Compatibility fans realizing graph associahedra

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Journées du GDR-IM CombAlg
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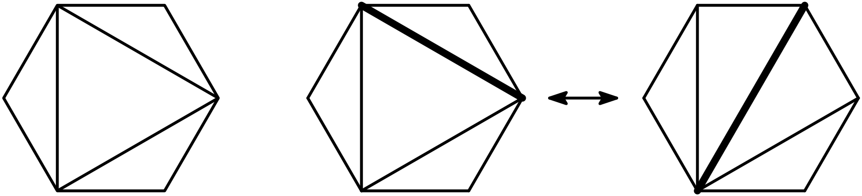
The flip operation

Flip graph on the triangulations of the polygon:

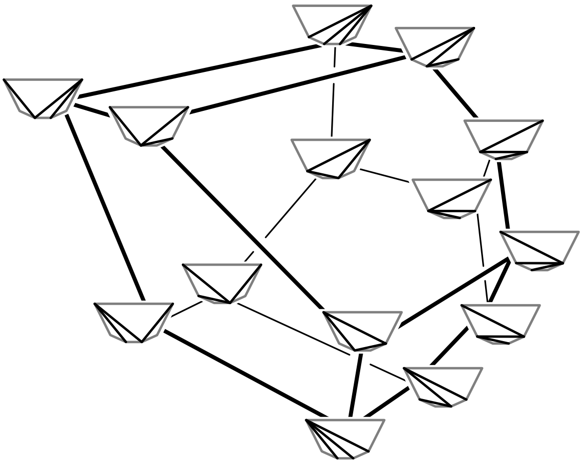
Vertices: *triangulations* Edges: *flips*



$(n+3)$-gon $\Rightarrow$ n diagonals $\Rightarrow$ the flip graph is n-regular.

Definition

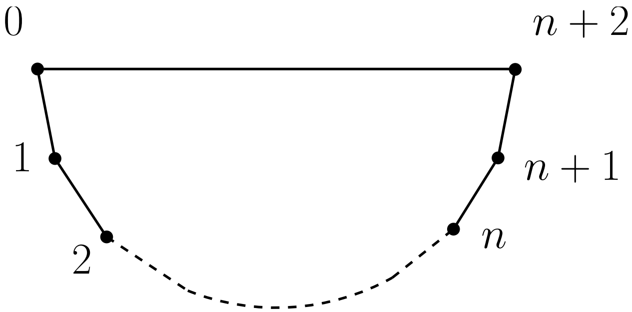
An ***associahedron*** is a polytope whose graph is the flip graph of triangulations of a convex polygon.



Faces ↔ dissections of the polygon

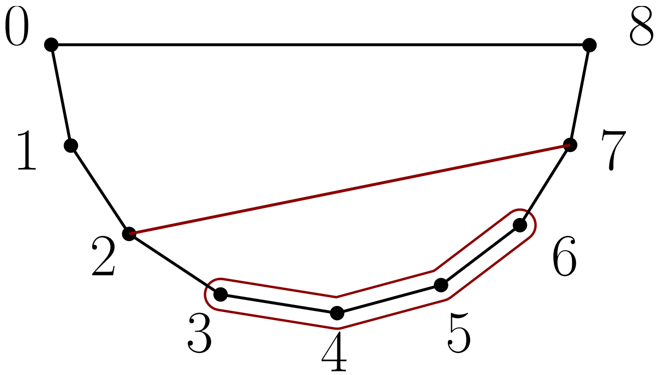
Useful configuration (Loday's)

$G_{n+3} =$



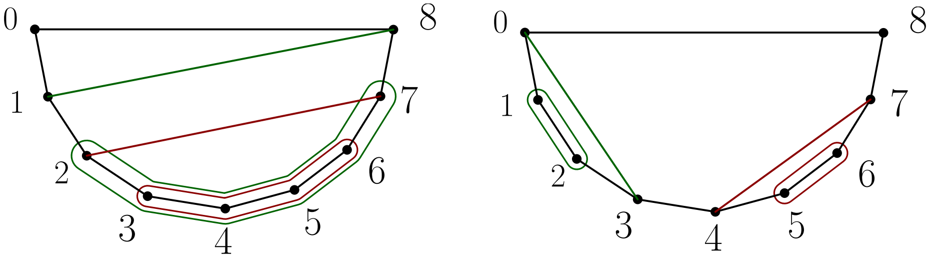
Graph point of view

$\{\text{diagonals of } G_{n+3}\} \longleftrightarrow \{\text{strict subpaths of the path } [n+1]\}$



Non-crossing diagonals

Two ways to be non-crossing in Loday's configuration:

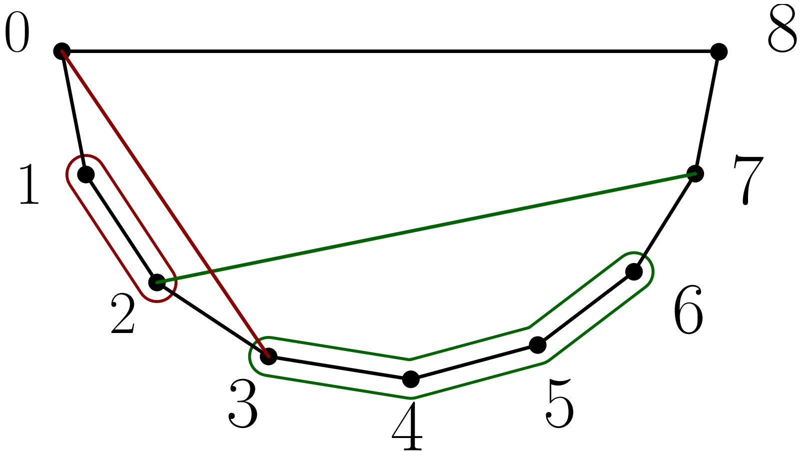


nested subpaths

non-adjacent subpaths

Caution with the second case:

The right condition is indeed *non-adjacent*, disjoint is not enough!



Now do it on graphs

$G = (V, E)$ a (connected) graph.

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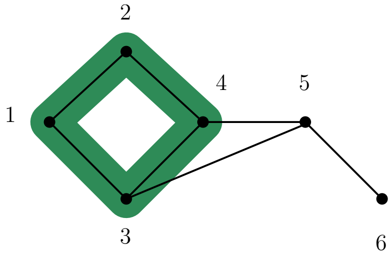
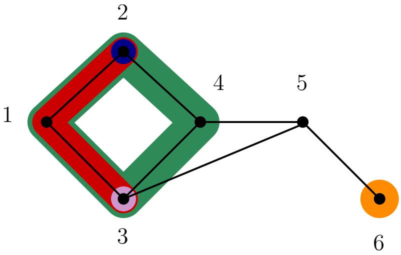
Definition

- A ***tube*** of G is a proper subset $t \subseteq V$ inducing a connected subgraph of G;
- t and t' are ***compatible*** if they are nested or non-adjacent;
- A ***tubing*** on G is a set of pairwise compatible tubes of G.



A tube

(generalizes a diagonal)



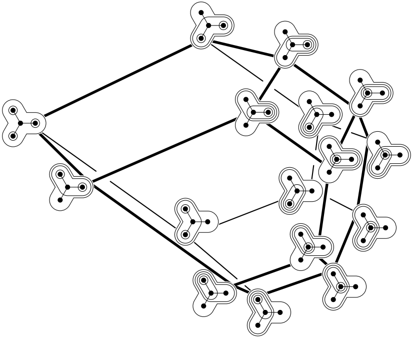
A maximal tubing

(generalizes a triangulation)

Graph associahedra

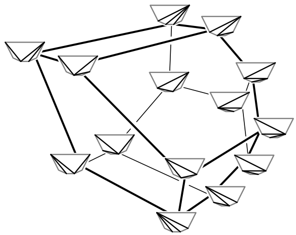
Theorem (Carr-Devadoss '06)

There exists a polytope $\textbf{Asso}_G$*, the* ***graph associahedron*** *of* G*, realizing the complex of tubings on* G*.*

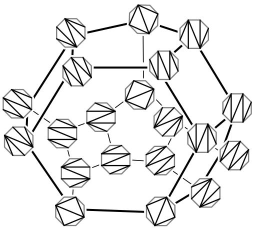


Faces $\leftrightarrow$ tubings of G.

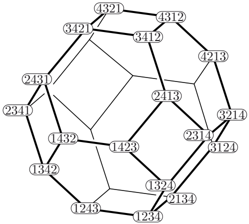
Some classical polytopes...



The associahedron

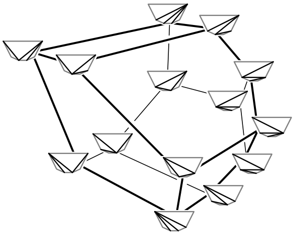


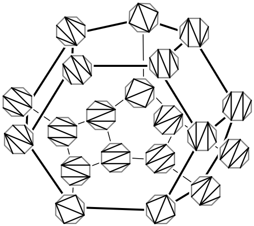
The cyclohedron

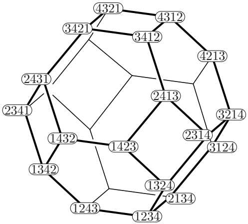


The permutahedron

...can be seen as graph associahedra







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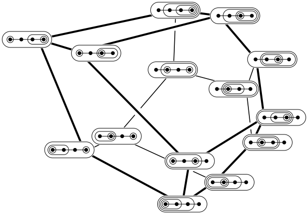

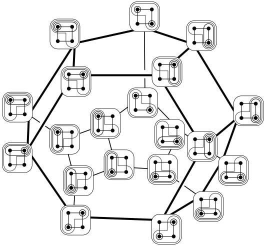

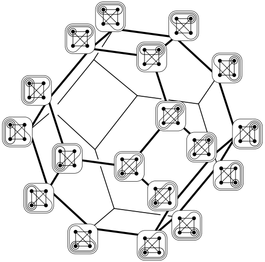

Many different associahedra

Hohlweg-Lange [HL]: $O(2^n)$

Ceballos-Santos-Ziegler [CSZ] (Santos): $O(Cat(n))$

[HL] $\bigcap$ [CSZ] $=$ Chapoton-Fomin-Zelewinsky [CFZ] (type A): 1

Few graph associahedra

Carr-Devadoss [CD]: 1 $\subset$ Postnikov [P]: 1

Volodin [Vol]: ???
Probably many, but not explicit.

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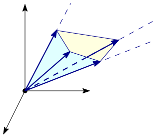
Fans

Polyhedral Cone: positive span of finitely many vectors.



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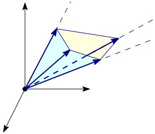
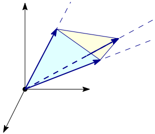


Simplicial Cone: positive span of independent vectors.



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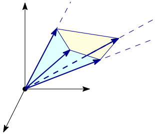

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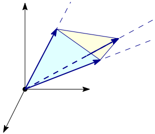
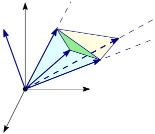


Fan = set of polyhedral cones intersecting properly.



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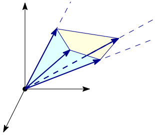

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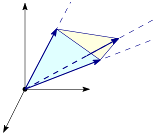

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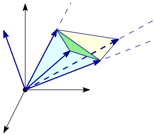
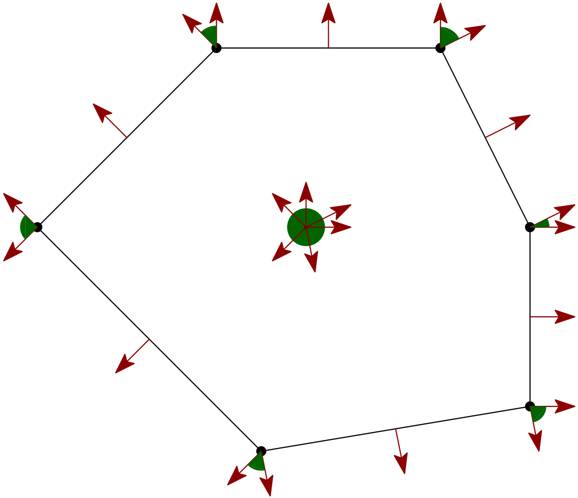


Simplicial Fan: fan whose cones all are simplicial.

Complete Fan: fan whose cones cover the whole space.

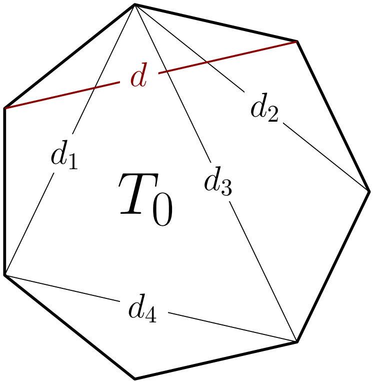
polytope ⇒ complete fan (*normal fan*).

simple polytope ⇒ complete simplicial fan.



Santos' construction for the fan

→ choose an initial triangulation T_0 of the polygon.



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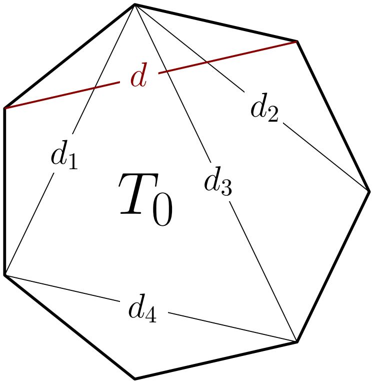

set $u_{d_i} = -e_i$

$\rightarrow$ for a diagonal $d \notin T_0$, define $u_d = (\mathbb{1}_{d \text{ crosses } d_i})_{d_i \in T_0}$.

$\rightarrow$ for a triangulation T, define $C(T) = cone(u_d | d \in T)$.

$\rightarrow$ Define $\mathcal{F} = \{C(T) | T \text{ triangulation}\}$.

Theorem (Ceballos-Santos-Ziegler 13)

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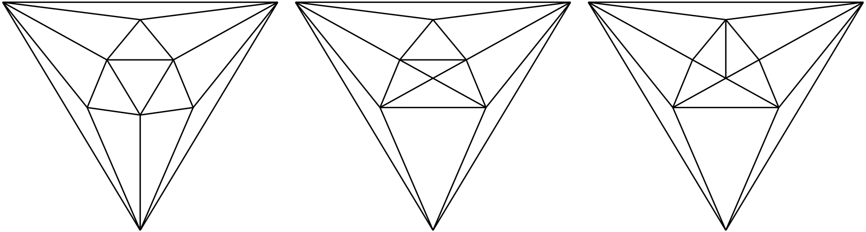
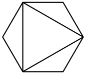
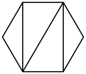
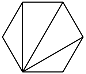


$T_0 =$ 

$T_0 =$ 

$T_0 =$ 

Idea of the proof

$\rightarrow$ The cone $C(T_0)$ is the negative orthant.

$\Rightarrow$ full-dimensional and simplicial

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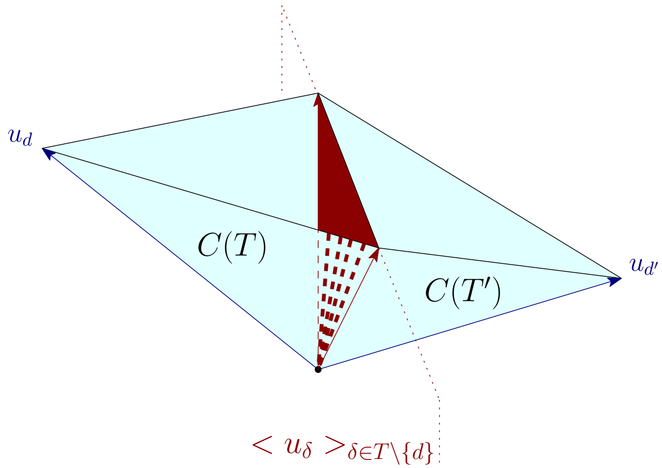
$\Rightarrow$ full-dimensional and simplicial

→ Local condition on flips $T \leftrightarrow T' = T \setminus \{d\} \cup \{d'\}$.



Checking local conditions

$\rightarrow$ Formulation: $\alpha u_d + \alpha' u_{d'} + \sum_{\delta \in T \setminus \{d\}} \beta_\delta u_\delta = 0 \Rightarrow \alpha.\alpha' > 0.$

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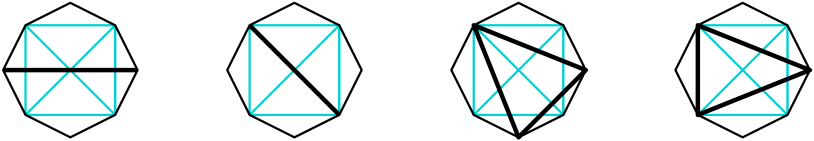
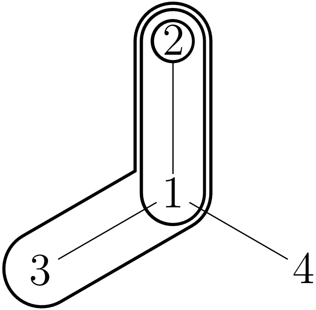
→ Reduction:



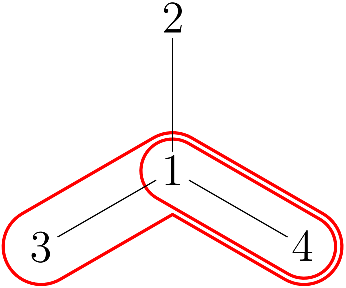
→ Finite number of linear dependences to check explicitly.

■

For graphs?



T_0



$\rightarrow$ impossible to choose $-1, 0, 1$ coordinates.

The compatibility degree

$\rightarrow$ notion of compatibility degree between two tubes $(t \parallel t')$.

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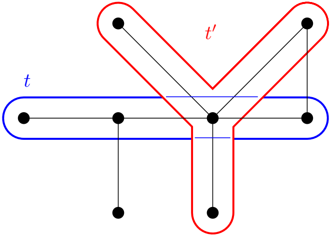
$$(t \parallel t') = \begin{cases} -1 \text{ if } t = t', \\ \#(\text{neighbors of } t' \text{ in } t \smallsetminus t') \text{ if } t' \not\subseteq t, \\ 0 \text{ otherwise.} \end{cases}$$

→ Counts compatibility obstructions.



$(t \parallel t') = 2$
$(t' \parallel t) = 3$

The result!

- $\to$ Define $u_t = ((t \parallel t_1), \ldots, (t \parallel t_n))$
- $\to$ For a maximal tubing T, define $C(T) = cone(u_t | t \in T)$.
- $\to$ Define $\mathcal{F}_G = \{C(T) | T \text{ triangulation}\}$.

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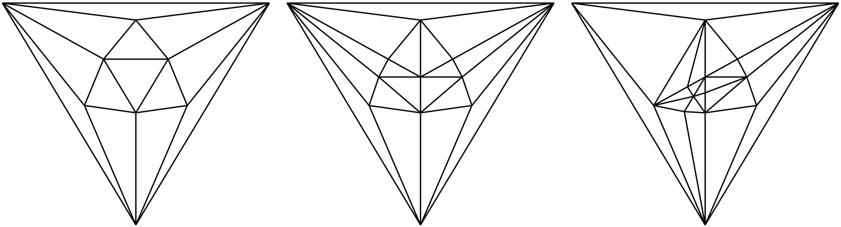
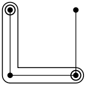
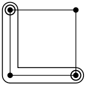
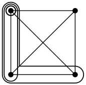


$T_0 =$ 

$T_0 =$ 

$T_0 =$ 

Link with cluster complexes

→ [CFZ]: compatibility degrees between roots in finite types to construct generalized associahedra.

$$\{\text{Generalized Associahedra}\} \bigcap \{\text{Graph Associahedra}\} = A, B, C.$$

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type	graph
A	path
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type	graph
A	path
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roots	tubes
$(\alpha \parallel \alpha')$	$(t \parallel t')$
$(\alpha \parallel \alpha')$	$(t \parallel t')$
$(\alpha \parallel \alpha')$	$(t' \parallel t)$

THANK YOU FOR YOUR AMAZED ATTENTION!