Random subgroups of a free group

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Joint work with Armando Martino, Cyril Nicaud, Enric Ventura et Pascal Weil

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Introduction

- Any group is isomorphic to a quotient group of some free group.
- Study of algebraic properties of free groups by combinatorial methods
  - Graphical representation of subgroups : Stallings graphs
  - Combinatorial interpretation of parameters or properties like the rank, malnormality, Whitehead minimality, ...
- Quantitative study of finitely generated subgroups of a free group and analysis of related algorithms

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I. Free Group

Free group : a definition

- A group F is *free* if there is a subset A of F such that any element of F can be uniquely written as a finite product of elements of A and their inverses.
- The cardinality of A is the *rank* of the free group.
- Apart from the existence of inverses no other relation exists between the generators of a free group.

Basic properties

- The subgroups of a free group are free (Nielsen-Schreier Theorem).
- A free group with finite rank contains subgroups with any countable rank.

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Free groups and reduced words

- Let A be a **finite** alphabet and $F = F(A)$ be the free group over A.
- The elements of $F(A)$ are uniquely represented by the *reduced* words over $A \cup A^{-1}$ where $A^{-1} = \{a^{-1} \mid a \in A\}$,
- A word is *reduced* if it does not contain factors of the form aa^{-1}
- Examples : $ab^{-1}b^{-1}aaba^{-1}$ is reduced,
  $aab^{-1}a^{-1}abcca^{-1}$ is not reduced
- Reduction of a word : replace **in any order** all occurrences of aa^{-1} by the empty word ϵ.
- Example :

$$aab^{-1}a^{-1}abcca^{-1} = aab^{-1}bcca^{-1} = aacca^{-1}$$

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Finitely generated subgroups

We are interested in finitely generated free subgroups, *i.e.*, obtained from a finite set of generators.

- Finitely generated free subgroups can be represented in a unique way by a finite graph called its **Stallings graph** (Stallings 1983).
- This description is very useful, some properties of the subgroup can be directly obtained from its graph representation.

A 1st goal

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Stallings foldings

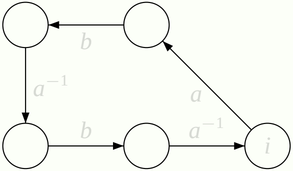
Let $Y = \{aba^{-1}ba^{-1}, b^2a^{-1}, b^3a^{-1}b^{-1}\}$.

Goal

Build a directed graph representing the free subgroup generated by Y

First step

Build a directed cycle labeled with $aba^{-1}ba^{-1}$ the first element of Y



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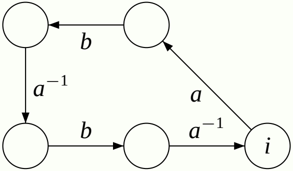
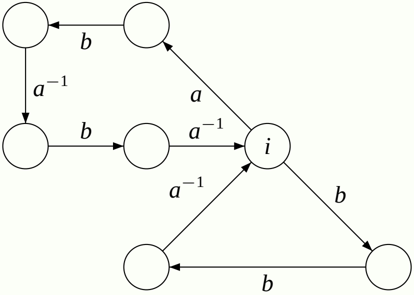


Stallings foldings

Second step

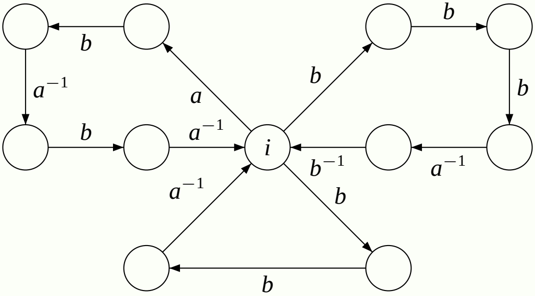
Build from the same vertex i a directed cycle labeled with b^2a^{-1} the second element of Y.



Stallings foldings

Third step

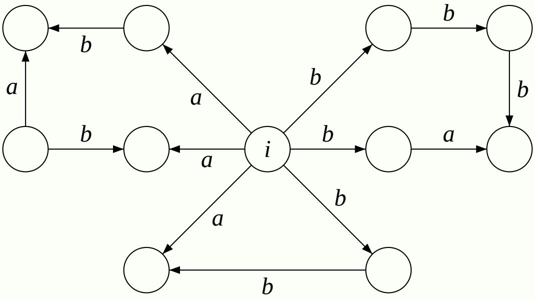
Build from the same vertex i a directed cycle labeled with $b^3a^{-1}b^{-1}$ the third and last element of Y.



Stallings foldings

Formal inverses

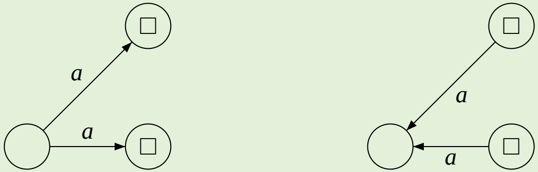
Reverse all edges labeled by a^{-1} are and replace their label by a.



Stallings foldings

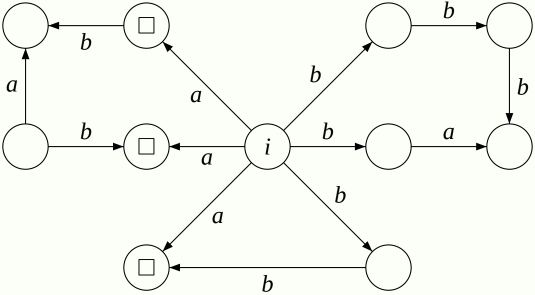
Foldings to obtain determinism and codeterminism

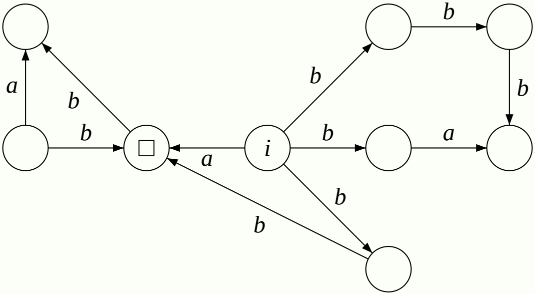
Apply as many times as possible the following rules of merging (or folding) :



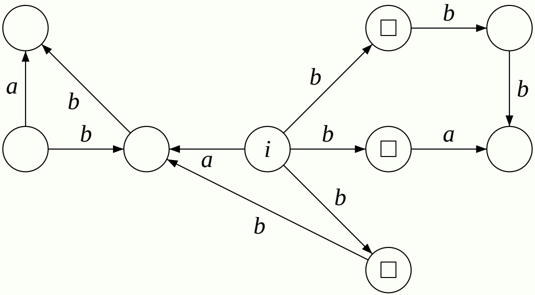
The result does not depend on the order in which the transformations are performed.

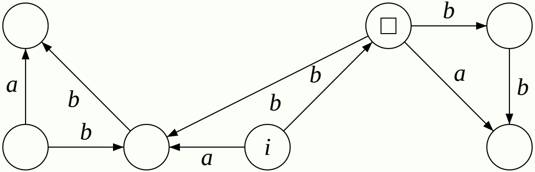
Stallings foldings - 1st folding



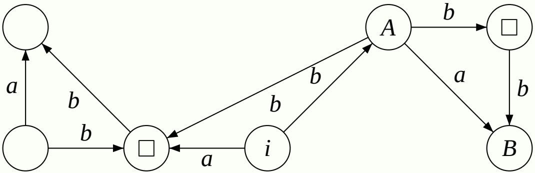


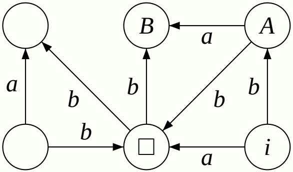
Stallings foldings - 2nd folding



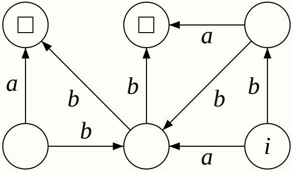


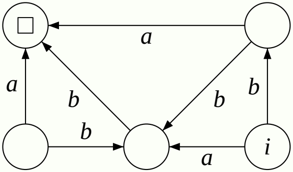
Stallings foldings - 3rd folding



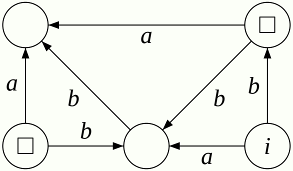


Stallings foldings - 4th folding



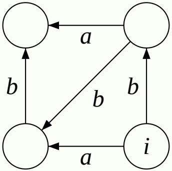


Stallings foldings - Last folding and Stallings graph



The Stallings graph representing the free subgroup generated by

$$Y = \{aba^{-1}ba^{-1}, b^2a^{-1}, b^3a^{-1}b^{-1}\}.$$



Stallings graphs : a definition

The graph (with a distinguished vertex *i*) obtained is a *Stallings graph*.

Stallings graph

- It is deterministic and co-deterministic : each letter acts like a **partial injection** on the set of states.
- it is connected
- all but the distinguished state *i* have degree at least two

Unicity of the representation

A Stallings graph represents in a unique way a finitely generated subgroup of the free group generated by the alphabet of the labels.

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Stallings graphs – examples of use

- One can check whether a (reduced) word belongs the subgroup or not.
  Check if there exists a cycle labeled by the word beginning in i
- One can compute a basis and the rank of the subgroup

$$rank = |E| - (|V| - 1)$$

  To obtain a basis, choose a spanning tree of the Stallings graph. Each edge e that is not in the tree corresponds to a generator of the base : the label of a cycle beginning in i using e and edges in the spanning tree.
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  All letters act like permutations on the set of vertices

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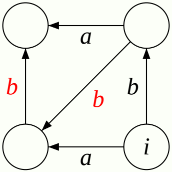
Example for the rank

The Stallings graph of the subgroup genrated by
$Y = \{aba^{-1}ba^{-1}, b^2a^{-1}, b^3a^{-1}b^{-1}\}$:



Therefore $\{b^2a^{-1}, aba^{-1}b^{-1}\}$ is a basis of the subgroup and the rank is 2.

Stallings graphs – algorithmic point of view

- Stalling foldings can be computed in $O(n \log^* n)$ where n is the total length of the generators. The algorithm due Touikan (2006) makes use of ”Union and Find”.
- The intersection (resp. union) of two subgroups can be computed in time and space $O(n_1 \times n_2)$ where n_1 (resp. n_2) is the size (here the number of vertices) of the first (resp. second) Stallings graph.

II. Distributions on Subgroups

A graph-based distribution on subgroups

- A random subgroup is given by choosing uniformly at random a **Stallings graph of size** n
- Studied by Bassino, Nicaud, Weil (2008, 2013, 2015)
- What does the Stallings graph of such a random subgroup look like ?

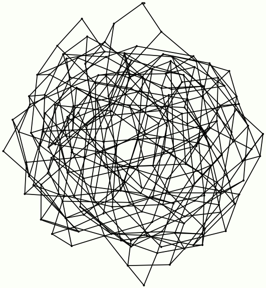


FIGURE: A random subgroup with 200 vertices for the graph-based distribution (The alphabet is of size 2).

The classical word-based distribution on subgroups

- A random subgroup is given by choosing randomly and uniformly k generators of length at most n, where k is fixed
- Studied by Gromov (1987), Arzhantseva and Ol'shanskiĭ (1996), Jitsukawa (2002), ...
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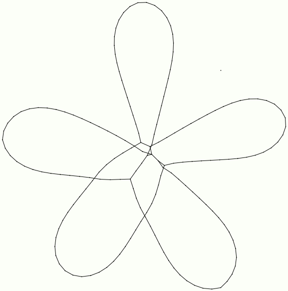

FIGURE: A random subgroup for the word-based distribution with 5 words of lengths at most 40 (The alphabet is of size 2.)

A word-based distribution (few generators)

- Fix the number k of generators and the maximal length n of each generator.
- Consider the uniform distribution over the k-tuples of reduced words of length at most n.
- Let R_n the number of reduced words of length n,

$$R_n = 2r(2r-1)^{n-1}$$

- The length of word in a random k-tuple is near to n.

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A word-based distribution (few generators)

Length, prefixes and suffixes

- Let $0 < \alpha < 1$. A reduced word in R_n has length greater than αn with probability that tends toward 1 when n tends toward $+\infty$.
- Let $0 < \beta < \alpha/2$. A k-uple of reduced words of R_n is such that the prefixes of length βn of all words and their inverses are pairwise distinct with probability that tends toward 1 when n tends toward $+\infty$.

Consequence

Each of the k reduced words has an outer loop of length at least $n(\alpha - 2\beta)$ with probability that tends to 1 when n tends to $+\infty$.

A graph-based distribution : Probabilistic results

Theorem (Bassino, Nicaud, Weil 2008)

The probability for a random r-tuple of partial injections of size n to form a Stallings graph tends toward 1 when n tends toward $+\infty$.

Stallings graph

- It is deterministic and co-deterministic : each letter acts like a **partial injection** on the set of states.
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The proof

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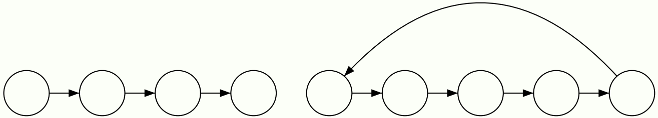


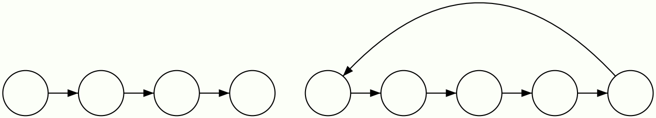
- A partial injection can be seen as a set of cycles and of non-empty sequences.
- Set(Cycle or non-empty Sequences)
- With the symbolic method :

$$I(z) = \sum_{n \geq 0} \frac{I_n}{n!} z^n = \exp\left(\log \frac{1}{1-z} + \frac{z}{1-z}\right) = \frac{1}{1-z} e^{z/(1-z)}$$

- With the saddle point method :

$$\frac{I_n}{n!} \sim \frac{e^{-\frac{1}{2}}}{2\sqrt{\pi}} e^{2\sqrt{n}} n^{-\frac{1}{4}}$$

A graph-based distribution : Partial injections



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Connectedness

Theorem

The probability for r partial injections of size n to form a connected graph is

$$p_n = 1 - \frac{2^r}{n^{r-1}} + o\left(\frac{1}{n^{r-1}}\right)$$

Proof

Let $J(z) = \sum_{n>0} j_n z^n = \sum_{n>0} I_n^r z^n / n!$.
Then $1 + J(z) = \exp(C(z))$ and $C(z) = \log(1 + J(z))$.
From a Bender theorem (1974) it is enough to check that $j_n = o(j_{n-1})$ and that for some $s \geq 1$, $\sum_{k=s}^{n-s} |j_k j_{n-k}| = O(j_{n-s})$, to obtain that

$$c_n = j_n \left(1 - \frac{2^r}{n^{r-1}} + o\left(\frac{1}{n^{r-1}}\right)\right)$$

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Vertices with zero or one outgoing or ingoing edge

- If x is a vertex with 0 or 1 edge, then x must be **isolated** for $r-1$ injections and **an endpoint** for the remaining injection.
- The probability it is isolated for an injection is $\frac{I_{n-1}}{I_n}$, which is smaller than $\frac{1}{n}$.
- Let $I_{n,k}$ be the number of size-n injections **having** k **sequences**, and let $I(z,u)$ be the bivariate generating function defined by :

$$I(z,u) = \exp\left(\frac{zu}{1-z} + \log\left(\frac{1}{1-z}\right)\right) = \frac{1}{1-z}\exp\left(\frac{zu}{1-z}\right)$$

Using the **saddle point theorem** we obtain that the expected number of sequences is $\frac{1}{\sqrt{n}}$ and that the probability that a given vertex is an endpoint is in $\mathcal{O}(\frac{1}{\sqrt{n}})$.

Trimness

Therefore

- A given vertex has degree 0 or 1 with probability $\mathcal{O}(n^{-r+1/2})$,
- there is such a vertex with probability $\mathcal{O}(n^{-r+3/2})$
- **with probability at least $\mathcal{O}(n^{-1/2})$ the graph has no such vertex.**

IV. How to compare the two distributions

Méthod

- A property P is *generic* for (X_n) when the probability for an element of X_n to satisfy P tends toward 1 when n tends toward ∞.
- A property P is *negligible* for (X_n) when the probability for an element of X_n to satisfy P tends toward O when n tends toward ∞.
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Experimental results

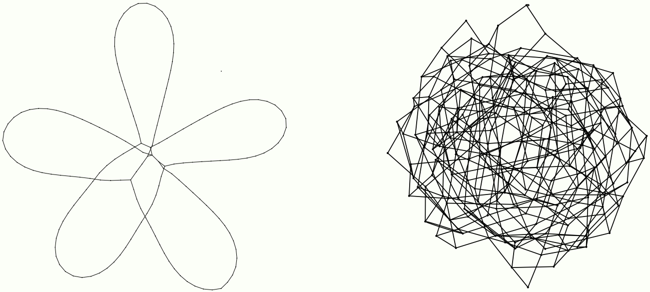

FIGURE: On the left, a random subgroup for the word-based distribution with 5 words of lengths at most 40. On the right, a random subgroup with 200 vertices for the graph-based distribution (The alphabet is of size 2).

Rank

- One can compute the rank of a finitely generated subgroup from its Stallings graph

$$rank = |E| - (|V| - 1)$$

- In the word based distribution (k words of maximal length n), the average rank is k
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Malnormality

- A subgroup H of G is **normal** when for any $g \in G$, $g^{-1}Hg = H$.
- A subgroup is **malnormal** when for any $g \notin H$, $g^{-1}Hg \cap H = 1$.

Theorem (combinatorial characterization)

A subgroup of a free group is **non-malnormal** if and only, in its Stallings graph, if there exists two vertices $x \neq y$ and a non-empty reduced word u, such that

u is the label of a loop on x and of a loop on y.

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Theorem

For the word-based distribution, malnormality is **generic**, but it is **negligible** for the graph-based.

Proof

- The proof in the word-based distribution is due to Jitsukawa (2002). Basically loops are long enough to be distinct with high probability.
- The probability that a partial injection contains at most one cycle and that the length of this cycle is 1 is $\sim \frac{e}{\sqrt{n}}$.

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Finite presentation

- The idea is to quotient the free group by a normal finitely generated subgroup. Let E be an arbitrary subset, and $N(E)$ be its normal closure, that is the smallest normal subgroup containing E.
- Equivalently each word x of E becomes a relator $x = 1$.
- In the word-based distribution generically the quotient subgroup is infinite (Jitsukawa, 2002).
- But in the graph-based distribution, the quotient group is generically trivial.

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Generically the gcd of the lengths of the cycles of a **partial injection** of size n is 1.

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Generically the permutation part of a size n injection is greater than $n^{1/3}$ and the gcd of the length of the cycles is 1.

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Thank you for your attention !