A bijection relating q-Eulerian polynomials

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1. Introduction
   - Eulerian polynomials
   - Combinatorial interpretations
   - q-Eulerian polynomials

2. Construction of $\varphi : \mathfrak{S}_n \to \mathfrak{S}_n$
   - Skeleton (graph) of $\varphi(\sigma)$
   - Labelling of the graph
     - Labelling of the circles (excedance values of $\varphi(\tau)$)
     - Labelling of the dots (non excedance values of $\varphi(\sigma)$)

3. Thereafter
   - Extension
   - Open problem

Eulerian polynomials

The sequence of Eulerian polynomials $(A_n(t))_{n\geq 1}$ can be defined by

$$\sum_{n\geq 1} A_n(t)\frac{x^n}{n!} = \frac{t-1}{t-e^{(t-1)x}}.$$

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The first values of $A_n(t)$:

$$\begin{aligned} A_1(t) &= 1, \\ A_2(t) &= 1+t, \\ A_3(t) &= 1+4t+t^2, \\ A_4(t) &= 1+11t+11t^2+t^3. \end{aligned}$$

Eulerian statistics

Proposition (MacMahon)

We have

$$A_n(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{des(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} t^{exc(\sigma)}$$

where $\mathfrak{S}_n$ *is the set of permutations on* $[n] := \{1, 2, \dots, n\}$ *and*

$$des(\sigma) = \#\{i \in [n-1], \sigma(i) > \sigma(i+1)\},$$
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Example : $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix} \in \mathfrak{S}_4$ has $\mathrm{des}(\sigma) = 2$ descents 2 and 3 and $\mathrm{exc}(\sigma) = 2$ exceedances 1 and 2.

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A statistic equidistributed with des or exc is said to be *Eulerian*.
Example : ides defined by $\mathrm{ides}(\sigma) = \mathrm{des}(\sigma^{-1})$.

Mahonian statistics

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The pairs (i,j) such that $i < j$ and $\sigma(i) > \sigma(j)$ are named *inversions.*

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2-versions of the previous statistics

Let $\sigma \in \mathfrak{S}_n$.

- A 2-*descent* of σ is an integer $i \in [n-1]$ such that $\sigma(i) \geq \sigma(i+1) + 2$.

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- A 2-*inversion* of σ is a pair $(i,j) \in [n]^2$ such that $i < j$ and $0 \leq \sigma(i) - \sigma(j) < 2$ (*i.e.* $\sigma(i) = \sigma(j) + 1$).

$$\mathrm{inv}_2(\sigma) := \text{number of 2-inversions of } \sigma.$$

Two pairs of statistics

Consider the pairs of statistics $(\mathrm{maj}_2, \mathrm{inv}_2)$ and $(\mathrm{maj} - \mathrm{exc}, \mathrm{exc})$ where, for all $\sigma \in \mathfrak{S}_n$.

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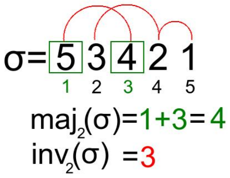

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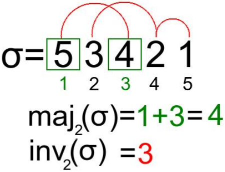

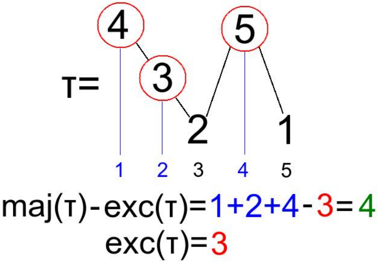

q-Eulerian polynomials

Let $A_n(q,t)$ and $A_n^{(2)}(q,t)$ be the q-Eulerian polynomials

$$A_n(q,t) = \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}(\sigma) - \mathrm{exc}(\sigma)} t^{\mathrm{exc}(\sigma)},$$

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Theorem 1 (Shareshian and Wachs, 2014)

For all $n \geq 1$, we have

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The proof relies on quasisymmetric function techniques.

Main result

Theorem 2 (B.,2015)

There exists a bijection $\varphi : \mathfrak{S}_n \to \mathfrak{S}_n$ such that

$$(maj_2(\sigma), inv_2(\sigma)) = (maj(\tau) - exc(\tau), exc(\tau))$$

for all $\sigma \in \mathfrak{S}_n$ and $\tau = \varphi(\sigma)$.

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This proves combinatorially Theorem 1.

Computation of a sequence

Let $\sigma = 425736981 \in \mathfrak{S}_9$ ($\mathrm{des}_2(\sigma) = 3$ and $\mathrm{inv}_2(\sigma) = 4$).

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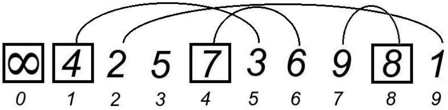

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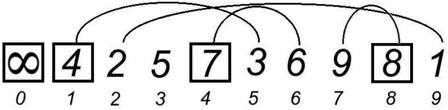

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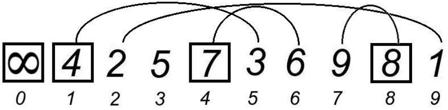

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Main principle : for k from $\mathrm{des}_2(\sigma)$ down to 1, counting the maximal number of 2-inversions $(i_1, j_1), \dots, (i_p, j_p)$ such that $d_2^k \leq i_1 < \dots < i_p$ and $\sigma(i_1) < \dots < \sigma(i_p)$.

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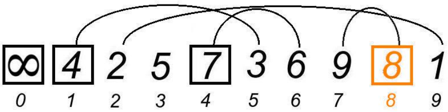

$c_3 = 0$

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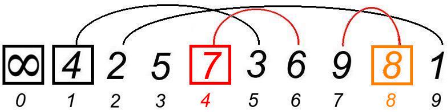

$$(c_2, c_3) = (2, 0)$$

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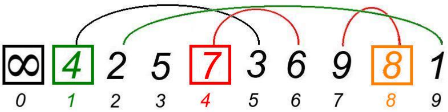

$$(c_1, c_2, c_3) = (1, 2, 0)$$

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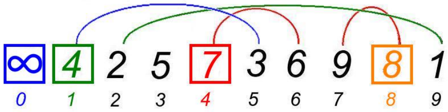

$$(c_0, c_1, c_2, c_3) = (1, 1, 2, 0)$$

Skeleton of $\varphi(\sigma)$

We plot $\mathrm{inv}_2(\sigma)$ circles corresponding with the sequence $(c_0, c_1, \ldots, c_{\mathrm{des}_2(\sigma)})$.

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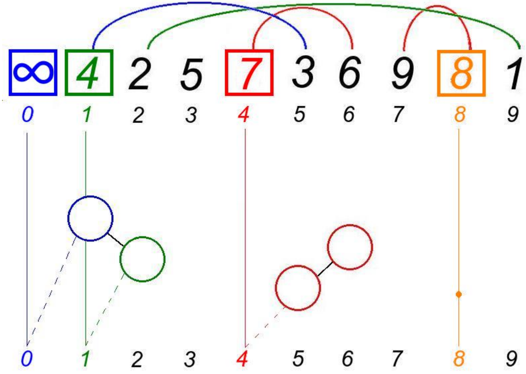

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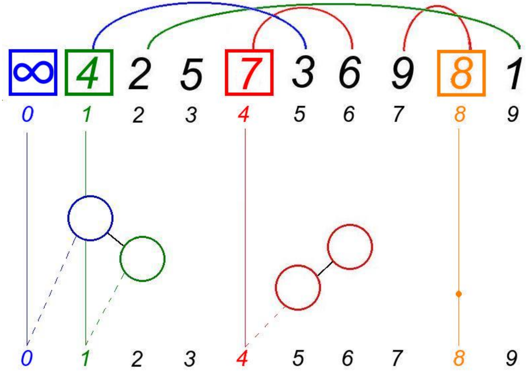

They will be labelled with the excedant values of $\varphi(\sigma)$.

Skeleton of $\varphi(\sigma)$

We complete the graph by ploting dots following the descents of the word $\sigma(i_1)\sigma(i_2)\ldots\sigma(i_{n-\mathrm{des}_2(\sigma)})$ where i_k is not the beginning of an arc of circle for all k.

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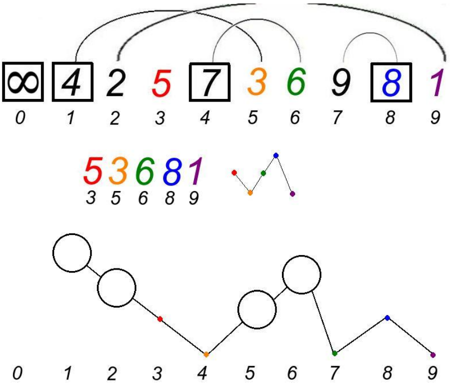

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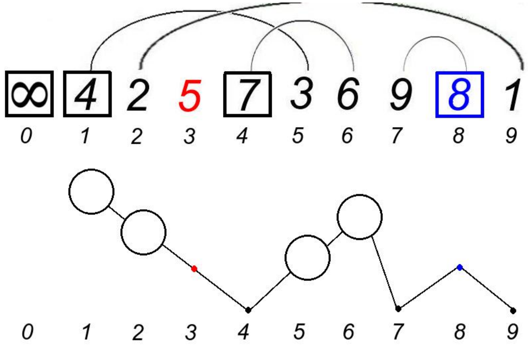

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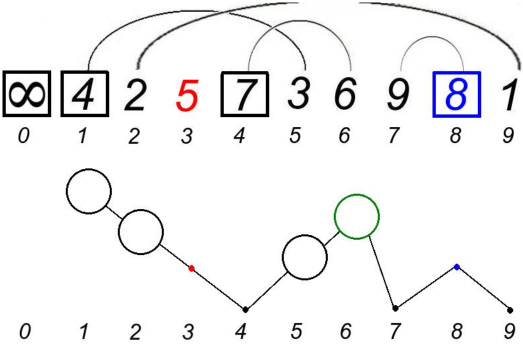

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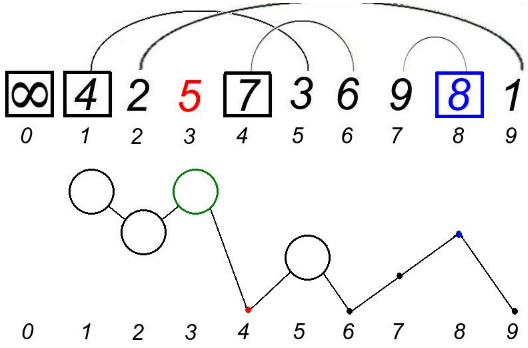

Labelling of the circles (excedances of $\varphi(\tau)$)

We label the circles of the graph with the integers where the arcs of circles are ended, while respecting the descents of the graph, the fact that each circle should be an excedance value, and the order of the arcs of circles.

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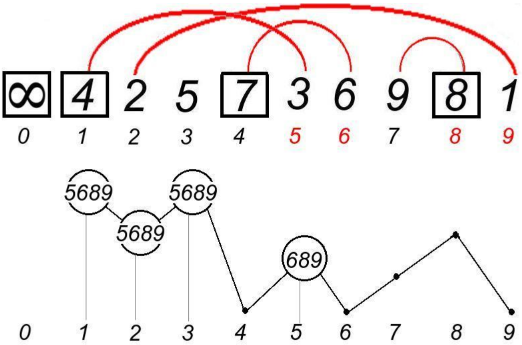

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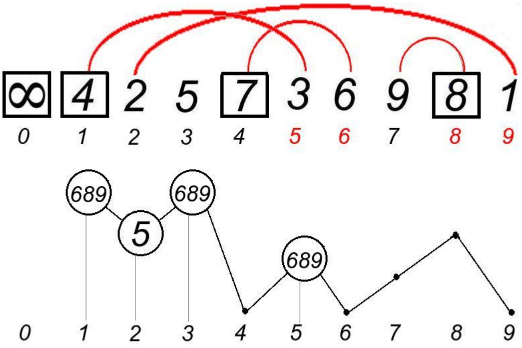

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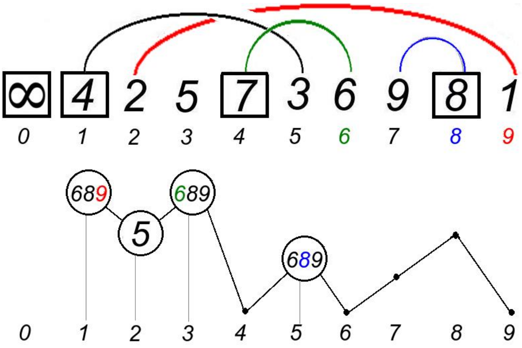

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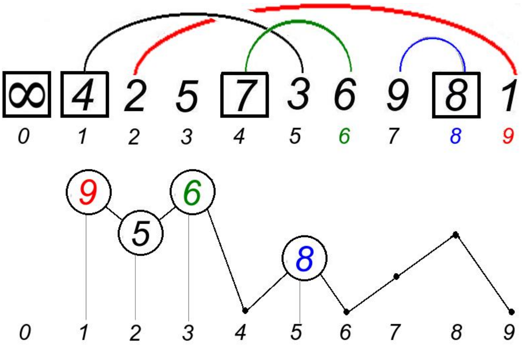

Labelling of the dots (non excedance values of $\varphi(\sigma)$)

We label the dots of the graph with the remaining integers while respecting the descents of the graph, the fact that each dot should not be an excedance value, and the order induced by the word $\sigma(i_1)\sigma(i_2)\ldots\sigma(i_{n-\mathrm{des}_2(\sigma)})$ where i_k is not the beginning of an arc of circle for all k.

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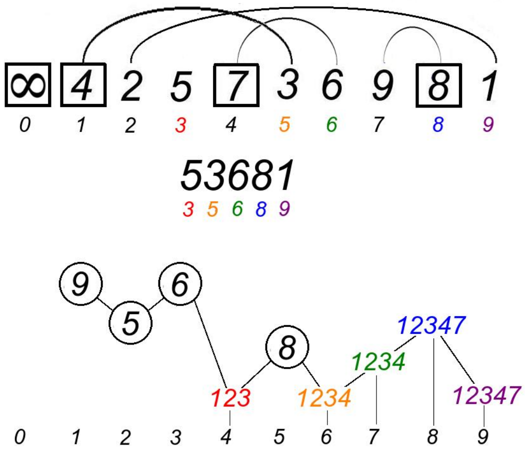

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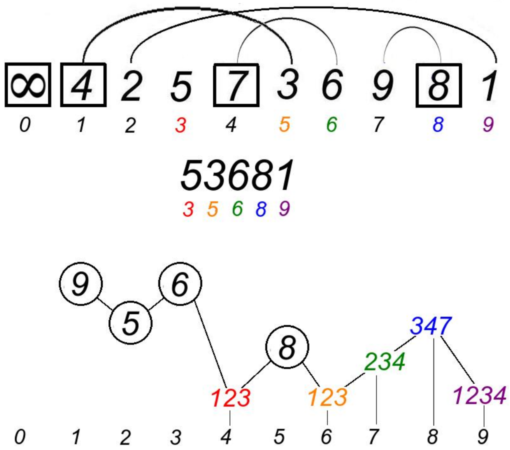

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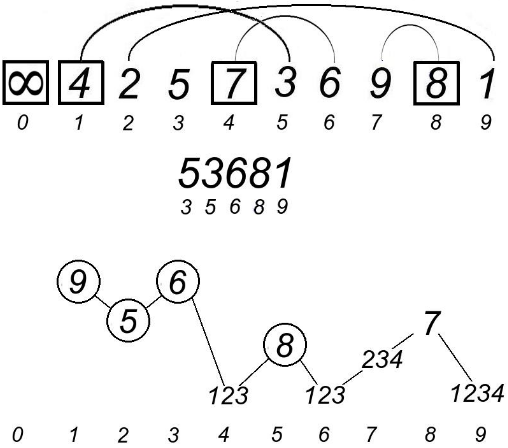

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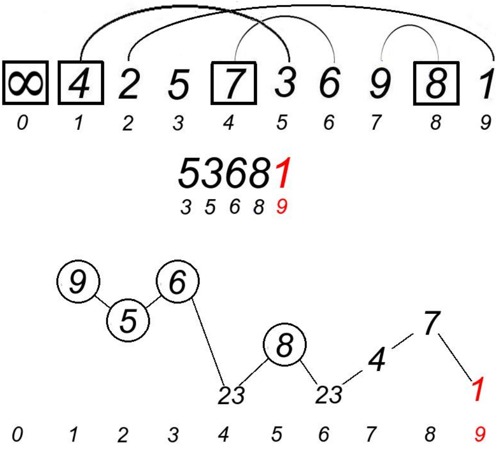
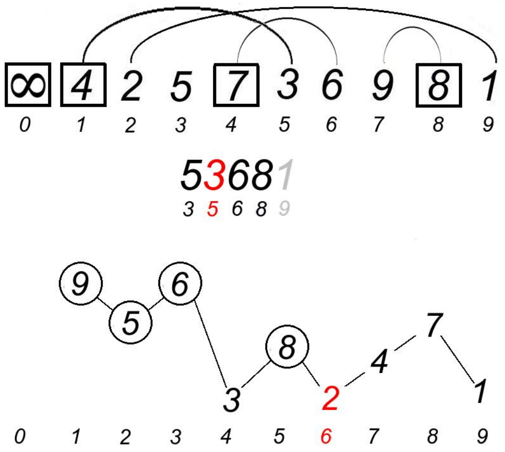


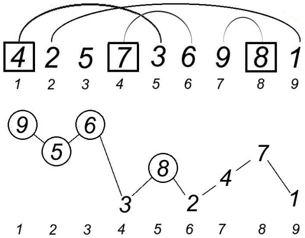
Labelling of the dots (non excedance values of $\varphi(\sigma)$)



Definition of $\varphi(\sigma)$

$$\sigma = 425736981 \in \mathfrak{S}_9,$$

$$(\mathrm{maj}_2(\sigma), \mathrm{inv}_2(\sigma)) = (1+4+8, 4),$$



$$\varphi(\sigma) := 956382471 \in \mathfrak{S}_9,$$

$$\mathrm{maj}(\varphi(\sigma)) - \mathrm{exc}(\varphi(\sigma)), \mathrm{exc}(\varphi(\sigma))) = (1-1+3-2+5-1+8-0, 4)$$
$$= (1+4+8, 4).$$

Extension

By using the same quasisymmetric function methode as Shareshian and Wachs, Hance and Li proved that

$$\sum_{\sigma\in\mathfrak{S}_n} x^{\mathrm{amaj}_2(\sigma)} y^{\mathrm{ides}(\sigma)} z^{\widetilde{\mathrm{asc}_2}(\sigma)} = \sum_{\sigma\in\mathfrak{S}_n} x^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)} y^{\mathrm{exc}(\sigma)} z^{\mathrm{des}(\sigma)}$$

where

$$\mathrm{asc}_2(\sigma) = \#\{i\in[n-1], \sigma(i) < \sigma(i+1)+1\} \text{ (number of 2-}\textit{ascents}\text{)},$$

$$\widetilde{\mathrm{asc}_2}(\sigma) = \begin{cases} \mathrm{asc}_2(\sigma) & \text{if } \sigma(1)=1, \\ \mathrm{asc}_2(\sigma)+1 & \text{if } \sigma(1)\neq 1, \end{cases}$$

$$\mathrm{amaj}(\sigma) = \sum_{\sigma(i)<\sigma(i+1)+1} i.$$

Extension

The bijection $\varphi : \mathfrak{S}_n \to \mathfrak{S}_n$ provides the equality

$$\sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{maj}_2(\sigma)} y^{\mathrm{inv}_2(\sigma)} z^{\widetilde{\mathrm{des}_2}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{maj}(\sigma) - \mathrm{exc}(\sigma)} y^{\mathrm{exc}(\sigma)} z^{\mathrm{des}(\sigma)}$$

where

$$\widetilde{\mathrm{des}_2}(\sigma) = \begin{cases} \mathrm{des}_2(\sigma) & \text{under certain conditions,} \\ \mathrm{des}_2(\sigma) + 1 & \text{otherwise.} \end{cases}$$

Extension

The bijection $\varphi : \mathfrak{S}_n \to \mathfrak{S}_n$ provides the equality

$$\sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{maj}_2(\sigma)} y^{\mathrm{inv}_2(\sigma)} z^{\widetilde{\mathrm{des}_2}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{maj}(\sigma) - \mathrm{exc}(\sigma)} y^{\mathrm{exc}(\sigma)} z^{\mathrm{des}(\sigma)}$$

where

$$\widetilde{\mathrm{des}_2}(\sigma) = \begin{cases} \mathrm{des}_2(\sigma) & \text{under certain conditions,} \\ \mathrm{des}_2(\sigma) + 1 & \text{otherwise.} \end{cases}$$

Open problem : prove combinatorially the equality of Hance and Li by composing φ with a bijection mapping $(\mathrm{maj}_2, \widetilde{\mathrm{des}_2}, \mathrm{inv}_2)$ to $(\mathrm{amaj}_2, \widetilde{\mathrm{asc}_2}, \mathrm{ides})$.

Open problem

$$\sum_{\sigma\in\mathfrak{S}_n} x^{\mathrm{maj}_2(\sigma)} y^{\mathrm{inv}_2(\sigma)} z^{?} = \sum_{\sigma\in\mathfrak{S}_n} x^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)} y^{\mathrm{exc}(\sigma)} z^{\mathrm{inv}(\sigma)}$$