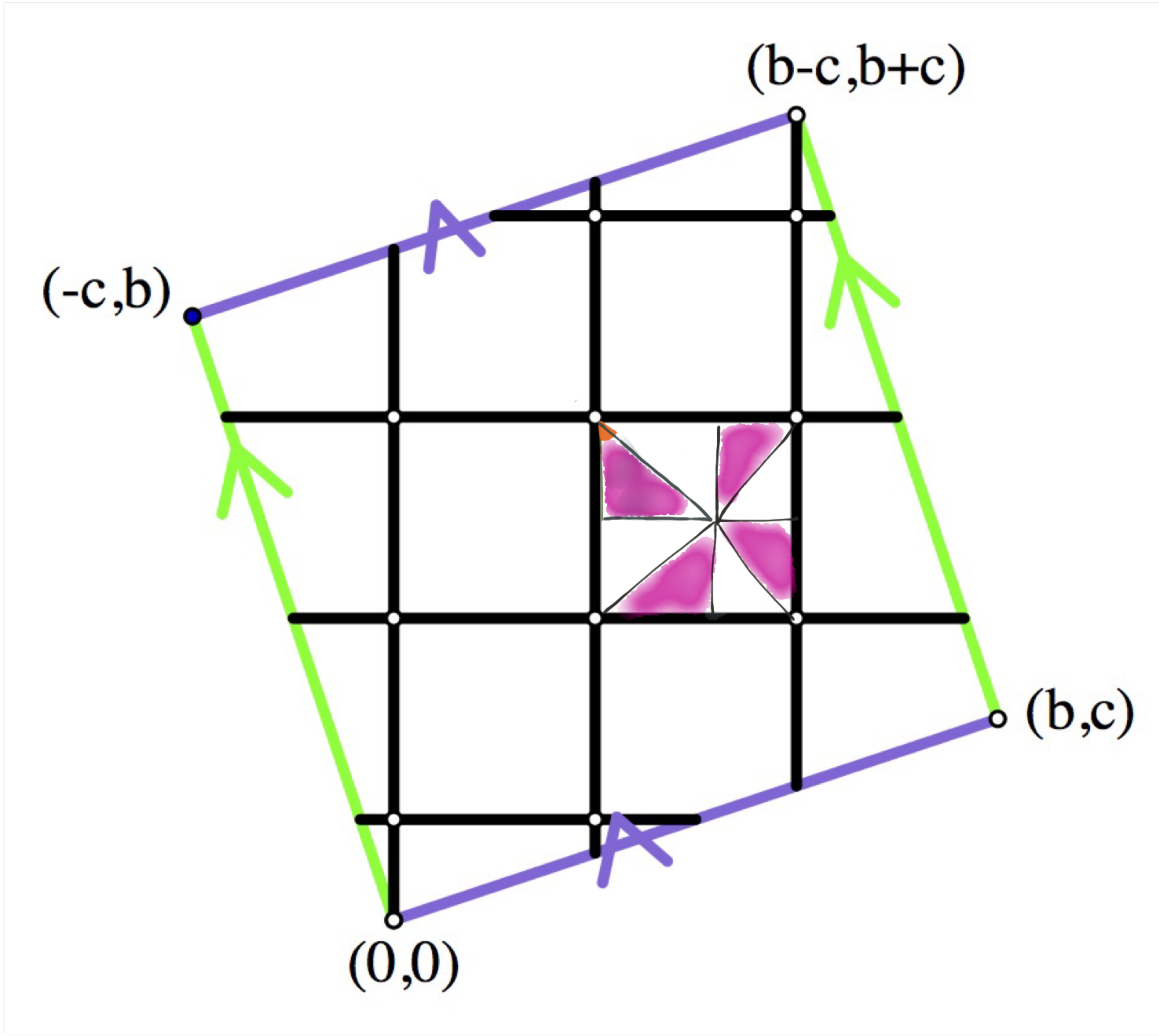

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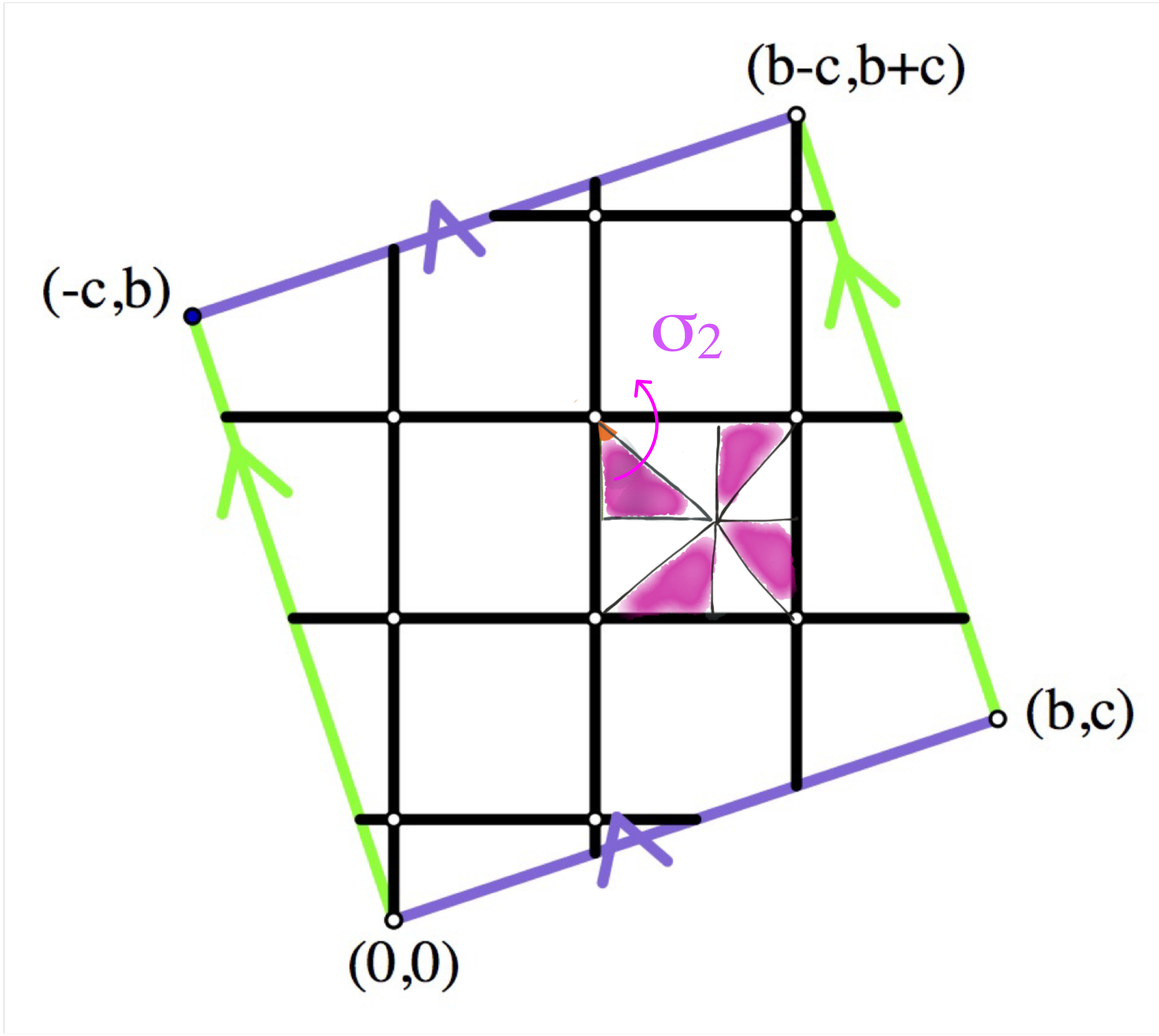

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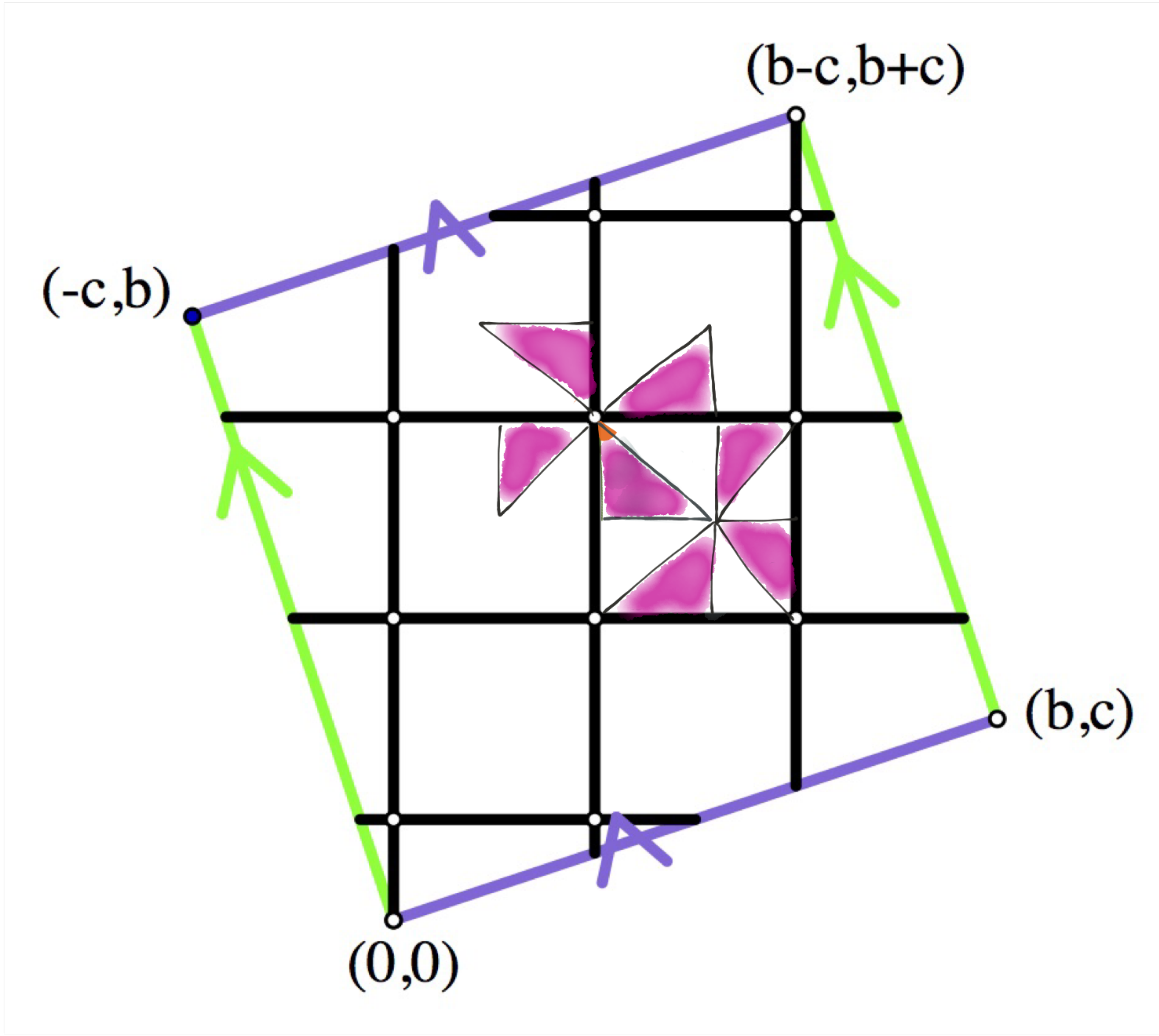

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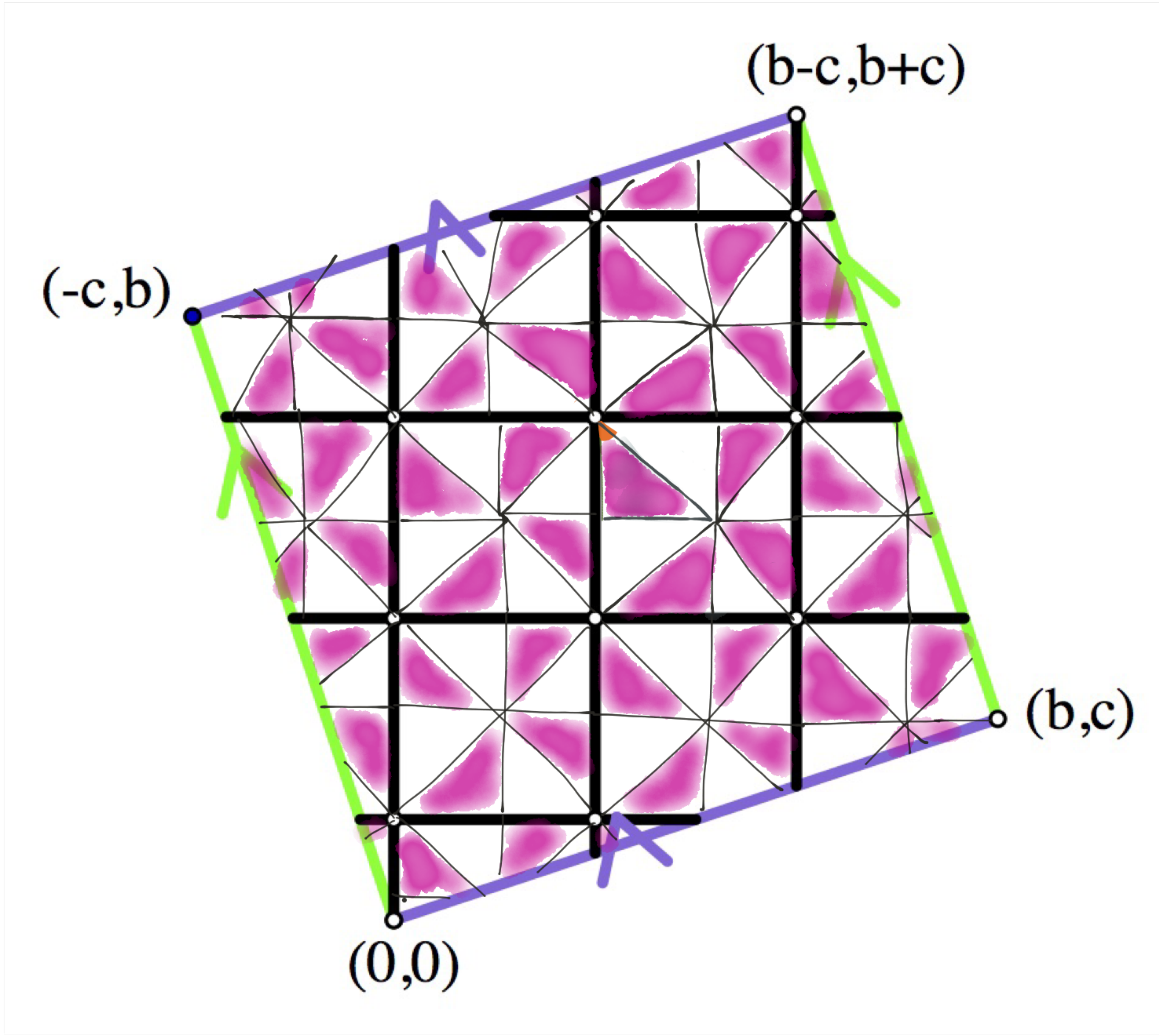

Rank 4

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During the 1990´s, Monson, Nostrand, Schulte, Weiss constructed infinite families

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Ranks 6-8
2009 Conder, Devillers

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2014 Cunningham, Pellicer.
Constructed chiral $(n+1)$-polytopes provided they have chiral n-polytopes with regular facets

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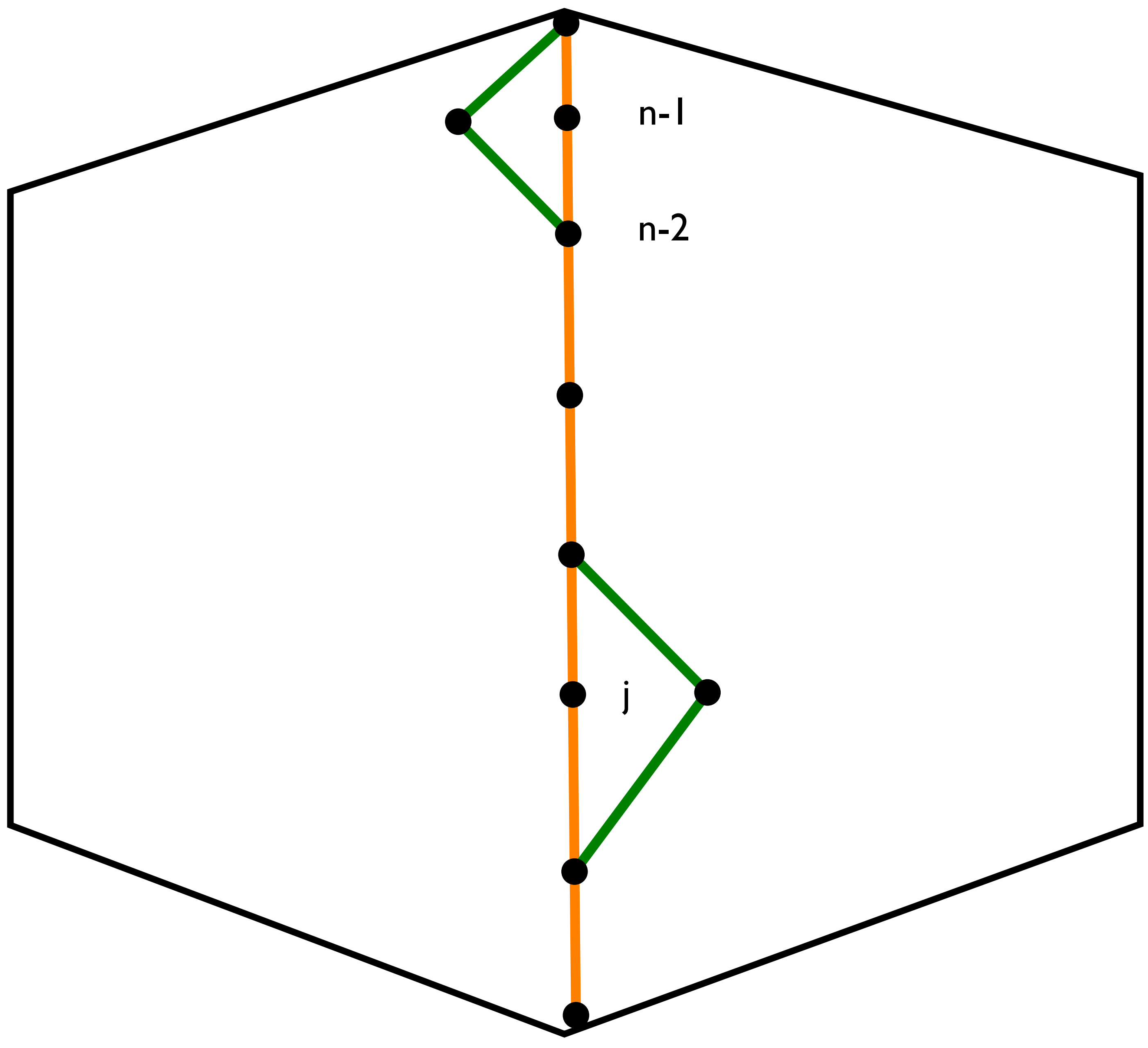

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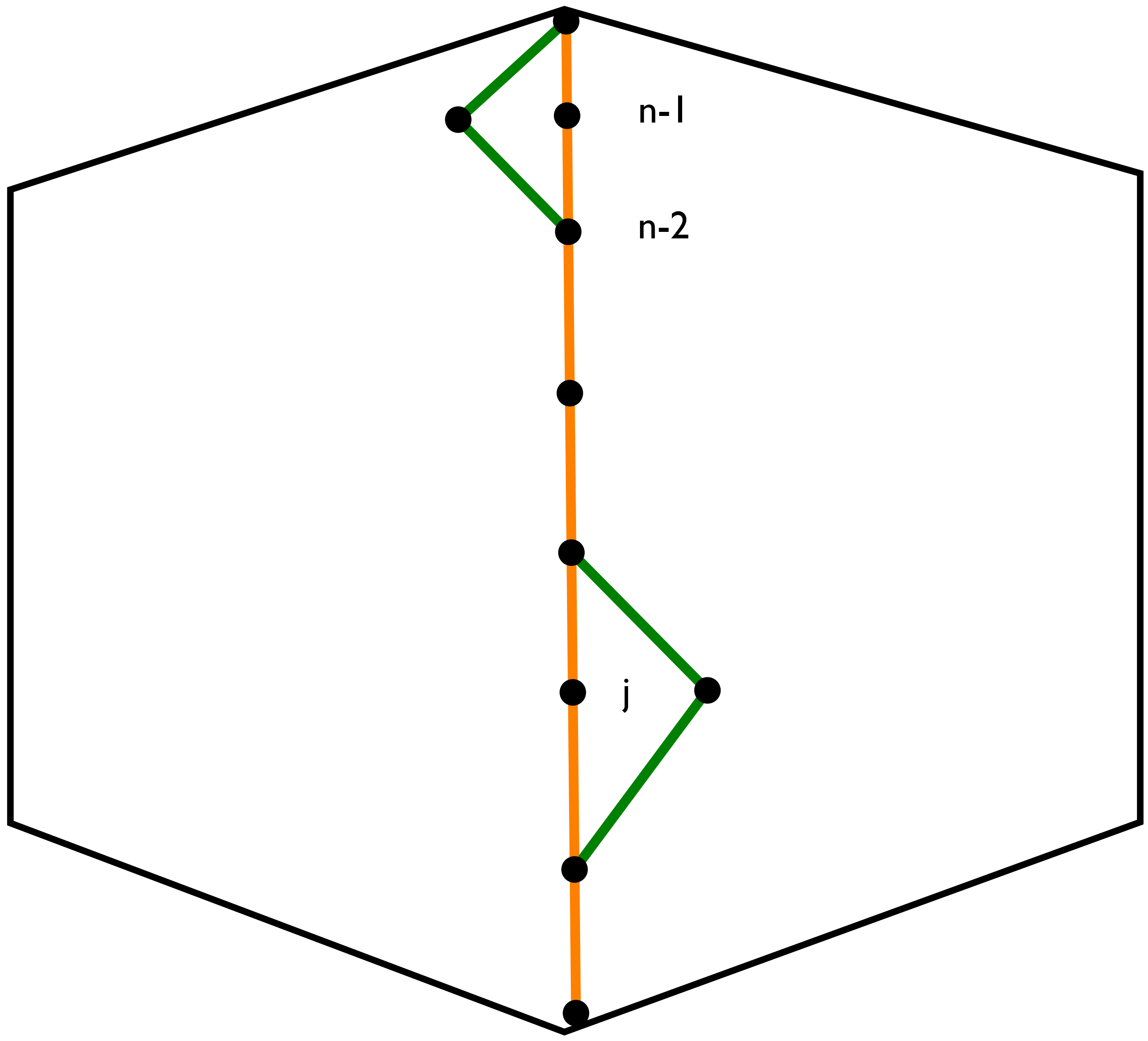

The faces of rank n-2 are always regular

A polytope is chiral if its automorphism group has two orbits on flags with adjacent flags in different orbits

The automorphism of a chiral n-polytope P can be generated by $\sigma_1, \ldots, \sigma_{n-1}$ such that

$$(\sigma_i \ldots \sigma_j)^2 = \varepsilon \quad \text{for} \quad i<j$$

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1991. Schulte and Weiss
Given a group Γ with distinguished generators

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and satisfying certain “intersection conditions”, one can construct an n-polytope with Γ acting on it.

The resulting polytope is either chiral or regular

Some open questions

- Given a (finite) group Γ, is there a chiral polytope having Γ as its automorphism group?

- Given a (finite/simple) group Γ, can one determine all chiral polytopes having Γ as automorphism group?

- Given a (finite) regular n-polytope P, is there a (finite) chiral polytope whose facets are all isomorphic to P? Can one classify them all?

Some open questions

- For each dimension n, is there a finite "geometrically chiral" n-polytope in R^n? Can one classify them all?

- Can one classify all chiral (n-1)-polytopes in R^n?

- Given a graph G, is there a chiral polytope having G as its 1-skeleton? Can one classify them all?

Some open questions

- The smallest chiral polytopes are known for ranks 3, 4 and 5. What are is the smallest chiral polytope of rank 6? Of rank n?

- How prevalent is chirality (vs. regularity) among n-chiral polytopes? (or among polytopes with certain properties, for example, with a given automorphism group or with a given 1-skeleton)

- For each dimension n, is there a finite "geometrically chiral" n-polytope in R^n?

In a work with Javier Bracho and Daniel Pellicer, we found the first example of a chiral 4-polytope in R^4. (The one on the video!)

The polytope is combinatorially regular, but geometrically chiral.

It's 1-skeleton is the hypercube.

The facets are double covers of a cube.

The automorphism group is the rotational group of the hyper-cube.

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(work with Marston Conder, Eugenia O'Reilly and Daniel Pellicer)

Recently we showed that:

For all but finitely many n, both S_n and A_n are the automorphism group of a chiral 4-polytope

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Recently we showed that:
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We are working on showing that that:
Given d>4, for infinitely many n, both S_n and A_n are the automorphism group of a chiral d-polytope

- How prevalent is chirality (vs. regularity) among n-chiral polytopes with Suzuki simple groups $Sz(q)$?

In a work with Dimitri Leemans we showed that:

- there are no chiral n-polytopes for $n>4$, with automorphism group $Sz(q)$.
- if $a(q)$ is the number of regular 3-polytopes with $Sz(q)$, and $b(q)$ the number of chiral ones, then

$$b(q) = O(q \cdot a(q))$$

THANK YOU!