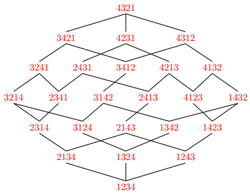
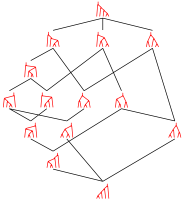
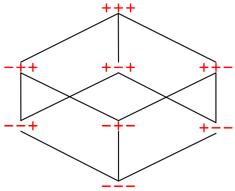
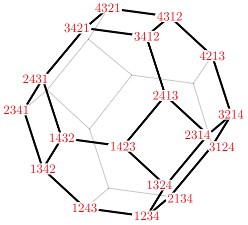
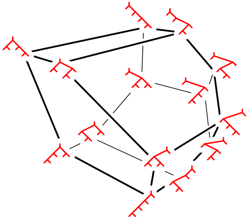
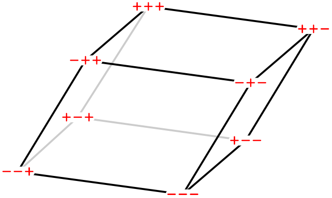
The Cambrian Hopf Algebra

G. Châtel

September 21st, 2015

Joint work with V. Pilaud

	permutations	binary trees	binary sequences
Combinatorics			
Algebra	Malvenuto-Reutenauer algebra $\mathsf{FQSym} = \mathsf{vect}\,\langle\, \mathbb{F}_\tau \mid \tau \in \mathfrak{S} \,\rangle$	Loday-Ronco algebra $\mathsf{PBT} = \mathsf{vect}\,\langle\, \mathbb{P}_{\mathrm{T}} \mid \mathrm{T} \in \mathcal{BT} \,\rangle$	Solomon algebra $\mathsf{Rec} = \mathsf{vect}\,\langle\, \mathbb{X}_\eta \mid \eta \in \pm^* \,\rangle$
Geometry			

1. Combinatorics
   - Binary trees
   - Cambrian trees
   - Cambrian lattices

2. Algebra
   - FQSym
   - The Cambrian algebra

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- Binary trees
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- Cambrian lattices

2 Algebra
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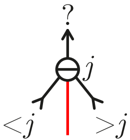
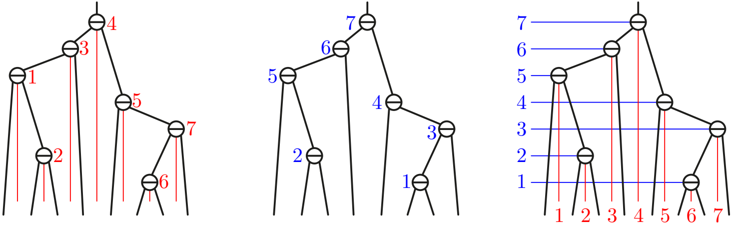
Binary trees

Binary search tree = directed and labeled tree such that



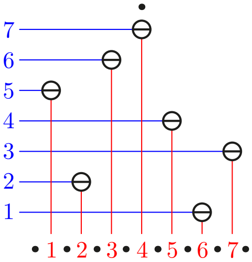
increasing tree = directed and labeled tree such that labels increase along arcs
leveled binary tree = directed tree with a binary search tree labeling and an increasing labeling



Permutations to leveled binary trees

The sylvester correspondence = permutations $\longmapsto$ leveled binary trees.

Exm: permutation 6275134



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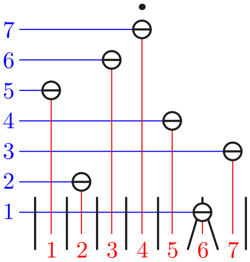

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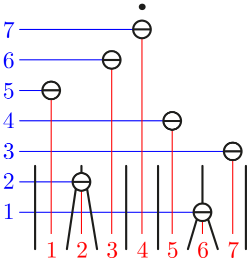

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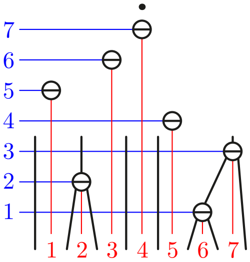

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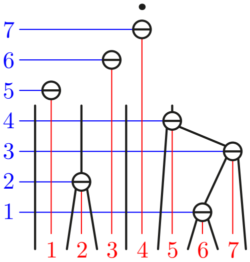

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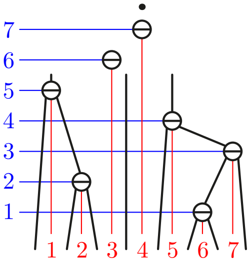

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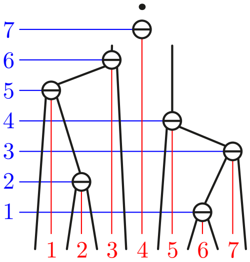

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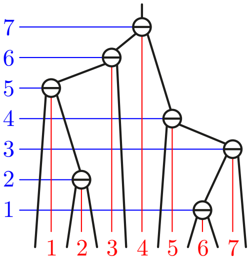
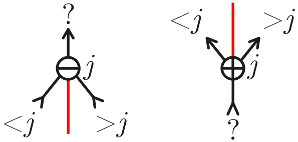
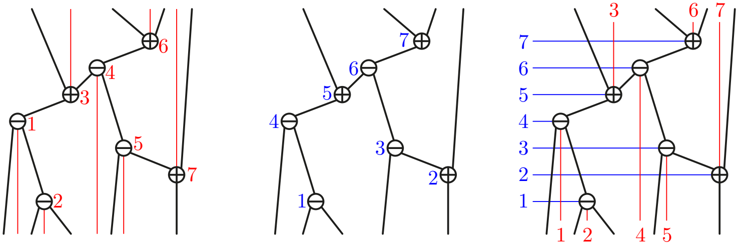


Cambrian trees

Cambrian tree = directed and labeled tree such that

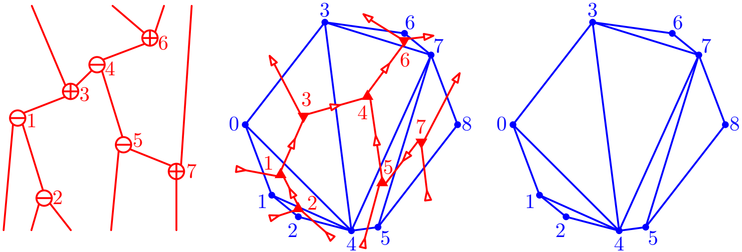


increasing tree = directed and labeled tree such that labels increase along arcs
leveled Cambrian tree = directed tree with a Cambrian labeling and an increasing labeling



Cambrian trees and triangulations

Cambrian trees are dual to triangulations of polygons



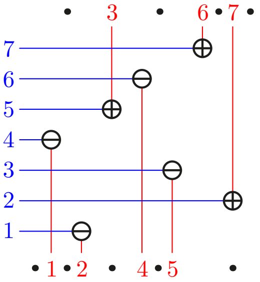
signature	$\longleftrightarrow$	vertices above or below $[0, 8]$
node j	$\longleftrightarrow$	triangle $i < j < k$

For any signature ε, there are $C_n = \frac{1}{n+1}\binom{2n}{n}$ ε-Cambrian trees.

Signed permutations to Cambrian trees

Cambrian correspondence = signed permutation $\longmapsto$ leveled Cambrian tree.

Exm: signed permutation $\underline{2}\overline{7}\underline{5}\underline{1}\overline{3}\underline{4}\overline{6}$



Reading. Cambrian lattices. 2006
Lange-Pilaud. Associahedra via spines. 2015

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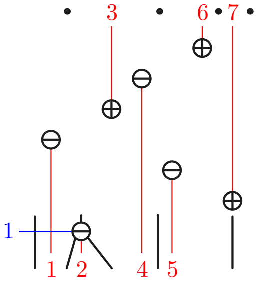

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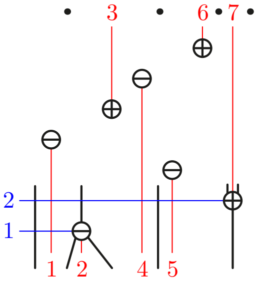

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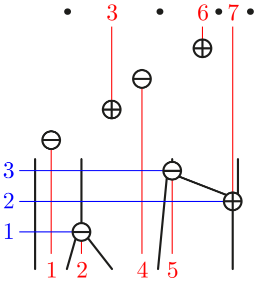

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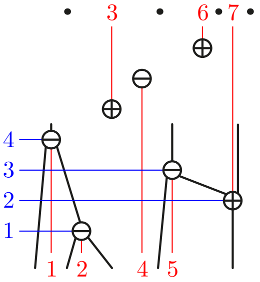

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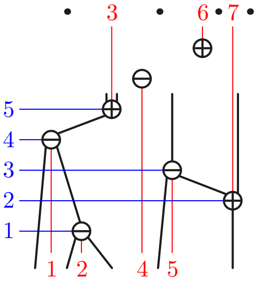

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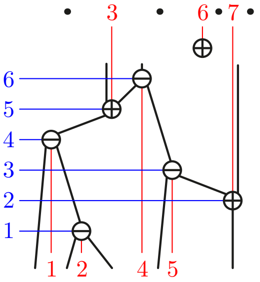

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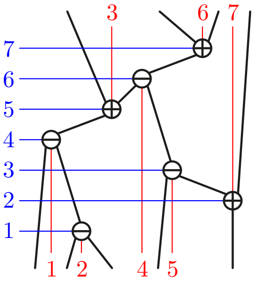

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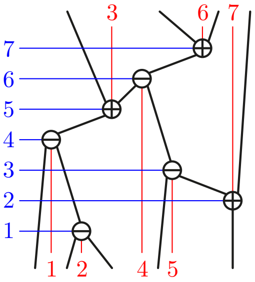


$\mathbf{P}(\tau)$ = $\mathbf{P}$-symbol of τ = Cambrian tree produced by the Cambrian corresp.
$\mathbf{Q}(\tau)$ = $\mathbf{Q}$-symbol of τ = increasing tree produced by the Cambrian corresp.

(analogous to the Robinson-Schensted algorithm)

The Cambrian congruence

ε-Cambrian congruence = transitive closure of the rewriting rules

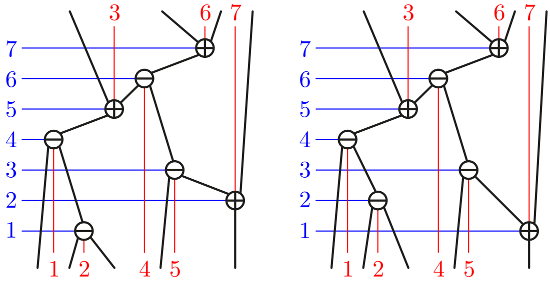
$$\cdots ac \cdots \underline{b} \cdots \equiv_\varepsilon \cdots ca \cdots \underline{b} \cdots \quad \text{if } a < b < c \text{ and } \varepsilon_b = -$$

$$\cdots \overline{b} \cdots ac \cdots \equiv_\varepsilon \cdots \overline{b} \cdots ca \cdots \quad \text{if } a < b < c \text{ and } \varepsilon_b = +$$

where a, b, c are elements of $[n]$.

Proposition [reformulating Reading 2006]

$$\tau \equiv_\varepsilon \tau' \iff \mathbf{P}(\tau) = \mathbf{P}(\tau')$$

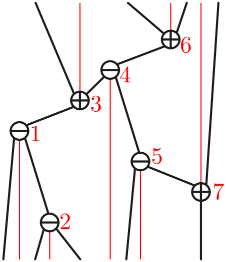


$$\underline{2}\overline{7}\underline{51}\overline{3}\underline{4}\overline{6} \equiv \overline{7}\underline{2}\underline{51}\overline{3}\underline{4}\overline{6}$$

Cambrian trees to signed permutations

Proposition [reformulating Reading 2006]

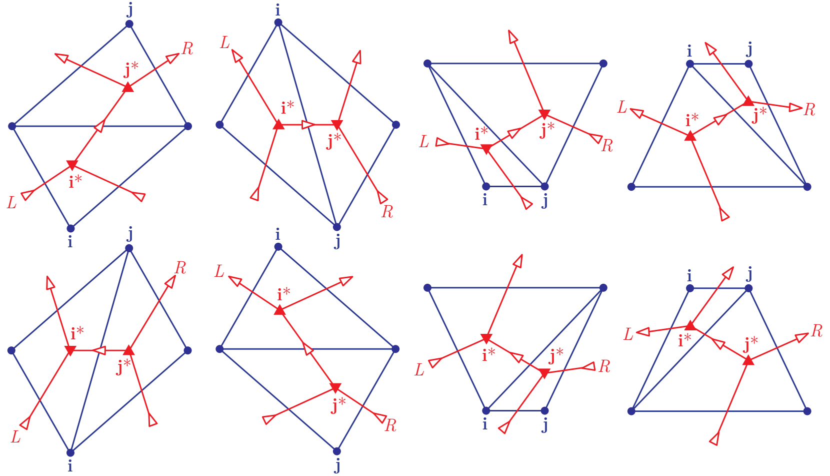
$$\mathbf{P}^{-1}(T) = \mathcal{L}(T)$$

$T =$ 

$$\mathbf{P}^{-1}(T) = \mathcal{L}(T) = \{\underline{21}\overline{37}\underline{54}\overline{6},\ \underline{21}\overline{73}\underline{54}\overline{6},\ \underline{21}\overline{7}\underline{5}\overline{3}\underline{4}\overline{6},\ \underline{2}\overline{7}\underline{1}\overline{3}\underline{54}\overline{6},\ \underline{2}\overline{7}\underline{15}\overline{3}\underline{4}\overline{6},\ \underline{2}\overline{7}\underline{51}\overline{3}\underline{4}\overline{6},\ \overline{7}\underline{21}\overline{3}\underline{54}\overline{6},\ \overline{7}\underline{215}\overline{3}\underline{4}\overline{6},\ \overline{7}\underline{251}\overline{3}\underline{4}\overline{6},\ \overline{7}\underline{521}\overline{3}\underline{4}\overline{6}\}$$

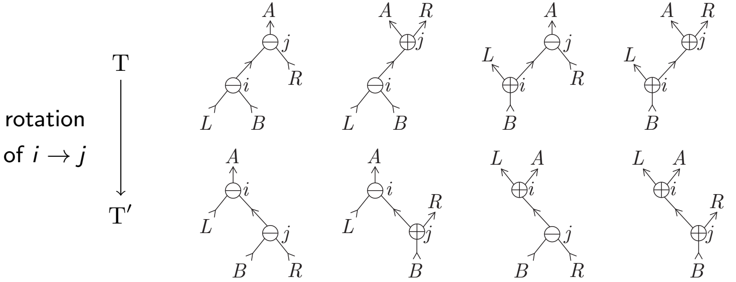
Rotations and flips

Rotation on Cambrian trees $\longleftrightarrow$ flips on triangulations.



Rotation and Cambrian lattices

Rotation operation preserves Cambrian trees:

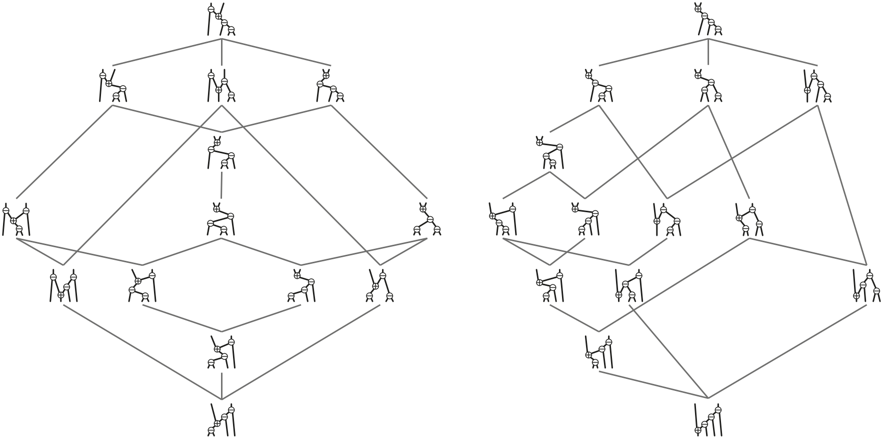


increasing rotation $=$ rotation of edge $i \to j$ where $i < j$

Proposition [reformulating Reading 2006]

The transitive closure of the increasing rotation graph is the Cambrian lattice.
P defines a lattice homomorphism from the weak order to the Cambrian lattice.

Rotations and Cambrian lattices



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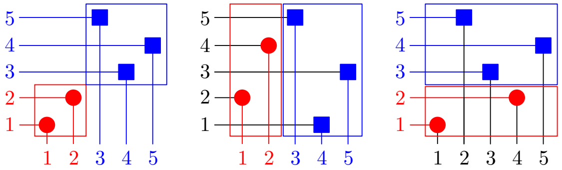
Two products on permutations

For $\tau \in \mathfrak{S}_n$ and $\tau' \in \mathfrak{S}_{n'}$ with $a \in [n], b \in [n']$, define

shifted concatenation $\tau\bar{\tau}' = [\tau(1), \dots, \tau(n), \tau'(1) + n, \dots, \tau'(n') + n] \in \mathfrak{S}_{n+n'}$
shifted shuffle product $\tau \,\bar{\sqcup\!\sqcup}\, \tau' = au \sqcup\!\sqcup bv = a(u \sqcup\!\sqcup bv) + (b + |au|)(au \sqcup\!\sqcup v)$
convolution product $\tau \star \tau' = (\tau^{-1} \,\bar{\sqcup\!\sqcup}\, \tau'^{-1})^{-1}$

When we compute products of permutations, there is no multiplicities so we can consider that the output of the shuffle is a set of permutations.

$$\begin{aligned} 12 \,\bar{\sqcup\!\sqcup}\, 231 &= \{12453, 14253, 14523, 14532, 41253, 41523, 41532, 45123, 45132, 45312\} \\ 12 \star 231 &= \{12453, 13452, 14352, 15342, 23451, 24351, 25341, 34251, 35241, 45231\} \end{aligned}$$



concatenation | shuffle | convolution

The Malvenuto-Reutenauer algebra

The Malvenuto-Reutenauer algebra = Hopf algebra FQSym with basis $(\mathbb{F}_\tau)_{\tau \in \mathfrak{S}}$ and where

$$\mathbb{F}_\tau \cdot \mathbb{F}_{\tau'} = \sum_{\sigma \in \tau \,\bar{\sqcup}\, \tau'} \mathbb{F}_\sigma \qquad \text{and} \qquad \triangle \mathbb{F}_\sigma = \sum_{\sigma \in \tau \star \tau'} \mathbb{F}_\tau \otimes \mathbb{F}_{\tau'}$$

Malvenuto-Reutenauer. Duality between Quasi-Symmetric functions and the Solomon Descent Algebra. 1995

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Definition: Combinatorial Hopf Algebras

A Combinatorial Hopf Algebra = combinatorial vector space $\mathcal{B}$ endowed with

$$\text{product } \cdot : \mathcal{B} \otimes \mathcal{B} \to \mathcal{B}$$
$$\text{coproduct } \triangle : \mathcal{B} \to \mathcal{B} \otimes \mathcal{B}$$

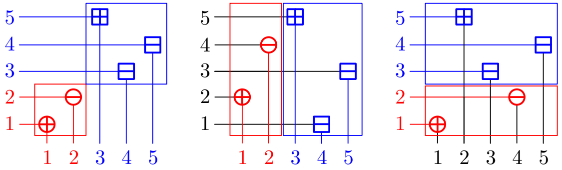
which are “compatible”, *i.e.*,

$$\triangle(f \cdot g) = \triangle(f) \cdot \triangle(g)$$

Two products on signed permutations

For signed permutations:

$$\begin{aligned}
\bar{1}2 \;\bar{\shuffle}\; 23\bar{1} &= \{\bar{1}245\bar{3}, \bar{1}425\bar{3}, \bar{1}452\bar{3}, \bar{1}45\bar{3}2, 4\bar{1}25\bar{3}, 4\bar{1}52\bar{3}, 4\bar{1}5\bar{3}2, 45\bar{1}2\bar{3}, 45\bar{1}\bar{3}2, 45\bar{3}\bar{1}2\},\\
\bar{1}2 \star 23\bar{1} &= \{\bar{1}245\bar{3}, \bar{1}345\bar{2}, \bar{1}435\bar{2}, \bar{1}534\bar{2}, \bar{2}345\bar{1}, \bar{2}435\bar{1}, \bar{2}534\bar{1}, \bar{3}425\bar{1}, \bar{3}524\bar{1}, \bar{4}523\bar{1}\}.
\end{aligned}$$



concatenation — shuffle — convolution

Signed analog of Malvenuto-Reutenauer [Novelli, Thibon 2010]

$\mathrm{FQSym}_{\pm}$ = Hopf algebra with basis $(\mathbb{F}_{\tau})_{\tau \in \mathfrak{S}_{\pm}}$ and where

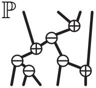
$$\mathbb{F}_{\tau} \cdot \mathbb{F}_{\tau'} = \sum_{\sigma \in \tau \,\bar{\shuffle}\, \tau'} \mathbb{F}_{\sigma} \qquad \text{and} \qquad \triangle \mathbb{F}_{\sigma} = \sum_{\sigma \in \tau \star \tau'} \mathbb{F}_{\tau} \otimes \mathbb{F}_{\tau'}$$

The Cambrian algebra as a subalgebra of $\mathsf{FQSym}_{\pm}$

The Cambrian algebra = subspace Camb of $\mathsf{FQSym}_{\pm}$ generated by

$$\mathbb{P}_{\mathrm{T}} := \sum_{\substack{\tau \in \mathfrak{S}_{\pm} \\ \mathbf{P}(\tau) = \mathrm{T}}} \mathbb{F}_{\tau} = \sum_{\tau \in \mathcal{L}(\mathrm{T})} \mathbb{F}_{\tau},$$

for all Cambrian trees T.

$\mathbb{P}$  $=$

$$\begin{aligned} & \mathbb{F}_{\underline{2}\underline{1}\overline{3}\overline{7}\underline{5}\underline{4}\overline{6}} + \mathbb{F}_{\underline{2}\underline{1}\overline{7}\overline{3}\underline{5}\underline{4}\overline{6}} + \mathbb{F}_{\underline{2}\underline{1}\overline{7}\underline{5}\overline{3}\underline{4}\overline{6}} + \mathbb{F}_{\underline{2}\overline{7}\underline{1}\overline{3}\underline{5}\underline{4}\overline{6}} + \mathbb{F}_{\underline{2}\overline{7}\underline{1}\underline{5}\overline{3}\underline{4}\overline{6}} \\ + & \mathbb{F}_{\underline{2}\overline{7}\underline{5}\underline{1}\overline{3}\underline{4}\overline{6}} + \mathbb{F}_{\overline{7}\underline{2}\underline{1}\overline{3}\underline{5}\underline{4}\overline{6}} + \mathbb{F}_{\overline{7}\underline{2}\underline{1}\underline{5}\overline{3}\underline{4}\overline{6}} + \mathbb{F}_{\overline{7}\underline{2}\underline{5}\underline{1}\overline{3}\underline{4}\overline{6}} + \mathbb{F}_{\overline{7}\underline{5}\underline{2}\underline{1}\overline{3}\underline{4}\overline{6}} \end{aligned}$$

Theorem [C.-Pilaud]

Camb is a Hopf subalgebra of $\mathsf{FQSym}_{\pm}$.

(*i.e.*, the Cambrian congruence is "compatible" with the product and coproduct in $\mathsf{FQSym}_{\pm}$)

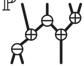
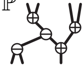
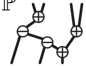
GAME: Explain the product and coproduct directly on the Cambrian trees...

Product in the Cambrian algebra

$\mathbb{P}$  $\cdot\, \mathbb{P}$ 

$$= \mathbb{F}_{\underline{1}\overline{2}} \cdot \left(\mathbb{F}_{\overline{2}\underline{1}\overline{3}} + \mathbb{F}_{\overline{23}\underline{1}}\right)$$

$$= \begin{pmatrix} \mathbb{F}_{\underline{1}\overline{24}\underline{3}\overline{5}} + \mathbb{F}_{\underline{1}\overline{245}\underline{3}} + \mathbb{F}_{\underline{1}\overline{42}\underline{3}\overline{5}} \\ + \mathbb{F}_{\underline{1}\overline{425}\underline{3}} + \mathbb{F}_{\underline{1}\overline{452}\underline{3}} + \mathbb{F}_{\overline{4}\underline{1}\overline{2}\underline{3}\overline{5}} \\ + \mathbb{F}_{\overline{4}\underline{1}\overline{25}\underline{3}} + \mathbb{F}_{\overline{4}\underline{1}\overline{52}\underline{3}} + \mathbb{F}_{\overline{45}\underline{1}\overline{2}\underline{3}} \end{pmatrix} + \begin{pmatrix} \mathbb{F}_{\underline{1}\overline{4}\underline{3}\overline{25}} + \mathbb{F}_{\underline{1}\overline{4}\underline{3}\overline{52}} \\ + \mathbb{F}_{\underline{1}\overline{45}\underline{3}\overline{2}} + \mathbb{F}_{\overline{4}\underline{13}\overline{25}} \\ + \mathbb{F}_{\overline{4}\underline{13}\overline{52}} + \mathbb{F}_{\overline{4}\underline{1}\overline{5}\underline{3}\overline{2}} \\ + \mathbb{F}_{\overline{45}\underline{13}\overline{2}} \end{pmatrix} + \begin{pmatrix} \mathbb{F}_{\overline{4}\underline{31}\overline{25}} + \mathbb{F}_{\overline{4}\underline{31}\overline{52}} \\ + \mathbb{F}_{\overline{4}\underline{3}\overline{5}\underline{1}\overline{2}} + \mathbb{F}_{\overline{45}\underline{31}\overline{2}} \end{pmatrix}$$

$= \mathbb{P}$  $+ \ \mathbb{P}$  $+ \ \mathbb{P}$ 

Proposition [C.-Pilaud]

For any Cambrian trees T and T',

$$\mathbb{P}_{\mathrm{T}} \cdot \mathbb{P}_{\mathrm{T}'} = \sum_{{}_{\mathrm{T}}\nearrow^{\overline{\mathrm{T}'}} \ \leq_{\text{Camb}} \ \mathrm{S} \ \leq_{\text{Camb}} \ {}^{\mathrm{T}}\nwarrow_{\overline{\mathrm{T}'}}} \mathbb{P}_{\mathrm{S}}$$

Product in the Cambrian algebra

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- For every Cambrian tree T, $\mathcal{L}(T)$ is an interval of the weak order.

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- $[\sigma, \tau] \, \overline{\sqcup\!\sqcup} \, [\sigma', \tau'] = [\sigma\overline{\sigma'}, \overline{\tau'}\tau]$.

Product in the Cambrian algebra

Proposition [C.-Pilaud]

For any Cambrian trees T and T',

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- For every Cambrian tree T, $\mathcal{L}(T)$ is an interval of the weak order.
- $[\sigma, \tau] \,\overline{\sqcup\!\sqcup}\, [\sigma', \tau'] = [\sigma\overline{\sigma'}, \overline{\tau'}\tau]$.
- $\mathcal{L}(T) = [\sigma, \tau], \mathcal{L}(T') = [\sigma', \tau']$

$$\mathbf{P}(\sigma\overline{\sigma'}) = T \nearrow \overline{T'} \qquad \mathbf{P}(\overline{\tau'}\tau) = \overline{T'} \nwarrow T$$

Product in the Cambrian algebra

Proposition [C.-Pilaud]

For any Cambrian trees T and T',

$$\mathbb{P}_{\mathrm{T}} \cdot \mathbb{P}_{\mathrm{T}'} = \sum_{\begin{smallmatrix} & & \overline{\mathrm{T}'} \\ & \nearrow & \\ \mathrm{T} & & \end{smallmatrix} \;\leq_{\mathsf{Camb}}\; \mathrm{S} \;\leq_{\mathsf{Camb}}\; \begin{smallmatrix} \mathrm{T} & & \\ & \nwarrow & \\ & & \overline{\mathrm{T}'} \end{smallmatrix}} \mathbb{P}_{\mathrm{S}}$$

- For every Cambrian tree T, $\mathcal{L}(T)$ is an interval of the weak order.
- $[\sigma, \tau] \,\overline{\sqcup\!\sqcup}\, [\sigma', \tau'] = [\sigma\overline{\sigma'}, \overline{\tau'}\tau]$.
- $\mathcal{L}(T) = [\sigma, \tau], \mathcal{L}(T') = [\sigma', \tau']$

$$\mathbf{P}(\sigma\overline{\sigma'}) = \begin{smallmatrix} & & \overline{T'} \\ & \nearrow & \\ T & & \end{smallmatrix} \qquad \mathbf{P}(\overline{\tau'}\tau) = \begin{smallmatrix} T & & \\ & \nwarrow & \\ & & \overline{T'} \end{smallmatrix}$$

- Permutations in $[\sigma\overline{\sigma'}, \overline{\tau'}\tau]$ form a union of Cambrian classes because Camb is a subalgebra of $\mathrm{FQSym}_{\pm}$.

Product in the Cambrian algebra

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For any Cambrian trees T and T',

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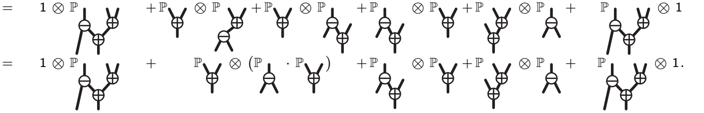
- Permutations in $[\sigma\overline{\sigma'}, \overline{\tau'}\tau]$ form a union of Cambrian classes because Camb is a subalgebra of $\mathrm{FQSym}_{\pm}$.
- These trees form an interval because **P** is a lattice homomorphism.

Coproduct in the Cambrian algebra

$\triangle\mathbb{P}$  $= \triangle\left(\mathbb{F}_{\bar{2}\underline{1}\bar{3}} + \mathbb{F}_{\bar{2}\bar{3}\underline{1}}\right)$

$= 1 \otimes \left(\mathbb{F}_{\bar{2}\underline{1}\bar{3}} + \mathbb{F}_{\bar{2}\bar{3}\underline{1}}\right) + \mathbb{F}_{\bar{1}} \otimes \mathbb{F}_{\underline{1}\bar{2}} + \mathbb{F}_{\bar{1}} \otimes \mathbb{F}_{\bar{2}\underline{1}} + \mathbb{F}_{\bar{2}\underline{1}} \otimes \mathbb{F}_{\bar{1}} + \mathbb{F}_{\bar{1}\bar{2}} \otimes \mathbb{F}_{\underline{1}} + \left(\mathbb{F}_{\bar{2}\underline{1}\bar{3}} + \mathbb{F}_{\bar{2}\bar{3}\underline{1}}\right) \otimes 1$

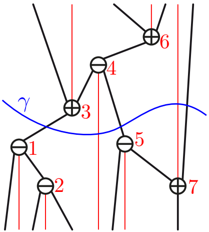


Proposition [C.-Pilaud]

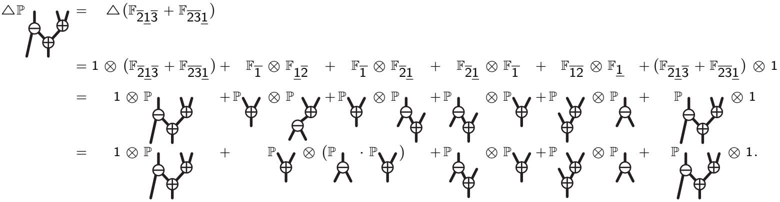
For any Cambrian tree S,

$$\triangle\mathbb{P}_{\mathrm{S}} = \sum_{\gamma} \left(\prod_{\mathrm{T} \in B(\mathrm{S},\gamma)} \mathbb{P}_{\mathrm{T}} \right) \otimes \left(\prod_{\mathrm{T}' \in A(\mathrm{S},\gamma)} \mathbb{P}_{\mathrm{T}'} \right)$$

where γ runs over all cuts of S, and $A(\mathrm{S},\gamma)$ and $B(\mathrm{S},\gamma)$ denote the Cambrian forests above and below γ respectively.



Coproduct in the Cambrian algebra



Proposition [C.-Pilaud]

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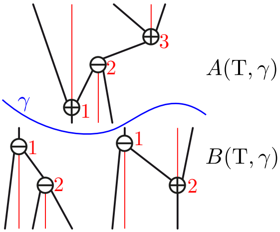
where γ runs over all cuts of S, and $A(\mathrm{S},\gamma)$ and $B(\mathrm{S},\gamma)$ denote the Cambrian forests above and below γ respectively.



Extensions

- Study of multiplicative basis.
- Hopf algebra on twin Cambrian trees.
- Hopf algebra on Schröder-Cambrian trees.