Des formules de Nekrasov-Okounkov en types affines $\widetilde{\mathrm{C}}$ et $\widetilde{C}\check{}$

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Plan

1. Partitions and Macdonald's formula
2. A Nekrasov–Okounkov formula in types $\tilde{\mathrm{C}}$ and $\widetilde{C}^{\vee}$
3. Generalizations through Littlewood decomposition

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2. A Nekrasov–Okounkov formula in types $\tilde{C}$ and $\widetilde{C}\check{}$
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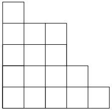

Figure: The Ferrers diagram of λ=(5,4,3,3,1)

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4	2	1		
5	3	2		
7	5	4	1	
9	7	6	3	1

Figure: The Ferrers diagram of λ=(5,4,3,3,1) and its hook lengths

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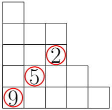

Figure: The Ferrers diagram of λ=(5,4,3,3,1) and its principal hook lengths

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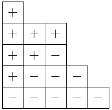

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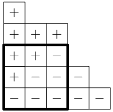

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$\delta_\lambda = \begin{cases} +1 & \text{if the Durfee square of } \lambda \text{ is even} \\ -1 & \text{else} \end{cases}$

$\mathcal{H}_t(\lambda)$ the multi-set of hook lengths which are multiple of t

t-cores

Let $t \geq 2$ be an integer. A partition is a *t-core* if its hook lengths set does not contain t. It is equivalent to the fact that the hook lengths set does not contain any integral multiple of t, *i.e.* $\mathcal{H}_t(\lambda) = \emptyset$.

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Han (2009): expansion of η function in terms of hooks

Dedekind η function

We define Dedekind eta function by $\eta(x) = x^{1/24} \prod_{i \geq 1} (1 - x^i)$.

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Lehmer's conjecture (1947)

Coefficents of expansion of η^{24} are nonzero.

Macdonald formula in type $\tilde{A}$

Theorem (Macdonald, 1972)

For any odd integer t, we have:

$$\eta(x)^{t^2-1} = c_0 \sum_{(v_0,v_1,\ldots,v_{t-1})} \prod_{i<j} (v_i - v_j) x^{(v_0^2+v_1^2+\cdots+v_{t-1}^2)/2t}, \tag{1}$$

where the sum is over t-tuples of integers $(v_0, \ldots, v_{t-1}) \in \mathbb{Z}^t$ such that $v_0 + \cdots v_{t-1} = 0$ and $v_i \equiv i \bmod t$.

Nekrasov-Okounkov formula in type $\tilde{A}$

Theorem (Nekrasov-Okounkov, 2003; Han, 2009)

For any complex number z we have

$$\prod_{k\geq 1}(1-x^k)^{z-1} = \sum_{\lambda\in\mathcal{P}} x^{|\lambda|} \prod_{h\in\mathcal{H}(\lambda)} \left(1-\frac{z}{h^2}\right).$$

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Macdonald in type $\tilde{C}^{\vee}$

Theorem (Macdonald, 1972)

For any integer $t \geq 2$, we have:

$$\left(\frac{\eta(x^2)^{t+1}}{\eta(x)}\right)^{2t-1} = c_1 \sum_{\mathbf{v}} x^{\|v\|^2/8t} \prod_{i<j} (v_i^2 - v_j^2),$$

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We write $v_i = 4tn_i + 2i - 1$.

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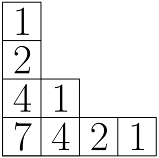

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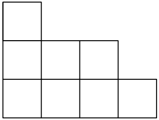

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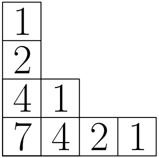

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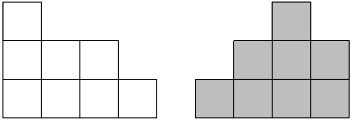

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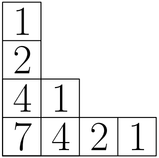

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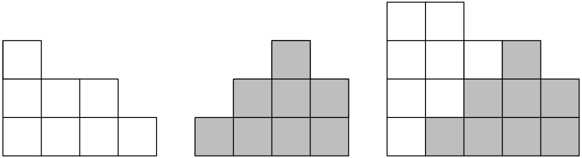

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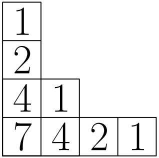

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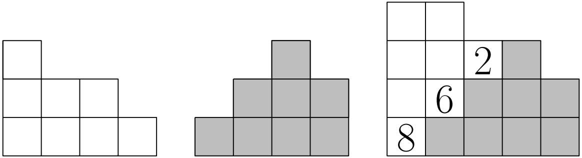

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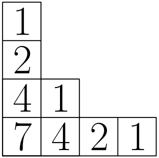

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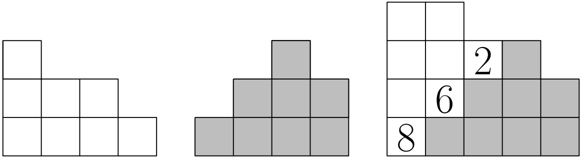

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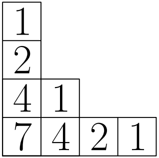

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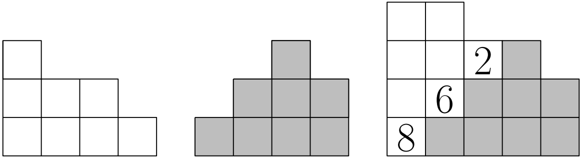

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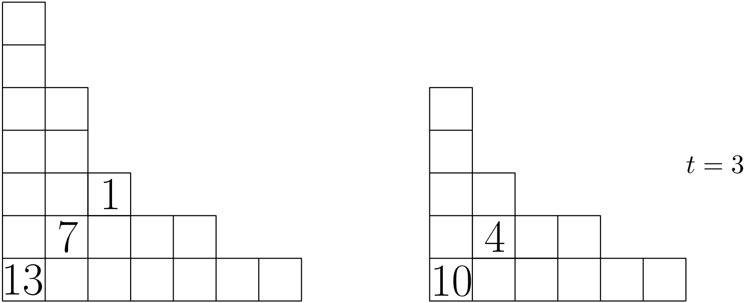

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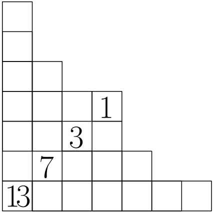

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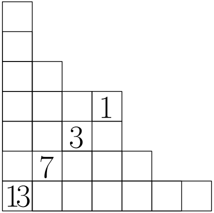

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$\Delta_1 = \max\{h, h \equiv \color{red}{\pm 1 - 6} \mod 12\}$

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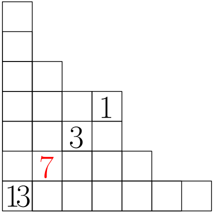

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$\Delta_1 = \max\{h, h \equiv \pm 1 - 6 \mod 12\} = \max\{7\} = 7$

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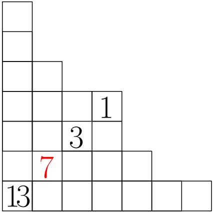

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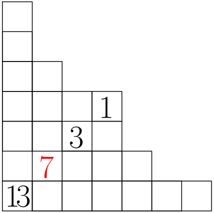

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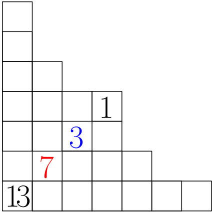

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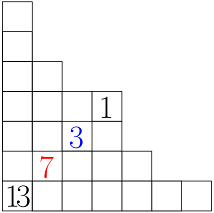

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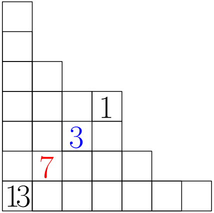

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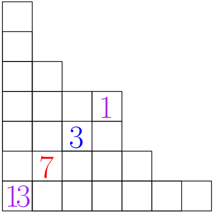
For $1 \leq i \leq t$, write $\Delta_{2i-1} = \max\{h, h \equiv \pm(2i-1) - 2t \mod 4t\}$.
We define $n_i := \dfrac{\pm(2t + \Delta_{2i-1}) - (2i-1))}{4t}$.



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$\Delta_1 = \max\{h, h \equiv \pm 1 - 6 \mod 12\} = \max\{7\} = 7 \Rightarrow n_1 = \frac{+(6+7)-1}{12} = 1$

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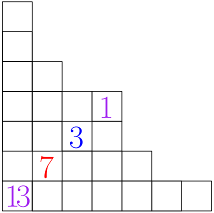

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Definition of bijection φ

Theorem (P., 2014)

Let t an integer ≥ 2. The map

$$\begin{array}{cccc} \varphi : & SC_{(2t)} & \to & \mathbb{Z}^t \\ & \lambda & \mapsto & (n_1, \ldots, n_t) \end{array}$$

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Recall :

$$\left(\frac{\eta(x^2)^{t+1}}{\eta(x)}\right)^{2t-1} = c_1 \sum_{\mathbf{v}} x^{\|v\|^2/8t} \prod_{i<j} \left[(4tn_i + 2i - 1)^2 - (4tn_j + 2j - 1)^2\right]$$

Theorem (P., 2014–2015)

For any complex number z we have

$$\left(\prod_{i\geq 1}\frac{(1-x^{2i})^{z+1}}{1-x^i}\right)^{2z-1} = \sum_{\lambda\in SC}\delta_\lambda\, x^{|\lambda|}\prod_{h\in\mathcal{H}(\lambda)}\left(1-\frac{2z}{h\,\varepsilon_h}\right), \qquad \textit{type } \widetilde{C}\check{}$$

A Nekrasov–Okounkov formula in types $\tilde{C}$ and $\widetilde{C}\check{}$

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$$\prod_{k\geq 1}(1-x^k)^{2z^2+z} = \sum_{\lambda\in DD}\delta_\lambda\, x^{|\lambda|/2}\prod_{h\in\mathcal{H}(\lambda)}\left(1-\frac{2z+2}{h\,\varepsilon_h}\right), \qquad \textit{type } \widetilde{C}$$

Some applications

- For any positive integer n,

$$\sum_{\substack{\lambda \in DD \\ |\lambda| = 2n}} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{h} = \frac{1}{2^n n!}$$

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Theorem (P., 2014)

Let k be a positive integer and s be a real number such that $s>k-1$. Then $(-1)^k f_k(2s^2+s)>0$.

Extensions to other types

Theorem

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2. A Nekrasov–Okounkov formula in types $\tilde{C}$ and $\widetilde{C}^{\vee}$
3. Generalizations through Littlewood decomposition

A generalization of Nekrasov-Okounkov formula in type $\tilde{C}$

Theorem (P., 2015)

Let $t = 2t' + 1$ be an odd positive integer. For any complex numbers y and z we have

$$\sum_{\lambda \in DD} \delta_\lambda x^{|\lambda|/2} \prod_{h \in \mathcal{H}_t(\lambda)} \left(y - \frac{yt(2z+2)}{\varepsilon_h \, h} \right)$$
$$= \prod_{k \geq 1} (1 - x^k)(1 - x^{kt})^{t'-1} \left(1 - x^{tk} y^{2k}\right)^{(2z+1)(zt+3t')}$$

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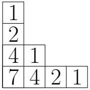

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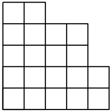

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\cline{1-2} 4 & 1 \\
\hline 7 & 4 & 2 & 1 \\
\hline
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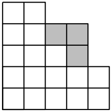

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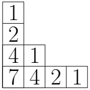

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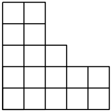

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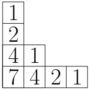

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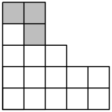

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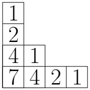

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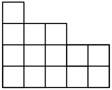

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2			
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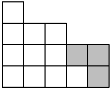

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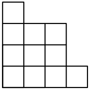

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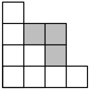

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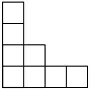

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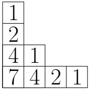

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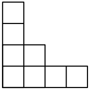

Fact : the t-core of a partition is a t-core.

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Theorem (Littlewood, 1951, probably)

The Littlewood decomposition maps a partition λ to $(\tilde{\lambda}, \lambda^0, \lambda^1, \ldots, \lambda^{t-1})$ such that:

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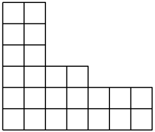

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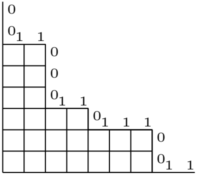

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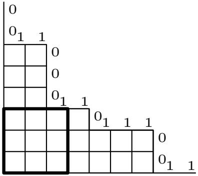

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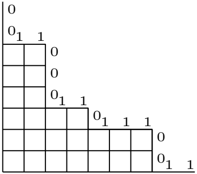

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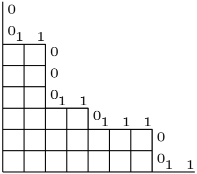

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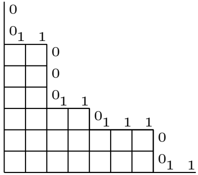

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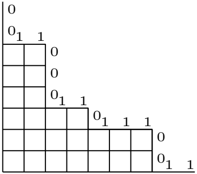

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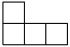
$\lambda^0 =$ 

$\lambda^1 =$ 

$\lambda^2 =$ 

Littlewood decomposition

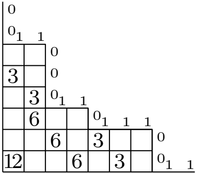
Theorem (Littlewood, 1951, probably)

The Littlewood decomposition *maps a partition λ to $(\tilde{\lambda}, \lambda^0, \lambda^1, \dots, \lambda^{t-1})$ such that:*

(i) $\tilde{\lambda}$ *is the t-core of* λ *and* $\lambda^0, \lambda^1, \dots, \lambda^{t-1}$ *are partitions;*

(ii) $|\lambda| = |\tilde{\lambda}| + t(|\lambda^0| + |\lambda^1| + \cdots + |\lambda^{t-1}|)$

(iii) $\{h/t, h \in \mathcal{H}_t(\lambda)\} = \mathcal{H}(\lambda^0) \cup \mathcal{H}(\lambda^1) \cup \cdots \cup \mathcal{H}(\lambda^{t-1})$.



$$
\begin{aligned}
w &= \cdots 00110001.101110011 \cdots \\
w_0 &= \cdots \ 1 \ 0 \ 1 \ 1 \ 0 \ \cdots \\
w_1 &= \cdots 0 \ 1 \ 0 \ 0 \ 1 \ 1 \ \cdots \\
w_2 &= \cdots \ 0 \ 0 \ 1 \ 1 \ 0 \ 1 \cdots
\end{aligned}
$$

$$\lambda^0 = \begin{array}{|c|c|c|}\hline 1 \\ \hline 4 & 2 & 1 \\ \hline\end{array} \qquad \lambda^1 = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \qquad \lambda^2 = \begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array}$$

New properties of Littlewood decomposition

When $\lambda \in DD$, its Littlewood decomposition $(\tilde{\lambda}, \lambda^0, \lambda^1, \ldots, \lambda^{t-1})$ satisfies:

(i) $\tilde{\lambda}$ and λ^0 are doubled distinct partitions

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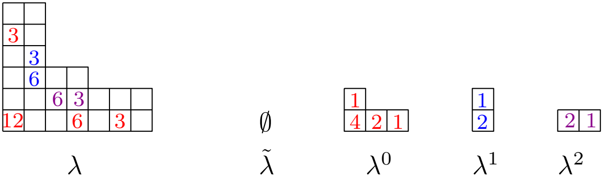

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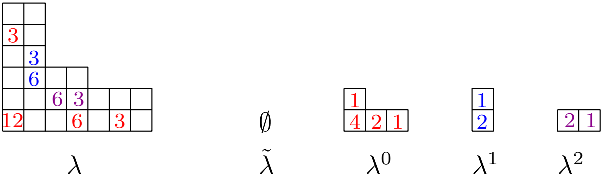

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(iv) two properties about the relative position of the boxes

Proof of our generalization

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$$\times \delta_{\lambda^0} \, x^{t|\lambda^0|/2} \prod_{h \in \mathcal{H}(\lambda^0)} \left(y - \frac{y(2z+2)}{\varepsilon_h \; h} \right)$$
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- And sum over all doubled distinct partitions.

Some consequences

Corollary (P., 2015)

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We have:

$$\sum_{\lambda \in DD} \delta_\lambda x^{|\lambda|/2} \prod_{h \in \mathcal{H}_t(\lambda)} \frac{bt}{h\,\varepsilon_h} = \exp(-tb^2x^t/2) \prod_{k \geq 1} (1 - x^k)(1 - x^{kt})^{t'-1}$$

A new hook formula

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We have:

$$\sum_{\substack{\lambda \in DD,\ |\lambda|=2tn \\ \#\mathcal{H}_t(\lambda)=2n}} \delta_\lambda \prod_{h \in \mathcal{H}_t(\lambda)} \frac{1}{h\,\varepsilon_h} = \frac{(-1)^n}{n!\,t^n 2^n}$$

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Question: can we prove this by using the RSK algorithm?

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- Link with representation theory?
- Other affine types (as $\widetilde{D}$)?

Thank you for your attention!