Reflections in Persistence and Quiver Theory

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(co-)homology functors connect the topological level to the algebraic level. Conversely, the algebraic level provides stable signatures to be exploited at the topological level.

**Persistence**

- Morse theory
- Spectral sequences
- Clustering
- Hierarchical mode analysis
- Size theory
- Representation theory
  - Polynomial rings
  - Quivers / Path algebras
Quiver theory
**Exploratory data analysis**

**Setup:** $K \subset \mathbb{R}^d$ a compact set, $p_1, \cdots, p_n$ data points sampled along (or close to) $K$

**Goal:** recover structural information about $K$, knowing only $p_1, \cdots, p_n$
Challenges in data analysis

① Scale
Challenges in data analysis

① Scale
Inferring the topology of data

algebraic invariants for classification

\[ \beta_0 = \beta_2 = 1 \]
\[ \beta_1 = 2 \]

algebraic signatures for inference

A.T. in the 20th century

A.T. in the 21st century
Intuitive viewpoint: hierarchical clustering
Intuitive viewpoint: hierarchical clustering

(single-linkage)
Intuitive viewpoint: hierarchical clustering

(single-linkage)
Intuitive viewpoint: hierarchical clustering

(single-linkage)
Intuitive viewpoint: hierarchical clustering

(single-linkage)
Intuitive viewpoint: hierarchical clustering
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(single-linkage)
Intuitive viewpoint: hierarchical clustering
Intuitive viewpoint: hierarchical clustering

(single-linkage)
Intuitive viewpoint: hierarchical clustering

dendrogram is:
- informative
- unstable

(single-linkage)
Intuitive viewpoint: hierarchical clustering

dendrogram → barcode
Intuitive viewpoint: hierarchical clustering
Intuitive viewpoint: hierarchical clustering

dendrogram → barcode
Intuitive viewpoint: hierarchical clustering

barcode is:
- less (but still) informative
- more stable
Intuitive viewpoint: hierarchical clustering

Barcode is:
- less (but still) informative
- more stable
- generalizable
Intuitive viewpoint: hierarchical clustering

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- generalizable
Mathematical viewpoint

Filtration: $F_1 \subseteq F_2 \subseteq F_3 \subseteq F_4 \subseteq F_5 \cdots$

Example 1: offsets filtration (nested family of unions of balls, cf. previous slide)
Mathematical viewpoint

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Example 1: *offsets filtration* (nested family of unions of balls, cf. previous slide)

Example 2: *simplicial filtration* (nested family of simplicial complexes)
**Mathematical viewpoint**

Filtration: \( F_1 \subseteq F_2 \subseteq F_3 \subseteq F_4 \subseteq F_5 \cdots \)

Example 1: *offsets filtration* (nested family of unions of balls, cf. previous slide)

Example 2: *simplicial filtration* (nested family of simplicial complexes)

Example 3: *sublevel-sets filtration* (family of sublevel sets of a function \( f : X \to \mathbb{R} \))

\[ F_\alpha := f^{-1}((-\infty, \alpha]) \]
Mathematical viewpoint

Filtration: \( F_1 \subseteq F_2 \subseteq F_3 \subseteq F_4 \subseteq F_5 \cdots \)

Persistence module: \( H_*(F_1) \rightarrow H_*(F_2) \rightarrow H_*(F_3) \rightarrow H_*(F_4) \rightarrow H_*(F_5) \cdots \)
Mathematical viewpoint

Zigzag: \( F_1 \subseteq F_2 \supseteq F_3 \supseteq F_4 \subseteq F_5 \cdots \)

Zigzag module: \( H_\ast(F_1) \rightarrow H_\ast(F_2) \leftarrow H_\ast(F_3) \leftarrow H_\ast(F_4) \rightarrow H_\ast(F_5) \cdots \)
Mathematical viewpoint

Example:

\[
\begin{array}{cccc}
\triangle & \subseteq & \triangle & \subseteq \\
\subseteq & \subseteq & \triangle & \subseteq \\
\end{array}
\]

(1-homology functor)

\[
k \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} k^2 \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} k \xrightarrow{\begin{pmatrix} 0 \end{pmatrix}} k^2 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} k^2 \ldots
\]
Example:

\[
\begin{align*}
\mathfrak{k} & \subseteq \mathfrak{k}^2 \\
\mathfrak{k} & \subseteq \mathfrak{k}^2 \\
\mathfrak{k} & \subseteq \mathfrak{k}^2 \\
\mathfrak{k} & \subseteq \mathfrak{k}^2
\end{align*}
\]

(1-homology functor)

\[
\mathfrak{k} \xrightarrow{0} \mathfrak{k}^2 \xleftarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} \mathfrak{k} \xleftarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} \mathfrak{k}^2 \xrightarrow{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} \mathfrak{k}^2 \ldots
\]
**Theorem.** Let $\mathbb{V}$ be a persistence/zigzag module over an index set $T \subseteq \mathbb{R}$. Then, $\mathbb{V}$ decomposes as a direct sum of interval modules $\mathbb{I}[b^*, d^*]$: $\mathbb{V} \cong \bigoplus_{j \in J} \mathbb{I}[b_j^*, d_j^*]$. (the barcode is a complete descriptor of the algebraic structure of $\mathbb{V}$)
**Theorem.** Let $\mathbb{V}$ be a persistence/zigzag module over an index set $T \subseteq \mathbb{R}$. Then, $\mathbb{V}$ decomposes as a direct sum of interval modules $\mathbb{I}[b^*, d^*]:$

\[
\begin{array}{cccccccccccc}
0 & 0 & \cdots & 0 & 0 & k & 1 & \cdots & 1 & k & 0 & 0 & \cdots & 0 & 0 \\
\hline
i < b^* & \quad & \quad & \quad & \quad & [b^*, d^*] & \quad & \quad & \quad & \quad & i > d^*
\end{array}
\]

in the following cases:

- $T$ is finite [Gabriel 1972] [Auslander 1974],
- all arrows are forward and $\mathbb{V}$ is **pointwise finite-dimensional** (i.e. every space $V_t$ has finite dimension) [Webb 1985] [Crawley-Boevey 2012].

Moreover, when it exists, the decomposition is **unique** up to isomorphism and permutation of the terms [Azumaya 1950].

(Note: this is independent of the choice of field $k$.)
Persistence Modules vs. Quiver Representations

\( k \): field of coefficients

persistence/zigzag module:  
\[
\begin{align*}
\mathbb{k} & \xrightarrow{0} \mathbb{k}^2 & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\mathbb{k} & \xleftrightarrow{0 \ 1} & \mathbb{k}^2 & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \rightarrow \mathbb{k}^2
\end{align*}
\]

module type:  
\[
\begin{array}{c}
\bullet \xrightarrow{a} \bullet \xleftrightarrow{b} \bullet \xleftrightarrow{c} \bullet \xrightarrow{d} \bullet \\
1 & \xrightarrow{a} & 2 & \xleftrightarrow{b} & 3 & \xleftrightarrow{c} & 4 & \xrightarrow{d} & 5
\end{array}
\]
Persistence Modules vs. Quiver Representations

$k$: field of coefficients

quiver representation:

\[
\begin{array}{c}
k \\ & \xrightarrow{0} \rightarrow & k^2 \\
& \xleftarrow{(1 \ 0)} \leftarrow \\
& \xleftarrow{(0 \ 1)} \leftarrow k^2 \\
& \xrightarrow{(0 \ 1)} \rightarrow \\
\end{array}
\]

(path) quiver:

\[
\begin{array}{c}
1 \\ & a \rightarrow \\
2 & \xleftarrow{b} \leftarrow \\
3 & \xleftarrow{c} \leftarrow \\
4 & \xrightarrow{d} \rightarrow \\
5
\end{array}
\]
Outline

• quivers and representations, classification, Gabriel’s theorem

• reflection functors, proof of Gabriel’s theorem ($A_n$ case)

• application 1: computing persistence for zigzags

• application 2: zigzags for topological inference
Quivers and Representations

Definition: A quiver $Q$ consists of two sets $Q_0, Q_1$ and two maps $s, t : Q_1 \to Q_0$. The elements in $Q_0$ are called the vertices of $Q$, while those of $Q_1$ are called the arrows. The source map $s$ assigns a source $s_a$ to every arrow $a \in Q_1$, while the target map $t$ assigns a target $t_a$. 

\begin{figure}
\centering
\begin{tikzpicture}

\node (a) at (0,0) {$2$};
\node (b) at (1,1) {$1$};
\node (c) at (2,0) {$3$};

\draw[-latex] (a) -- (b) node[midway,above] {$a$};
\draw[-latex] (b) -- (c) node[midway,above] {$c$};
\draw[-latex] (c) -- (a) node[midway,above] {$d$};
\draw[-latex] (b) edge[loop below] node[left] {$b$} (b);
\draw[-latex] (a) edge[loop above] node[right] {$e$} (a);
\end{tikzpicture}
\end{figure}
**Quivers and Representations**

**Definition:** A quiver $Q$ consists of two sets $Q_0, Q_1$ and two maps $s, t : Q_1 \rightarrow Q_0$. The elements in $Q_0$ are called the vertices of $Q$, while those of $Q_1$ are called the arrows. The source map $s$ assigns a source $s_a$ to every arrow $a \in Q_1$, while the target map $t$ assigns a target $t_a$.

**Dynkin quivers:**

$A_n(n \geq 1)$

$D_n(n \geq 4)$

$E_6$

$E_7$

$E_8$
Quivers and Representations

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Dynkin quivers:

\[
\begin{align*}
A_n(n \geq 1) & \quad \begin{array}{c}
\bullet \\
1 & 2 & \cdots & n-1 & n
\end{array} \\
D_n(n \geq 4) & \quad \begin{array}{c}
\bullet \\
1 & 2 & \cdots & n-2 \\
\bullet & \bullet & \cdots & \bullet
\end{array}
\end{align*}
\]
**Definition:** A representation of $Q$ over a field $k$ is a pair $\mathcal{V} = (V_i, v_a)$ consisting of a set of $k$-vector spaces $\{V_i \mid i \in Q_0\}$ together with a set of $k$-linear maps $\{v_a : V_{s_a} \to V_{t_a} \mid a \in Q_1\}$.
Quivers and Representations

Definition: A representation of $Q$ over a field $k$ is a pair $V = (V_i, v_a)$ consisting of a set of $k$-vector spaces $\{V_i \mid i \in Q_0\}$ together with a set of $k$-linear maps $\{v_a : V_{s_a} \to V_{t_a} \mid a \in Q_1\}$.

Note: diagram commutativity is not required
Quivers and Representations

**Definition:** A *morphism* $\phi$ between two $k$-representations $V, W$ of $Q$ is a set of $k$-linear maps $\phi_i : V_i \to W_i$ such that $w_a \circ \phi_{s_a} = \phi_{t_a} \circ v_a$ for every arrow $a \in Q_1$. 

\[ \begin{array}{c}
\text{Q} \\
2 \rightarrow c \rightarrow 3
\end{array} \]
Quivers and Representations

Definition: A morphism $\phi$ between two $k$-representations $V, W$ of $Q$ is a set of $k$-linear maps $\phi_i : V_i \to W_i$ such that $w_u \circ \phi_s = \phi_t \circ v_a$ for every arrow $a \in Q_1$.

\[
\begin{align*}
\text{every quadrangle commutes}
\end{align*}
\]
Definition: A morphism $\phi$ between two $k$-representations $V, W$ of $Q$ is a set of $k$-linear maps $\phi_i : V_i \to W_i$ such that $w_a \circ \phi_s = \phi_t \circ v_a$ for every arrow $a \in Q_1$.

Note: $\phi$ isomorphism iff every $\phi_i$ isomorphism
The Category of Representations

The representations of a quiver $Q = (Q_0, Q_1)$, together with their morphisms, form a category called $\text{Rep}_k(Q)$. This category is abelian:

- it contains a zero object, namely the trivial representation

\[ 0 \rightarrow 0 \leftarrow 0 \leftarrow 0 \rightarrow 0 \]
The Category of Representations

The representations of a quiver $Q = (Q_0, Q_1)$, together with their morphisms, form a category called $\text{Rep}_k(Q)$. This category is abelian:

- it contains a zero object, namely the trivial representation
- it has internal and external direct sums, defined pointwise. For any $V, W$, the representation $V \oplus W$ has spaces $V_i \oplus W_i$ for $i \in Q_0$ and maps $v_a \oplus w_a = \begin{pmatrix} v_a & 0 \\ 0 & w_a \end{pmatrix}$ for $a \in Q_1$

$$
\begin{align*}
\begin{array}{ccc}
k & \xrightarrow{(1 \ 0)} & k^2 \\
\oplus & & \\
k & \xleftarrow{(0 \ 1)} & k^2 \\
\end{array}
\end{align*}
$$

$$
\begin{align*}
k & \xrightarrow{0} & 0 \\
\oplus & & \\
k & \xleftarrow{1} & k & \xrightarrow{0} & 0 \\
\end{align*}
$$

$$
\begin{align*}
k^2 & \xrightarrow{(1 \ 0 \ 0)} & k^2 \\
\oplus & & \\
k^2 & \xleftarrow{(0 \ 1 \ 0)} & k^2 \\
\oplus & & \\
k^3 & \xrightarrow{(1 \ 0 \ 0 \ 0)} & k^2 \\
\end{align*}
$$
The Category of Representations

The representations of a quiver $Q = (Q_0, Q_1)$, together with their morphisms, form a category called $\text{Rep}_k(Q)$. This category is **abelian**: 

- it contains a zero object, namely the *trivial representation*
- it has internal and external direct sums, defined *pointwise*. For any $V, W$, the representation $V \oplus W$ has spaces $V_i \oplus W_i$ for $i \in Q_0$ and maps $v_a \oplus w_a = \begin{pmatrix} v_a & 0 \\ 0 & w_a \end{pmatrix}$ for $a \in Q_1$
- every morphism has a *kernel*, an *image* and a *cokernel*, defined *pointwise*.

→ a morphism $\phi$ is injective iff $\ker \phi = 0$, and surjective iff $\text{coker} \phi = 0$. 

$$
\begin{array}{cccccc}
\text{k} & \xrightarrow{(1)} & \text{k}^2 & \xleftarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} & \text{k} & \xleftarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} & \text{k}^2 & \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} & \text{k}^2 \\
\text{k} \xrightarrow{1} & 0 & \xleftarrow{-1} & \text{k} & \xleftarrow{\begin{pmatrix} 0 & -1 \end{pmatrix}} & \text{k} \xrightarrow{0} \text{k} \xrightarrow{0} \text{k} \xrightarrow{0} \text{k} \xrightarrow{0} \text{k} \\
\end{array}
$$

$$
\ker \phi = \begin{array}{cccccc}
0 & \xrightarrow{0} & \text{k}^2 & \xleftarrow{0} & \text{k} & \xleftarrow{0} & \text{k} & \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} & \text{k}^2 \\
\text{k} \xrightarrow{0} & 0 & \xleftarrow{0} & \text{k} & \xrightarrow{1} & \text{k} \xrightarrow{0} \text{k} \xrightarrow{0} \text{k} \xrightarrow{0} \text{k} \xrightarrow{0} \text{k} \\
\end{array}
$$

$$
\text{coker} \phi = 0
$$
The Category of Representations

The representations of a quiver $Q = (Q_0, Q_1)$, together with their morphisms, form a category called $\text{Rep}_k(Q)$. This category is **abelian**:

- it contains a zero object, namely the **trivial representation**
- it has internal and external direct sums, defined **pointwise**. For any $V, W$, the representation $V \oplus W$ has spaces $V_i \oplus W_i$ for $i \in Q_0$ and maps $v_a \oplus w_a = \begin{pmatrix} v_a & 0 \\ 0 & w_a \end{pmatrix}$ for $a \in Q_1$
- every morphism has a **kernel**, an **image** and a **cokernel**, defined **pointwise**.
  \[ \text{a morphism } \phi \text{ is injective iff } \ker \phi = 0, \text{ and surjective iff } \text{coker } \phi = 0. \]

**WARNING:** no semisimplicity (subrepresentations may not be summands)

\[
V = \begin{array}{c}
\mathbb{k} \\
\end{array} \xrightarrow{1} \begin{array}{c}
\mathbb{k} \\
\end{array}
\]

\[
W = \begin{array}{c}
0 \\
\end{array} \xrightarrow{0} \begin{array}{c}
\mathbb{k} \\
\end{array}
\]
The Classification Problem

**Goal:** Classify the representations of a given quiver $Q = (Q_0, Q_1)$ up to isomorphism.
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→ simplifying assumptions:

- \( Q \) is finite and connected
- study the subcategory \( \text{rep}_k(Q) \) of finite-dimensional representations

\[
\dim V = (\dim V_1, \cdots, \dim V_n)^\top, \\
\dim V = \| \dim V \|_1 = \sum_{i=1}^{n} \dim V_i.
\]
The Classification Problem

**Goal:** Classify the representations of a given quiver $Q = (Q_0, Q_1)$ up to isomorphism.

→ simplifying assumptions:

- $Q$ is finite and connected
- study the subcategory $\text{rep}_k(Q)$ of *finite-dimensional* representations

**Theorem:** [Krull-Remak-Schmidt-Azumaya]

$\forall V \in \text{rep}_k(Q), \exists V_1, \cdots, V_r$ indecomposable s.t. $V \cong V_1 \oplus \cdots \oplus V_r$. The decomposition is unique up to isomorphism and reordering.

note: $V$ indecomposable iff there are no $U, W \neq 0$ such that $V \cong U \oplus W$
The Classification Problem

Goal: Classify the representations of a given quiver $Q = (Q_0, Q_1)$ up to isomorphism.

→ simplifying assumptions:
  - $Q$ is finite and connected
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→ problem becomes to identify the indecomposable representations of $Q$
  ($\neq$ from identifying representations with no subrepresentations)
  (no semisimplicity)
Gabriel’s Theorem

**Theorem:** [Gabriel I]
Assuming $Q$ is finite and connected, there are finitely many isomorphism classes of indecomposable representations in $\text{rep}_k(Q)$ iff $Q$ is Dynkin.

- $A_n(n \geq 1)$
  - $1 \rightarrow 2 \rightarrow \cdots \rightarrow n-1 \rightarrow n$

- $D_n(n \geq 4)$
  - $1 \rightarrow 2 \rightarrow \cdots \rightarrow n-2$

- $E_6$
  - $1 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6$

- $E_7$
  - $1 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7$

- $E_8$
  - $1 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7 \rightarrow 8$
Gabriel’s Theorem

**Theorem:** [Gabriel I]
Assuming $Q$ is finite and connected, there are finitely many isomorphism classes of indecomposable representations in $\text{rep}_k(Q)$ iff $Q$ is Dynkin.

(does not depend on the choice of field and of arrow orientations)

\begin{align*}
A_n(n \geq 1) & \quad \begin{array}{c}
1 \quad 2 \quad \cdots \quad n-1 \quad n
\end{array} \\
E_6 & \quad \begin{array}{c}
1 \quad 3 \quad 4 \quad 5 \quad 6
\end{array} \\
E_7 & \quad \begin{array}{c}
1 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7
\end{array} \\
E_8 & \quad \begin{array}{c}
1 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8
\end{array}
\end{align*}
Gabriel’s Theorem

**Theorem:** [Gabriel I]
Assuming \( Q \) is finite and connected, there are finitely many isomorphism classes of indecomposable representations in \( \text{rep}_k(Q) \) iff \( Q \) is Dynkin.

Given that \( Q \) is Dynkin, how to identify indecomposable representations?
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Given that $Q$ is Dynkin, how to identify indecomposable representations?

**Theorem: [Gabriel II]**
Assuming $Q$ is Dynkin with $n$ vertices, the map $\mathcal{V} \mapsto \dim \mathcal{V}$ induces a bijection between the set of isomorphism classes of indecomposable representations of $Q$ and the set of positive roots of the Tits form of $Q$. 
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Assuming $Q$ is Dynkin with $n$ vertices, the map $\mathcal{V} \mapsto \dim \mathcal{V}$ induces a bijection between the set of isomorphism classes of indecomposable representations of $Q$ and the set of positive roots of the Tits form of $Q$.

(isom. classes of indecomposables are fully characterized by their dim. vectors)
Gabriel’s Theorem

Tits form: given \( Q = (Q_0, Q_1) \) with \( |Q_0| = n \) and \( x = (x_1, \cdots, x_n) \in \mathbb{Z}^n \),

\[
q_Q(x) = \sum_{i \in Q_0} x_i^2 - \sum_{a \in Q_1} x_{s_a} x_{t_a}.
\]

Proposition: \( q_Q \) is positive definite (\( q_Q(x) > 0 \ \forall x \neq 0 \)) iff \( Q \) is Dynkin.

example: \( Q \) of type \( A_n \):

\[
q_Q(x) = \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n-1} x_i x_{i+1}
\]

\[
= \sum_{i=1}^{n-1} \frac{1}{2} (x_i - x_{i+1})^2 + \frac{1}{2} x_1^2 + \frac{1}{2} x_n^2 \geq 0
\]

\[
= 0 \text{ iff } x = (0, \cdots, 0)
\]
Gabriel’s Theorem

**Tits form:** given $Q = (Q_0, Q_1)$ with $|Q_0| = n$ and $x = (x_1, \cdots, x_n) \in \mathbb{Z}^n$,

$$q_Q(x) = \sum_{i \in Q_0} x_i^2 - \sum_{a \in Q_1} x_s x_t a.$$

**Root:** $x \in \mathbb{Z}^n \setminus \{0\}$ is a root if $q_Q(x) \leq 1$. It is positive if $x_i \geq 0 \ \forall i$. 
Gabriel’s Theorem

**Tits form:** given $Q = (Q_0, Q_1)$ with $|Q_0| = n$ and $x = (x_1, \cdots, x_n) \in \mathbb{Z}^n$

$$q_Q(x) = \sum_{i \in Q_0} x_i^2 - \sum_{a \in Q_1} x_{s_a} x_{t_a}.$$  

**Root:** $x \in \mathbb{Z}^n \setminus \{0\}$ is a root if $q_Q(x) \leq 1$. It is positive if $x_i \geq 0 \ \forall i$.

**Proposition:** If $Q$ is Dynkin, then the set of positive roots of $q_Q$ is finite.
Gabriel’s Theorem

**Tits form:** given $Q = (Q_0, Q_1)$ with $|Q_0| = n$ and $x = (x_1, \cdots, x_n) \in \mathbb{Z}^n$,

$$q_Q(x) = \sum_{i \in Q_0} x_i^2 - \sum_{a \in Q_1} x_s a x_t a.$$

**Root:** $x \in \mathbb{Z}^n \setminus \{0\}$ is a root if $q_Q(x) \leq 1$. It is positive if $x_i \geq 0 \ \forall i$.

**Proposition:** If $Q$ is Dynkin, then the set of positive roots of $q_Q$ is finite.

**Theorem:** [Gabriel II]
Assuming $Q$ is Dynkin with $n$ vertices, the map $\mathbb{V} \mapsto \text{dim} \mathbb{V}$ induces a bijection between the set of isomorphism classes of indecomposable representations of $Q$ and the set of positive roots of the Tits form of $Q$. 
Gabriel’s Theorem

element: $Q$ of type $A_n$:

\[
q_Q(x) = \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n-1} x_i x_{i+1}
\]

\[
= \sum_{i=1}^{n-1} \frac{1}{2} (x_i - x_{i+1})^2 + \frac{1}{2} x_1^2 + \frac{1}{2} x_n^2
\]

\[
= 1 \text{ iff } x = (0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0)
\]
Gabriel’s Theorem

eexample: $Q$ of type $A_n$:

\[
\begin{array}{ccccccccccc}
\bullet & \bullet & \cdots & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet
\end{array}
\]

\[
q_Q(x) = \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n-1} x_i x_{i+1}
\]

\[
= \sum_{i=1}^{n-1} \frac{1}{2} (x_i - x_{i+1})^2 + \frac{1}{2} x_1^2 + \frac{1}{2} x_n^2
\]

\[
= 1 \text{ iff } x = (0, \cdots, 0, 1, \cdots, 1, 0, \cdots, 0)
\]

the corresponding indecomp. representations are isomorphic to $\mathbb{I}_Q[b, d]$:

\[
\begin{array}{cccccccccccc}
0 & 0 & \cdots & 0 & 0 & 0 & \bullet & \bullet & \cdots & \bullet & \bullet & \bullet & \bullet & 0
\end{array}
\]

\[
\begin{array}{cccccccccccc}
[1, b-1] & [b, d] & [d+1, n]
\end{array}
\]
Reflection Functors

**Advantage:** explains the fact that only the dimension vectors play a role in the identification of indecomposable representations. In particular, arrow orientations are irrelevant.

**Idea:** modify quivers by reversing arrows, and study the effect on their representations (peeling off summands).
Reflection Functors

\[ \begin{array}{ccc}
1 & \rightarrow & 2 \\
\uparrow & & \downarrow \\
\downarrow & & \uparrow \\
3 & \leftarrow & 4 \\
\uparrow & & \downarrow \\
\downarrow & & \uparrow \\
5 & \rightarrow & 1 \\
\end{array} \]

**Definition:** sink = only incoming arrows; source = only outgoing arrows
Reflection Functors

\[ \bullet \rightarrow \bullet \leftarrow \bullet \leftarrow \bullet \rightarrow \bullet \]

\[ s_1 \mathbb{Q} : \]

\[ \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \rightarrow \bullet \]

\[ s_2 \mathbb{Q} : \]

\[ \bullet \leftarrow \bullet \rightarrow \bullet \leftarrow \bullet \rightarrow \bullet \]

\[ s_4 \mathbb{Q} : \]

\[ \bullet \rightarrow \bullet \leftarrow \bullet \rightarrow \bullet \leftarrow \bullet \]

\[ s_5 \mathbb{Q} : \]

\[ \bullet \rightarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \]

**Definition:** reflection \( s_i \) = reverse all arrows incident to sink/source \( i \)
Reflection Functors

\[ s_1 Q : \]
\[
\begin{array}{cccccc}
1 & \longrightarrow & 2 & \leftarrow & 3 & \leftarrow & 4 & \longrightarrow & 5 \\
\end{array}
\]

\[ s_2 Q : \]
\[
\begin{array}{cccccc}
1 & \leftarrow & 2 & \longrightarrow & 3 & \leftarrow & 4 & \longrightarrow & 5 \\
\end{array}
\]

\[ s_4 Q : \]
\[
\begin{array}{cccccc}
1 & \longrightarrow & 2 & \leftarrow & 3 & \longrightarrow & 4 & \leftarrow & 5 \\
\end{array}
\]

\[ s_5 Q : \]
\[
\begin{array}{cccccc}
1 & \longrightarrow & 2 & \leftarrow & 3 & \leftarrow & 4 & \longrightarrow & 5 \\
\end{array}
\]

**Definition:** reflection functor \( R_i^\pm = \text{functor} \ \text{Rep}_k(Q) \rightarrow \text{Rep}_k(s_iQ) \)
Reflection Functors

Let $V = (V_i, v_a) \in \text{Rep}_k(Q)$, let $i$ be a sink
Reflection Functors

Let \( \mathbb{V} = (V_i, v_a) \in \text{Rep}_k(Q) \), let \( i \) be a sink

**Definition:** \( \mathcal{R}_i^+ \mathbb{V} = (W_i, w_a) \) is defined by:

- \( W_j = V_j \) for all \( j \neq i \)
- \( w_a = v_a \) for all \( a \notin Q_i^1 \) (arrows incident to \( i \))
Let $\mathcal{V} = (V_i, v_a) \in \text{Rep}_k(Q)$, let $i$ be a sink.

**Definition:** $\mathcal{R}_{i}^{+} \mathcal{V} = (W_i, w_a)$ is defined by:

- $W_j = V_j$ for all $j \neq i$

- $w_a = v_a$ for all $a \notin Q_i^1$

- $W_i = \ker \xi_i : \bigoplus_{a \in Q_i^1} V_{sa} \rightarrow V_i$

\[
W_i \mapsto \sum_{a \in Q_i^1} v_a(x_{sa})
\]
Reflection Functors

Let $V = (V_i, v_a) \in \text{Rep}_k(Q)$, let $i$ be a sink

**Definition:** $\mathcal{R}_i^+ V = (W_i, w_a)$ is defined by:

- $W_j = V_j$ for all $j \neq i$
- $w_a = v_a$ for all $a \notin Q_i$
- $W_i = \ker \xi_i : \bigoplus_{a \in Q_i} V_{sa} \rightarrow V_i$
  
  $$(x_{sa})_{a \in Q_i} \rightarrow \sum_{a \in Q_i} v_a(x_{sa})$$

- for $a \in Q_i$, let $b$ be the opposite arrow, and let $w_b$ be the composition:
  
  $$W_{sb} = W_i = \ker \xi_i \hookrightarrow \bigoplus_{c \in Q_i} V_{sc} \rightarrow V_{sa} = W_{sa} = W_{tb}$$

  (arrows incident to $i$)
  (canonical inclusion)
  (projection to component $V_{sa}$)
Reflection Functors

Let $\mathcal{V} = (V_i, v_a) \in \text{Rep}_k(Q)$, let $i$ be a sink

**Definition:** $\mathcal{R}_i^{+} \mathcal{V} = (W_i, w_a)$ is defined by:

- $W_j = V_j$ for all $j \neq i$
- $w_a = v_a$ for all $a \not\in Q^i_1$
- $W_i = \ker \xi_i : \bigoplus_{a \in Q^i_1} V_{sa} \rightarrow V_i$
  \[
  (x_{sa})_{a \in Q^i_1} \mapsto \sum_{a \in Q^i_1} v_a(x_{sa})
  \]
- for $a \in Q^i_1$, let $b$ be the opposite arrow, and let $w_b$ be the composition:
  \[
  W_{s_b} = W_i = \ker \xi_i \hookrightarrow \bigoplus_{c \in Q^i_1} V_{sc} \rightarrow V_{sa} = W_{s_a} = W_{t_b}
  \]

intuition: $W_i$ carries the information passing through $V_i$ in $\mathcal{V}$
Reflection Functors

Let $\mathbb{V} = (V_i, v_a) \in \text{Rep}_k(Q)$, let $i$ be a sink (or source).

**Definition:** $\mathcal{R}^+_{i} \mathbb{V} = (W_i, w_a)$ is defined by:

- $W_j = V_j$ for all $j \neq i$
- $w_a = v_a$ for all $a \notin Q^i_1$
- $W_i = \ker \xi_i : \bigoplus_{a \in Q^i_1} V_{s_a} \leftarrow V_i$
- $x_i \mapsto (v_a(x_i))_{a \in Q^i_1}$

- for $a \in Q^i_1$, let $b$ be the opposite arrow, and let $w_b$ be the composition:

\[
W_{s_b} = W_{t_a} = V_{t_a} \leftarrow \bigoplus_{c \in Q^i_1} V_{t_c} \rightarrow \text{coker } \xi_i = W_i = W_{t_b}
\]

(canonical inclusion) (quotient modulo $\text{im } \xi_i$)
Reflection Functors

Let $\mathcal{V} = (V_i, v_a) \in \text{Rep}_k(Q)$, let $i$ be a sink.

**Definition:** $\mathcal{R}_i^+ \mathcal{V} = (W_i, w_a)$ is defined by:

- $W_j = V_j$ for all $j \neq i$
- $w_a = v_a$ for all $a \notin Q_1^i$
- $W_i = \ker \xi_i : \bigoplus_{a \in Q_1^i} V_{s_a} \hookrightarrow V_i$

\[ x_i \mapsto (v_a(x_i))_{a \in Q_1^i} \]

- for $a \in Q_1^i$, let $b$ be the opposite arrow, and let $w_b$ be the composition:

$$W_{s_b} = W_{t_a} = V_{t_a} \hookrightarrow \bigoplus_{c \in Q_1^i} V_{t_c} \rightarrow \text{coker } \zeta_i = W_i = W_{t_b}$$

(intuition: this is the operation dual to the previous one (take $V_i = \ker \xi_i$))
Reflection Functors

\( \mathbb{V} : \quad V_1 \xrightarrow{v_a} V_2 \xleftarrow{v_b} V_3 \xleftarrow{v_c} V_4 \xrightarrow{v_d} V_5 \)

\( \mathcal{R}_5^+ \mathbb{V} : \quad V_1 \xrightarrow{v_a} V_2 \xleftarrow{v_b} V_3 \xleftarrow{v_c} V_4 \xleftarrow{\ker v_d} \)

\( \mathcal{R}_5^- \mathcal{R}_5^+ \mathbb{V} : \quad V_1 \xrightarrow{v_a} V_2 \xleftarrow{v_b} V_3 \xleftarrow{v_c} V_4 \xrightarrow{\mod \ker v_d} V_4 / \ker v_d \)
we have $R^{-5}R^5 V \sim V$ whenever $v_d$ is surjective. Otherwise, $V$ decomposes into $R^{-5}R^5 V$ plus a number of copies of the simple representation $S_5$ having $k$ at node 5 and 0 at every other node.

Reflection Functors

$V : V_1 \xrightarrow{v_a} V_2 \xleftarrow{v_b} V_3 \xleftarrow{v_c} V_4 \xrightarrow{v_d} V_5$

$R_5^+V : V_1 \xrightarrow{v_a} V_2 \xleftarrow{v_b} V_3 \xleftarrow{v_c} V_4 \xleftarrow{\circ} \ker v_d$

$R_5^-R_5^+V : V_1 \xrightarrow{v_a} V_2 \xleftarrow{v_b} V_3 \xleftarrow{v_c} V_4 \xrightarrow{\text{mod} \ ker v_d} V_4/\ker v_d \cong \text{im} v_d$

$V \cong R_5^-R_5^+V \oplus S_5^r$, where $r = \text{dim coker } v_d$
Reflection Functors

\[ \mathbb{V} : \quad V_1 \xrightarrow{\nu_a} V_2 \xleftarrow{\nu_b} V_3 \xleftarrow{\nu_c} V_4 \xrightarrow{\nu_d} V_5 \]

\[ R_{2}^{+}\mathbb{V} : \quad V_1 \xleftarrow{\ker \nu_a + \nu_b} V_3 \xleftarrow{\nu_c} V_4 \xrightarrow{\nu_d} V_5 \]
Reflection Functors

\[ V : \quad V_1 \xrightarrow{v_a} V_2 \leftarrow V_3 \xleftarrow{v_b} V_4 \xrightarrow{v_d} V_5 \]

\[ \mathcal{R}_2^- \mathcal{R}_2^+ V : \quad V_1 \xrightarrow{v_c} \frac{V_1 \oplus V_3}{\text{ker } v_a + v_b} \xleftarrow{v_c} V_3 \xleftarrow{v_c} V_4 \xrightarrow{v_d} V_5 \]

\[ \text{ker } v_a + v_b \xrightarrow{(0,-)} V_1 \oplus V_3 \xrightarrow{v_a+v_b} V_2 \]
Reflection Functors

\[ V : \quad V_1 \xrightarrow{v_a} V_2 \xleftarrow{v_b} V_3 \xleftarrow{v_c} V_4 \xrightarrow{v_d} V_5 \]

\[ \mathcal{R}_2^- \mathcal{R}_2^+ V : \quad V_1 \xrightarrow{v_a} V_2 \xleftarrow{v_b} V_3 \xleftarrow{v_c} V_4 \xrightarrow{v_d} V_5 \]

\[ \ker v_a + v_b \subseteq V_1 \oplus V_3 \xrightarrow{v_a+v_b} V_2 \]

\[ V \cong \mathcal{R}_2^- \mathcal{R}_2^+ V \oplus S_2^r, \text{ where } r = \dim \operatorname{coker} v_a + v_b \]
Reflection Functors

**Theorem:** [Bernstein, Gelfand, Ponomarev]

Let $Q$ be a finite connected quiver and let $V$ be a representation of $Q$. If $V \cong U \oplus W$, then for any source or sink $i \in Q_0$, $R^\pm_i V \cong R^\pm_i U \oplus R^\pm_i W$.

If now $V$ is indecomposable:

1. If $i \in Q_0$ is a sink, then two cases are possible:

   - $V \cong S_i$: in this case, $R^+_i V = 0$.
   - $V \not\cong S_i$: in this case, $R^+_i V$ is nonzero and indecomposable, $R^-_i R^+_i V \cong V$, and the dimension vectors $x$ of $V$ and $y$ of $R^+_i V$ are related to each other by the following formula:

     $$y_j = \begin{cases} 
     x_j & \text{if } j \neq i; \\
     -x_i + \sum_{a \in Q_1 \atop t_a = i} x_{s_a} & \text{if } j = i.
     \end{cases}$$
Theorem: [Bernstein, Gelfand, Ponomarev]
Let \( Q \) be a finite connected quiver and let \( \mathcal{V} \) be a representation of \( Q \). If \( \mathcal{V} \cong \mathcal{U} \oplus \mathcal{W} \), then for any source or sink \( i \in Q_0 \), \( \mathcal{R}_i^\pm \mathcal{V} \cong \mathcal{R}_i^\pm \mathcal{U} \oplus \mathcal{R}_i^\pm \mathcal{W} \).

If now \( \mathcal{V} \) is indecomposable:
2. If \( i \in Q_0 \) is a source, then two cases are possible:
   - \( \mathcal{V} \cong S_i \): in this case, \( \mathcal{R}_i^- \mathcal{V} = 0 \).
   - \( \mathcal{V} \not\cong S_i \): in this case, \( \mathcal{R}_i^- \mathcal{V} \) is nonzero and indecomposable, \( \mathcal{R}_i^+ \mathcal{R}_i^- \mathcal{V} \cong \mathcal{V} \), and the dimension vectors \( x \) of \( \mathcal{V} \) and \( y \) of \( \mathcal{R}_i^- \mathcal{V} \) are related to each other by the following formula:

\[
y_j = \begin{cases} 
x_j & \text{if } j \neq i; \\
-x_i + \sum_{a \in Q_1, s_a = i} x_{t_a} & \text{if } j = i.
\end{cases}
\]
Reflection Functors

**Theorem:** [Bernstein, Gelfand, Ponomarev]
Let $Q$ be a finite connected quiver and let $V$ be a representation of $Q$. If $V \cong U \oplus W$, then for any source or sink $i \in Q_0$, $R_i^\pm V \cong R_i^\pm U \oplus R_i^\pm W$. [...] 

**Corollary:** Reflection Functors preserve the Tits form values except at simple representations:

For $i$ source/sink and $V$ indecomposable,

- either $V \cong S_i$, in which case $q_{s_i,Q}(\text{dim } R_i^\pm V) = 0$,
- or $q_{s_i,Q}(\text{dim } R_i^\pm V) = q_Q(V)$.

For $V$ arbitrary,

$V \cong V_1 \oplus \cdots \oplus V_r \oplus S_i^s \implies q_{s_i,Q}(\text{dim } R_i^\pm V) = q_Q(\text{dim } V_1 \oplus \cdots \oplus V_r)$
Reflection Functors

Example: $Q$ of type $A_n$, $i$ sink, $\mathcal{V} \cong \bigoplus_{j=1}^{r} \mathbb{I}_Q[b_j, d_j] \in \text{rep}_k(Q)$:
Reflection Functors

Example: \( Q \) of type \( A_n \), \( i \) sink, \( \nabla \cong \bigoplus_{j=1}^{r} \mathbb{I}_Q[b_j, d_j] \in \text{rep}_k(Q) \):

\[
\begin{array}{c}
V_i \\
V_1 \cdots V_{i-1} \\
W_i \\
\downarrow \downarrow \\
V_{i+1} \cdots V_n \\
\end{array}
\]

\[\mathcal{R}^+_i \nabla \cong \bigoplus_{j=1}^{r} \mathcal{R}^+_i \mathbb{I}_Q[b_j, d_j], \text{ where}
\begin{align*}
\mathcal{R}^+_i \mathbb{I}_Q[b_j, d_j] &= \begin{cases} 
0 & \text{if } i = b_j = d_j; \\
\mathbb{I}_Q[b_{i+1}, d_j] & \text{if } i = b_j < d_j; \\
\mathbb{I}_Q[b_j, d_{i-1}] & \text{if } i + 1 = b_j \leq d_j; \\
\mathbb{I}_Q[b_j, i] & \text{if } b_j < d_j = i; \\
\mathbb{I}_Q[b_j, d_j] & \text{if } b_j \leq d_j = i - 1; \\
& \text{otherwise.}
\end{cases}
\end{align*}\]
Reflection Functors

Example: $Q$ of type $A_n$, $i$ sink, $\bigvee \cong \bigoplus_{j=1}^{r} \mathbb{I}_Q[b_j, d_j] \in \text{rep}_k(Q)$:

Diamond (forced exact by $\mathcal{R}_i^+$)

\[
\begin{array}{ccc}
V_i & \searrow & V_{i+1} \\
\downarrow & & \downarrow \\
V_1 & \cdots & V_{i-1} & \cdots & V_n \\
\nearrow & & \nearrow \\
W_i & & \\
\end{array}
\]

$\mathcal{R}_i^+ \bigvee \cong \bigoplus_{j=1}^{r} \mathcal{R}_i^+ \mathbb{I}_Q[b_j, d_j]$, where

$\mathcal{R}_i^+ \mathbb{I}_Q[b_j, d_j] = \begin{cases} 
0 & \text{if } i = b_j = d_j; \\
\mathbb{I}_{s_i}Q[i+1, d_j] & \text{if } i = b_j < d_j; \\
\mathbb{I}_{s_i}Q[i, d_j] & \text{if } i + 1 = b_j \leq d_j; \\
\mathbb{I}_{s_i}Q[b_j, i - 1] & \text{if } b_j < d_j = i; \\
\mathbb{I}_{s_i}Q[b_j, i] & \text{if } b_j \leq d_j = i - 1; \\
\mathbb{I}_{s_i}Q[b_j, d_j] & \text{otherwise.}
\end{cases}$

Diamond Principle [Carlsson, de Silva]
Proof of Gabriel’s Theorem ($A_n$ case)

**Theorem:** [Gabriel I, $A_n$ type]
Assuming $Q$ is of type $A_n$, every isomorphism class of indecomposable representations in $\text{rep}_k(Q)$ contains $\mathbb{I}_Q[b,d]$ for some $1 \leq b \leq d \leq n$. 

Proof of Gabriel’s Theorem ($A_n$ case)

**Theorem:** [Gabriel I, $A_n$ type]
Assuming $Q$ is of type $A_n$, every isomorphism class of indecomposable representations in $\text{rep}_k(Q)$ contains $\mathbb{I}_Q[b,d]$ for some $1 \leq b \leq d \leq n$.

What we are currently able to do:

- turn indecomposable representations of $Q$ into indecomposable representations of reflections of $Q$ (or zero)
- while doing so, preserve the value of the Tits form (or zero)
This is what we want to prove. This statement implies Gabriel’s Theorem I for $A_n$-type quivers because an interval representation cannot be in two isomorphism classes at a time (equivalence relation), and since there are finitely many interval representations, the number of isomorphism classes of indecomposable representations of $Q$ must then be finite.

Proof of Gabriel’s Theorem ($A_n$ case)

**Theorem:** [Gabriel I, $A_n$ type]
Assuming $Q$ is of type $A_n$, every isomorphism class of indecomposable representations in $\text{rep}_k(Q)$ contains $\mathbb{I}_Q[b,d]$ for some $1 \leq b \leq d \leq n$.

What we are currently able to do:

- turn indecomposable representations of $Q$ into indecomposable representations of reflections of $Q$ (or zero)

- while doing so, preserve the value of the Tits form (or zero)

→ idea: turn $Q$ into itself via sequences of reflections, and observe the evolution of the indecomposables and their Tits form values
Proof of Gabriel’s Theorem ($A_n$ case)

Special case: linear quiver $L_n$:  

\[ \bullet \rightarrow 1 \rightarrow \bullet \rightarrow 2 \rightarrow \cdots \rightarrow \bullet \rightarrow n-1 \rightarrow \bullet \rightarrow n \]

Let $V \in \text{rep}_k(L_n)$ indecomposable, $\dim V = (x_1, x_2, \cdots, x_{n-1}, x_n)^\top$

\[ \rightarrow \text{apply reflections } s_1s_2\cdots s_{n-1}s_nL_n \text{ and observe evolution of } \dim V \]
Proof of Gabriel’s Theorem ($A_n$ case)

Special case: linear quiver $L_n$:

Let $V \in \text{rep}_k(L_n)$ indecomposable, $\dim V = (x_1, x_2, \ldots, x_{n-1}, x_n)^\top$

$\dim R_n^+ V = 0$ or $(x_1, x_2, \ldots, x_{n-1}, x_{n-1} - x_n)^\top$

$\dim R_{n-1}^+ R_n^+ V = 0$ or $(x_1, x_2, \ldots, x_{n-2} - x_n, x_{n-1} - x_n)^\top$

$\ldots$

$\dim R_2^+ \cdots R_{n-1}^+ R_n^+ V = 0$ or $(x_1, x_1 - x_n, \ldots, x_{n-2} - x_n, x_{n-1} - x_n)^\top$

$\dim R_1^+ R_2^+ \cdots R_{n-1}^+ R_n^+ V = 0$ or $(-x_n, x_1 - x_n, \ldots, x_{n-2} - x_n, x_{n-1} - x_n)^\top$
Proof of Gabriel’s Theorem \((A_n\) case)

Special case: linear quiver \(L_n\):

\[
\begin{array}{ccccccc}
& & & & & \bullet & \\
\bullet & \rightarrow & \bullet & \rightarrow & \cdots & \rightarrow & \bullet & \rightarrow & \bullet
\end{array}
\]

Let \(V \in \text{rep}_k(L_n)\) indecomposable, \(\dim V = (x_1, x_2, \cdots, x_{n-1}, x_n)\top\)

\(\dim R_n^+ V = 0\) or \((x_1, x_2, \cdots, x_{n-1}, x_{n-1} - x_n)\top\)

\(\dim R_{n-1}^+ R_n^+ V = 0\) or \((x_1, x_2, \cdots, x_{n-2} - x_n, x_{n-1} - x_n)\top\)

\[\vdots\]

\(\dim R_2^+ \cdots R_{n-1}^+ R_n^+ V = 0\) or \((x_1, x_1 - x_n, \cdots, x_{n-2} - x_n, x_{n-1} - x_n)\top\)

\(\dim R_1^+ R_2^+ \cdots R_{n-1}^+ R_n^+ V = 0\) or \((x_n \xleftarrow{0} x_n, x_1 - x_n, \cdots, x_{n-2} - x_n, x_{n-1} - x_n)\top\)

\(\implies \mathcal{C}^+ V = R_1^+ R_2^+ \cdots R_{n-1}^+ R_n^+ V = 0\) or \(x_n = 0\)
Proof of Gabriel’s Theorem \( (A_n \text{ case}) \)

Special case: linear quiver \( L_n \):

\[
\begin{array}{ccccccccc}
\bullet & \rightarrow & \bullet & \rightarrow & \cdots & \rightarrow & \bullet & \rightarrow & \bullet \\
1 & & 2 & & \cdots & & n-1 & & n
\end{array}
\]

Let \( V \in \text{rep}_k(L_n) \) indecomposable, \( \dim V = (x_1, x_2, \cdots, x_{n-1}, x_n)^\top \)

\[
\dim C^+ V = 0 \text{ or } (0, x_1, x_2, \cdots, x_{n-2}, x_{n-1})^\top
\]

\[
\dim C^+ C^+ V = 0 \text{ or } (0, 0, x_1, \cdots, x_{n-3}, x_{n-2})^\top
\]

\[
\vdots
\]

\[
\dim \underbrace{C^+ \cdots C^+}_n V = 0 \text{ or } (0, 0, 0, \cdots, 0, x_1)^\top
\]

\[
\dim \underbrace{C^+ \cdots C^+}_n V = 0
\]
Proof of Gabriel’s Theorem ($A_n$ case)

Special case: linear quiver $L_n$:

\[
\begin{array}{ccccccc}
 & \bullet & \rightarrow & \bullet & \rightarrow & \cdots & \rightarrow & \bullet & \rightarrow & \bullet \\
1 & 2 & \cdots & n-1 & n
\end{array}
\]

Let $V \in \text{rep}_k(L_n)$ indecomposable, $\dim V = (x_1, x_2, \cdots, x_{n-1}, x_n)\top$

\[
\dim C^+ V = 0 \text{ or } (0, x_1, x_2, \cdots, x_{n-2}, x_{n-1})\top
\]

\[
\dim C^+ C^+ V = 0 \text{ or } (0, 0, x_1, \cdots, x_{n-3}, x_{n-2})\top
\]

\[
\dim \underbrace{C^+ \cdots C^+}_n V = 0 \text{ or } (0, 0, 0, \cdots, 0, x_1)\top
\]

\[
\Rightarrow \exists i_1, i_2, \cdots, i_{s-1}, i_s \text{ s.t. } \mathcal{R}_{i_s}^+ \mathcal{R}_{i_{s-1}}^+ \cdots \mathcal{R}_{i_2}^+ \mathcal{R}_{i_1}^+ V = 0
\]

\[
\dim \underbrace{C^+ \cdots C^+}_n V = 0 \quad \Rightarrow \mathcal{R}_{i_{s-1}}^+ \cdots \mathcal{R}_{i_2}^+ \mathcal{R}_{i_1}^+ V \neq 0
\]
Proof of Gabriel’s Theorem ($A_n$ case)

Special case: linear quiver $L_n$:

\[
\begin{array}{ccccccccc}
\bullet & \rightarrow & \bullet & \rightarrow & \cdots & \rightarrow & \bullet & \rightarrow & \bullet \\
1 & & 2 & & \cdots & & n-1 & & n
\end{array}
\]

Let $V \in \text{rep}_k(L_n)$ indecomposable, $\dim V = (x_1, x_2, \cdots, x_{n-1}, x_n)^\top$

\[
\exists i_1, i_2, \cdots, i_{s-1}, i_s \text{ s.t. } \mathcal{R}^+_{i_s} \mathcal{R}^+_{i_{s-1}} \cdots \mathcal{R}^+_{i_2} \mathcal{R}^+_{i_1} V = 0
\]

\[
\Rightarrow \mathcal{R}^+_{i_{s-1}} \cdots \mathcal{R}^+_{i_2} \mathcal{R}^+_{i_1} V \neq 0
\]

$\mathcal{R}^+_{i_{s-1}} \cdots \mathcal{R}^+_{i_2} \mathcal{R}^+_{i_1} V$ is indecomposable and isomorphic to $S_r$ for some $1 \leq r \leq n$ (Reflection Functor Thm)
Proof of Gabriel’s Theorem ($A_n$ case)

Special case: linear quiver $L_n$:

\[
\begin{array}{ccccccccc}
\bullet & \rightarrow & \bullet & \rightarrow & \cdots & \rightarrow & \bullet & \rightarrow & \bullet \\
1 & & 2 & & \cdots & & n-1 & & n
\end{array}
\]

Let $V \in \text{rep}_k(L_n)$ indecomposable, $\dim V = (x_1, x_2, \cdots, x_{n-1}, x_n)^\top$

\[
\exists i_1, i_2, \cdots, i_{s-1}, i_s \text{ s.t. } \mathcal{R}_{i_s}^+ \mathcal{R}_{i_{s-1}}^+ \cdots \mathcal{R}_{i_2}^+ \mathcal{R}_{i_1}^+ V = 0
\]

\[
\mathcal{R}_{i_{s-1}}^+ \cdots \mathcal{R}_{i_2}^+ \mathcal{R}_{i_1}^+ V \neq 0
\]

\[\implies \mathcal{R}_{i_{s-1}}^+ \cdots \mathcal{R}_{i_2}^+ \mathcal{R}_{i_1}^+ V \text{ is indecomposable and isomorphic to } S_r \text{ for some } 1 \leq r \leq n\]

\[\implies q_{L_n}(\dim V) = q_{s_{i_{s-1}} \cdots s_{i_1} L_n}(\dim \mathcal{R}_{i_{s-1}}^+ \cdots \mathcal{R}_{i_2}^+ \mathcal{R}_{i_1}^+ V) = q_{s_{i_{s-1}} \cdots s_{i_1} L_n}(\dim S_r) = 1\]

(Corollary)
Proof of Gabriel’s Theorem ($A_n$ case)

Special case: linear quiver $L_n: \bullet \rightarrow 2 \rightarrow \cdots \rightarrow n-1 \rightarrow n$

Let $V \in \text{rep}_k(L_n)$ indecomposable, $\dim V = (x_1, x_2, \cdots, x_{n-1}, x_n)^{\top}$

$\exists i_1, i_2, \cdots, i_{s-1}, i_s \text{ s.t. } \mathcal{R}_{i_s}^+ \mathcal{R}_{i_{s-1}}^+ \cdots \mathcal{R}_{i_2}^+ \mathcal{R}_{i_1}^+ V = 0$

$\mathcal{R}_{i_{s-1}}^+ \cdots \mathcal{R}_{i_2}^+ \mathcal{R}_{i_1}^+ V \neq 0$

$\implies \mathcal{R}_{i_{s-1}}^+ \cdots \mathcal{R}_{i_2}^+ \mathcal{R}_{i_1}^+ V$ is indecomposable and isomorphic to $S_r$ for some $1 \leq r \leq n$

$\implies q_{L_n}(\dim V) = q_{s_{i_{s-1}} \cdots s_{i_1} L_n}(\dim \mathcal{R}_{i_{s-1}}^+ \cdots \mathcal{R}_{i_2}^+ \mathcal{R}_{i_1}^+ V) = q_{s_{i_{s-1}} \cdots s_{i_1} L_n}(\dim S_r) = 1$

$\implies \dim V = \dim \mathbb{I}_{L_n}[b, d]$ for some $1 \leq b \leq d \leq n \implies V \cong \mathbb{I}_{L_n}[b, d]$ (Example)
Proof of Gabriel’s Theorem ($A_n$ case)

Special case: linear quiver $L_n$:

$$
\begin{align*}
\bullet & \rightarrow 
\bullet & \rightarrow 
\cdots & \rightarrow 
\bullet & \rightarrow 
\bullet
\end{align*}
$$

$1 \rightarrow 2 \rightarrow \cdots \rightarrow n-1 \rightarrow n$

Let $V \in \text{rep}_k(L_n)$ indecomposable, $\dim V = (x_1, x_2, \cdots, x_{n-1}, x_n)^\top$

$$
\exists i_1, i_2, \cdots, i_{s-1}, i_s \text{ s.t. } \mathcal{R}_{i_s}^+ \mathcal{R}_{i_{s-1}}^+ \cdots \mathcal{R}_{i_2}^+ \mathcal{R}_{i_1}^+ V = 0
$$

$$
\mathcal{R}_{i_{s-1}}^+ \cdots \mathcal{R}_{i_2}^+ \mathcal{R}_{i_1}^+ V \neq 0
$$

$$
\implies \mathcal{R}_{i_{s-1}}^+ \cdots \mathcal{R}_{i_2}^+ \mathcal{R}_{i_1}^+ V \text{ is indecomposable and isomorphic to } S_r \text{ for some } 1 \leq r \leq n
$$

$$
\implies q_{L_n}(\dim V) = q_{s_{i_{s-1}} \cdots i_1 L_n}(\dim \mathcal{R}_{i_{s-1}}^+ \cdots \mathcal{R}_{i_2}^+ \mathcal{R}_{i_1}^+ V) = q_{s_{i_{s-1}} \cdots i_1 L_n}(\dim S_r) = 1
$$

$$
\implies \dim V = \dim \mathbb{I}_{L_n}[b, d] \text{ for some } 1 \leq b \leq d \leq n \implies V \cong \mathbb{I}_{L_n}[b, d]
$$

Algo: apply Coxeter functor to peel off summands $\mathbb{I}_{L_n}[b_i, n]$ and to shift other summands to the right. Repeat until all summands have been peeled off.

Proof of Gabriel’s Theorem ($A_n$ case)

$A_n$-type quiver $Q$:

\[
\bullet_1 \longrightarrow \bullet_2 \longrightarrow \cdots \longrightarrow \bullet_{n-1} \longrightarrow \bullet_n
\]

→ goal: find a sequence of indices $i_1, i_2, \ldots, i_{s-1}, i_s$ s.t.

\[
\mathcal{R}_{i_s}^+ \mathcal{R}_{i_{s-1}}^+ \cdots \mathcal{R}_{i_2}^+ \mathcal{R}_{i_1}^+ \mathcal{V} = 0 \text{ for all } \mathcal{V} \in \text{rep}_k(Q)
\]
Proof of Gabriel’s Theorem ($A_n$ case)

$A_n$-type quiver $Q$: 

\[
\begin{array}{ccccccc}
1 & \rightarrow & 2 & \rightarrow & \cdots & \rightarrow & n-1 & \rightarrow & n \\
\end{array}
\]

→ goal: find a sequence of indices $i_1, i_2, \cdots, i_{s-1}, i_s$ s.t.

\[
\mathcal{R}_{i_s}^+ \mathcal{R}_{i_{s-1}}^+ \cdots \mathcal{R}_{i_2}^+ \mathcal{R}_{i_1}^+ \mathbf{V} = 0 \text{ for all } \mathbf{V} \in \text{rep}_k(Q)
\]

→ idea: turn $Q$ into $L_n$, then use the same sequence as before
Proof of Gabriel’s Theorem ($A_n$ case)

$A_n$-type quiver $Q$:

- embed $Q$ in a giant pyramid
Proof of Gabriel’s Theorem ($A_n$ case)

$A_n$-type quiver $Q$:

- embed $Q$ in a giant pyramid

- travel down the pyramid to its bottom $L_n$
  
  → travelling one level down reverses the leftmost backward arrow

  e.g. $s_1s_2s_3$ reverses $\bullet \leftarrow \bullet$
Proof of Gabriel’s Theorem \((A_n \text{ case})\)

\(A_n\)-type quiver \(Q\):

- embed \(Q\) in a giant pyramid

- travel down the pyramid to its bottom \(L_n\)

\[ \rightarrow \text{ travelling one level down reverses the leftmost backward arrow} \]

\[ \text{e.g. } s_1 s_2 s_3 \text{ reverses } 3 \quad \leftarrow \quad 4 \]

- each diamond

\[ i-1 \quad \bullet \quad i \quad \bullet \quad i+1 \]

is travelled down using \(R_i^+\)
Proof of Gabriel’s Theorem ($A_n$ case)

$A_n$-type quiver $Q$:

- embed $Q$ in a giant pyramid
- travel down the pyramid to its bottom $L_n$
  
  \[ \rightarrow \text{travelling one level down reverses the leftmost backward arrow} \]

\[ \text{e.g. } s_1 s_2 s_3 \text{ reverses } 3 \leftarrow 4 \]

- each diamond

\[ \begin{array}{c}
  \bullet \\
  \downarrow \quad \downarrow \\
  i-1 \quad i \quad i+1 \\
  \downarrow \quad \downarrow \\
  \bullet \\
\end{array} \]

\[ \rightarrow \text{travelling one level down reverses the leftmost backward arrow} \]

\[ \text{e.g. } s_1 s_2 s_3 \text{ reverses } 3 \leftarrow 4 \]

- algo. to compute zigzag persistence

(at the algebraic level $\rightarrow$ maintain bases)
Proof of Gabriel’s Theorem ($A_n$ case)

**Theorem:** [Gabriel II]
Assuming $Q$ is Dynkin with $n$ vertices, the map $\mathcal{V} \mapsto \dim \mathcal{V}$ induces a bijection between the set of isomorphism classes of indecomposable representations of $Q$ and the set of *positive roots* of the *Tits form* of $Q$. 
Proof of Gabriel’s Theorem ($A_n$ case)

**Theorem**: [Gabriel II]
Assuming $Q$ is Dynkin with $n$ vertices, the map $\mathbb{V} \mapsto \dim \mathbb{V}$ induces a bijection between the set of isomorphism classes of indecomposable representations of $Q$ and the set of *positive roots* of the *Tits form* of $Q$.

What we know:

- the positive roots of $q_Q$ are the dimension vectors of interval modules $\mathbb{I}_Q[b, d]$
- each isomorphism class $C$ of indecomposables contains $\geq 1$ interval module
Proof of Gabriel’s Theorem ($A_n$ case)

**Theorem:** [Gabriel II]
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What we know:

- the positive roots of $q_Q$ are the dimension vectors of interval modules $\mathbb{I}_Q[b, d]$

- each isomorphism class $C$ of indecomposables contains $\geq 1$ interval module

Additional observations:

- $\neq$ interval modules are $\not\sim$, therefore each class $C$ contains 1 interval module

- each interval module is indecomposable (endomorphism ring isom. to $k$)

□
Application 1: Persistence Computation

\[ K_1 \longrightarrow \cdots \longrightarrow K_i \overset{\sigma}{\longrightarrow} K_{i+1} \longrightarrow \cdots \longrightarrow K_n \]

\[ H(K_1) \longrightarrow \cdots \longrightarrow H(K_i) \overset{f}{\longrightarrow} H(K_{i+1}) \longrightarrow \cdots \longrightarrow H(K_n) \]

- every horizontal map is either forward or backward
- the \( K_i \) are simplicial complexes, the inclusions are \textit{elementary}
- the \( H(K_i) \) are vector spaces connected by linear maps (quiver representation)

\[ K \overset{\sigma}{\longrightarrow} K \cup \{\sigma\} \]

\[ H(K) \overset{f}{\longrightarrow} H(K \cup \{\sigma\}) \]

\( \ker f = [\partial \sigma] \)

\( f \text{ inj. of corank 1} \)

\( f \text{ surj. of nullity 1} \)
Application 1: Persistence Computation

$K_1 \quad \cdots \quad K_i \quad \quad \sigma \quad K_{i+1} \quad \cdots \quad K_n$

$\quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow$

$H(K_1) \quad \cdots \quad H(K_i) \quad f \quad H(K_{i+1}) \quad \cdots \quad H(K_n)$

- every horizontal map is either forward or backward
- the $K_i$ are simplicial complexes, the inclusions are elementary
- the $H(K_i)$ are vector spaces connected by linear maps (quiver representation)

Algorithms for when all maps are forward:

- Gaussian elimination: worst-case $O(n^3)$, highly optimized in practice
- Fast matrix multiplication: worst-case $O(n^\omega)$, not implemented

Algorithms for when maps can be forward or backward:

- Gaussian elimination + right filtration functor: worst-case $O(n^3)$, not optimized
Application 1: Persistence Computation

We compute of the persistent homology of:

\[ K_1 \quad K_2 \quad \cdots \quad K_i \quad K_{i+1} \quad \cdots \quad K_{n-1} \quad K_n \]
Application 1: Persistence Computation

We compute of the persistent homology of:

\[ K_1 \quad K_2 \quad \cdots \quad K_i \quad \sigma \quad K_{i+1} \quad \cdots \quad K_{n-1} \quad K_n \]

by maintaining a compatible homology basis for

\[ \underbrace{K_1 \quad \cdots \quad K_i}_{K[1; i]} \]

[Carlsson, de Silva '10], [C, deS, Morozov '09]
Application 1: Persistence Computation

We compute of the persistent homology of:

\[ K_1 \quad K_2 \quad \cdots \quad K_i \quad \sigma \quad K_{i+1} \quad \cdots \quad K_{n-1} \quad K_n \]

by maintaining a compatible homology basis for

\[ K_1 \quad \cdots \quad K_i = K'_m \quad \overset{\tau_m}{\leftarrow} \quad K'_{m-1} \quad \overset{\tau_{m-1}}{\leftarrow} \quad K'_{m-2} \quad \overset{\tau_{m-2}}{\leftarrow} \quad \cdots \quad \overset{\tau_1}{\leftarrow} \quad \emptyset \]

\[ K[1; i] \]

[Maria, O. '15]
Application 1: Persistence Computation

We compute the persistent homology of:

\[ K_1 \rightarrow K_2 \rightarrow \cdots \rightarrow K_i \xrightarrow{\sigma} K_{i+1} \rightarrow \cdots \rightarrow K_{n-1} \rightarrow K_n \]

by maintaining a compatible homology basis for

\[ K_1 \rightarrow \cdots \rightarrow K_i = K'_m \xleftarrow{\tau_m} K'_{m-1} \xleftarrow{\tau_{m-1}} K'_{m-2} \xleftarrow{\tau_{m-2}} \cdots \xleftarrow{\tau_1} \emptyset \]

K[1; i]

- arrow reflection if \( \xrightarrow{\sigma} \) is forward

- arrow transposition if \( \xleftarrow{\sigma} \) is backward

[Maria, O. '15]

\[ \xrightarrow{\sigma} \]

\[ \xleftarrow{\sigma} \]
Theorem: Exact Diamond Principle [Carlsson, de Silva ’10]
Under the exactness hypothesis on the diamond:

\[ W := V_1 \cdots V_{i-1} b \rightarrow W_i \leftarrow W_{i+1} \cdots V_n \]
\[ V := V_i \cdots V_{i+1} a \rightarrow V_i \leftarrow V_{i-1} \]

Interval decompositions of \( V, W \) are related as follows:
Application 1: Persistence Computation

**Theorem: Injective/Surjective Diamond Principles [Maria, O. ’15]**
For $f$ injective of corank 1 or surjective of nullity 1:

\[
\begin{align*}
\mathbb{W} := & \\
V_1 & \cdots \cdots V_{i-1} & W_i & \cdots & V_{i+1} & \cdots & V_n \\
\mathbb{V} := & \\
\end{align*}
\]

Interval decompositions of $\mathbb{V}, \mathbb{W}$ are related through *greedy rule*.

[Diagram showing interval decompositions and greedy rule]
Application 1: Persistence Computation

**Theorem:** Transposition Diamond Principle [Maria, O. '15]
For an exact diamond + morphisms inj. of corank 1 or surj. of nullity 1:

\[ \mathbb{W} := V_1 \rightarrow \cdots \rightarrow V_{i-1} \]
\[ \mathbb{V} := V_i \leftarrow \cdots \leftarrow V_{i+1} \]

Interval decompositions of \( \mathbb{V}, \mathbb{W} \) are related as follows:

![Diagram showing interval decompositions](image-url)
Application 1: Persistence Computation

Concluding remarks on this application:

- extensions of Exact Diamond Principle / Reflection Functors
  (cf. injective/surjective diamonds and transposition diamonds)
- same asymptotic complexity: $O(n^3)$ in the worst case
- better performances than [CdSM’09] in practice ($\times 0.2$)
- extension to cohomology $\rightarrow$ significant improvement expected
  ($\times 0.01$)
**Application 2: Zigzags for homology inference**

**Setup:** $K \subset \mathbb{R}^d$ a compact set, $p_1, \cdots, p_n$ data points sampled along (or close to) $K$

**Goal:** infer the topology (homology) of $K$, knowing only $p_1, \cdots, p_n$
Application 2: Zigzags for homology inference

Manufactured Data Set

$\mathbb{R}^2 \mod \mathbb{Z}^2$

$(u, v) \mapsto \frac{1}{\sqrt{2}} \left( \cos(2\pi u), \sin(2\pi u), \cos(2\pi v), \sin(2\pi v) \right)$

$\mathbb{R}^4$

$\cup S^3 \subset \mathbb{R}^4$

source: http://en.wikipedia.org/wiki/Clifford_torus
Application 2: Zigzags for homology inference

Manufactured Data Set

2000 data points

\[(u, v) \mapsto \frac{1}{\sqrt{2}} (\cos(2\pi u), \sin(2\pi u), \cos(2\pi v), \sin(2\pi v))\]

Application 2: Zigzags for homology inference

Manufactured Data Set

2000 data points

Betti_0
Betti_1
Betti_2
Betti_3

Rips
Application 2: Zigzags for homology inference

Manufactured Data Set

2000 data points

comput. limit

3-sphere \((37 \cdot 10^9\) simplices)

Betti_0

Betti_1

Betti_2

Betti_3

Rips

2000 data points

comput. limit

3-sphere \((37 \cdot 10^9\) simplices)

Betti_0

Betti_1

Betti_2

Betti_3

Rips

2000 data points

comput. limit

3-sphere \((37 \cdot 10^9\) simplices)

Betti_0

Betti_1

Betti_2

Betti_3

Rips
Application 2: Zigzags for homology inference

Manufactured Data Set

2000 data points

(12 \cdot 10^6 simplices)

mesh-based

[HMOS10]
Application 2: Zigzags for homology inference

Manufactured Data Set

2000 data points

(200 \cdot 10^3 \text{ simplices})

Betti_0

Betti_1

Betti_2

Betti_3

[OS14]
Application 2: Zigzags for homology inference

Manufactured Data Set

What we learn from this experiment:

- commonly used filtrations (Čech, Rips, alpha, witness, graph-induced) become huge at large scales and/or in high ambient dimensions: $2^n$, $n^{d/2}$, etc.

- approximations (mesh-based, sparse Rips, simplicial maps) may introduce defects in the barcodes: extra noise, over-simplification, etc.

- it is possible to take advantage of both worlds...
Application 2: Zigzags for homology inference

Input: $P \subset \mathbb{R}^d$ finite

$R_\alpha(P) = \text{clique complex of intersection graph of balls of radius } \alpha$

$\neq \text{nerve of union of balls of radius } \alpha$ (Čech complex)

Rips filtration: $\{R_\alpha(P)\}_{\alpha=0}^{+\infty}$
Application 2: Zigzags for homology inference

**Approach**

**Input:** $P \subset \mathbb{R}^d$ finite

**Params:** $\rho \geq \eta \geq 0$, ordering $p_1, \cdots, p_n$ of $P$ (e.g. furthest-point order)
Application 2: Zigzags for homology inference

Input: $P \subset \mathbb{R}^d$ finite

Params: $\rho \geq \eta \geq 0$, ordering $p_1, \ldots, p_n$ of $P$ (e.g. furthest-point order)

- let $P_i = \{p_1, \ldots, p_i\}$ be the i-th prefix, and $\varepsilon_i = d_H(P_i, P)$ the i-th scale
- $\forall i$, compute $\mathcal{R}_{\eta \varepsilon_i}(P_i)$ and $\mathcal{R}_{\rho \varepsilon_i}(P_i)$
Application 2: Zigzags for homology inference

**Input:** $P \subset \mathbb{R}^d$ finite

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- let $P_i = \{p_1, \cdots, p_i\}$ be the i-th prefix, and $\varepsilon_i = d_H(P_i, P)$ the i-th scale
- $\forall i$, compute $R_{\eta\varepsilon_i}(P_i)$ and $R_{\rho\varepsilon_i}(P_i)$
- [CO08] $\forall i$, compute $r_i^* = \text{rank } H R_{\eta\varepsilon_i}(P_i) \rightarrow H R_{\rho\varepsilon_i}(P_i)$

**Output:** plot $r_i^*$ against $\varepsilon_i$
Application 2: Zigzags for homology inference

**Input:** $P \subset \mathbb{R}^d$ finite

**Params:** $\rho \geq \eta \geq 0$, ordering $p_1, \cdots, p_n$ of $P$ (e.g. furthest-point order)

- let $P_i = \{p_1, \cdots, p_i\}$ be the i-th prefix, and $\varepsilon_i = d_H(P_i, P)$ the i-th scale
- $\forall i$, compute $R_{\eta \varepsilon_i}(P_i)$ and $R_{\rho \varepsilon_i}(P_i)$
- [OS14] relate the $R_{\eta \varepsilon_i}(P_i) \rightarrow R_{\rho \varepsilon_i}(P_i)$ through the following zigzag:
Application 2: Zigzags for homology inference

**Input:** $P \subset \mathbb{R}^d$ finite

**Params:** $\rho \geq \eta \geq 0$, ordering $p_1, \ldots, p_n$ of $P$ (e.g. furthest-point order)

- let $P_i = \{p_1, \ldots, p_i\}$ be the i-th prefix, and $\varepsilon_i = d_H(P_i, P)$ the i-th scale
- $\forall i$, compute $\mathcal{R}_{\eta\varepsilon_i}(P_i)$ and $\mathcal{R}_{\rho\varepsilon_i}(P_i)$
- [OS14] relate the $\mathcal{R}_{\eta\varepsilon_i}(P_i) \rightarrow \mathcal{R}_{\rho\varepsilon_i}(P_i)$ through the following zigzag:

$$\begin{align*}
\mathcal{R}_{\rho\varepsilon_1}(P_1) & \rightarrow \mathcal{R}_{\rho\varepsilon_2}(P_2) \\
\mathcal{R}_{\rho\varepsilon_2}(P_2) & \rightarrow \mathcal{R}_{\rho\varepsilon_3}(P_3) \\
\mathcal{R}_{\rho\varepsilon_3}(P_3) & \rightarrow \ldots
\end{align*}$$

---

**Application 2:** Zigzags for homology inference

**Approach**

**Input:** $P \subset \mathbb{R}^d$ finite
Application 2: Zigzags for homology inference

Input: $P \subset \mathbb{R}^d$ finite

Params: $\rho \geq \eta \geq 0$, ordering $p_1, \cdots, p_n$ of $P$ (e.g. furthest-point order)

- let $P_i = \{p_1, \cdots, p_i\}$ be the i-th prefix, and $\varepsilon_i = d_H(P_i, P)$ the i-th scale
- $\forall i$, compute $\mathcal{R}_{\eta \varepsilon_i}(P_i)$ and $\mathcal{R}_{\rho \varepsilon_i}(P_i)$
- [OS14] relate the $\mathcal{R}_{\eta \varepsilon_i}(P_i) \rightarrow \mathcal{R}_{\rho \varepsilon_i}(P_i)$ through the following zigzag:
Application 2: Zigzags for homology inference

**Thm:** ”If $P$ is $\varepsilon$-close to $X$ in the Hausdorff distance, with $\varepsilon < \Theta(1) \text{ wfs}(X)$, then there exists a sweet range of scales $[O(\varepsilon), \Omega(\text{wfs}(X))]$ such that the oR-ZZ restricted to this range has a persistence barcode made only of full-length intervals, revealing the homology of $X$, and of ephemeral (length zero) intervals.”
Application 2: Zigzags for homology inference

**Thm:** Let $\rho$ and $\eta$ be multipliers such that $\rho > 10$ and $\frac{3}{\vartheta_d} < \eta < \frac{\rho - 4}{2\vartheta_d}$. Let $X \subset \mathbb{R}^d$ be a compact set and let $P \subset \mathbb{R}^d$ be such that $d_H(P, X) < \varepsilon$ with

$$\varepsilon < \min \left\{ \frac{\vartheta_d \eta - 3}{6 \vartheta_d \eta}, \frac{\eta - 3/\vartheta_d}{3 \rho + \eta}, \frac{\rho - 2 \vartheta_d \eta - 4}{6 (\rho - 2 \vartheta_d \eta)}, \frac{\rho - 2 \vartheta_d \eta - 4}{(4 \vartheta_d + 1) \rho - 2 \vartheta_d \eta} \right\} \text{wfs}(X).$$

Then, for any $k < l$ such that

$$\max \left\{ \frac{3 \varepsilon}{\vartheta_d \eta - 3}, \frac{4 \varepsilon}{\rho - 2 \vartheta_d \eta - 4} \right\} \leq \varepsilon_k, \varepsilon_l < \min \left\{ \frac{1}{6} \text{wfs}(X) - \varepsilon, \frac{1}{\vartheta_d \rho + 1} (\text{wfs}(X) - \varepsilon) \right\},$$

the oR-ZZ restricted to $\mathcal{R}_{\rho \varepsilon_k} (P_{k+1}) \leftarrow \cdots \leftarrow \mathcal{R}_{\eta \varepsilon_l} (P_l)$ has a persistence barcode made only of full-length intervals and ephemeral (length zero) intervals, the number of full-length intervals being equal to the dimension of $H_*(X^\lambda)$ for any $\lambda \in (0, \text{wfs}(X))$. 
Application 2: Zigzags for homology inference

**Thm:** Let $\rho$ and $\eta$ be multipliers such that $\rho > 10$ and $\frac{3}{\vartheta_d} < \eta < \frac{\rho-4}{2\vartheta_d}$. Let $X \subset \mathbb{R}^d$ be a compact set and let $P \subset \mathbb{R}^d$ be such that $d_H(P, X) < \varepsilon$ with

$$\varepsilon < \min \left\{ \frac{\vartheta_d \eta - 3}{6 \vartheta_d \eta}, \frac{\eta - 3/\vartheta_d}{3 \rho + \eta}, \frac{\rho - 2 \vartheta_d \eta - 4}{6 (\rho - 2 \vartheta_d \eta)}, \frac{\rho - 2 \vartheta_d \eta - 4}{(4 \vartheta_d + 1) \rho - 2 \vartheta_d \eta} \right\} \text{wfs}(X).$$

Then, for any $k < l$ such that

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the oR-ZZ restricted to $\mathcal{R}_{\rho \varepsilon_k} (P_{k+1}) \leftarrow \cdots \leftarrow \mathcal{R}_{\eta \varepsilon_l} (P_l)$ has a persistence barcode made only of full-length intervals and ephemeral (length zero) intervals, the number of full-length intervals being equal to the dimension of $H_\ast(X^\lambda)$ for any $\lambda \in (0, \text{wfs}(X))$. 

$\varTheta(1)$
Application 2: Zigzags for homology inference

**Thm:** Let $\rho$ and $\eta$ be multipliers such that $\rho > 10$ and $\frac{3}{\vartheta_d} < \eta < \frac{\rho-4}{2\vartheta_d}$. Let $X \subset \mathbb{R}^d$ be a compact set and let $P \subset \mathbb{R}^d$ be such that $d_H(P, X) < \varepsilon$ with

$$\varepsilon < \min \left\{ \frac{\vartheta_d \eta - 3}{6\vartheta_d \eta}, \frac{\eta - 3/\vartheta_d}{3\rho + \eta}, \frac{\rho - 2\vartheta_d \eta - 4}{6(\rho - 2\vartheta_d \eta)}, \frac{\rho - 2\vartheta_d \eta - 4}{(4\vartheta_d + 1)\rho - 2\vartheta_d \eta} \right\} \text{wfs}(X).$$

Then, for any $k < l$ such that

$$\max \left\{ \frac{3\varepsilon}{\vartheta_d \eta - 3}, \frac{4\varepsilon}{\rho - 2\vartheta_d \eta - 4} \right\} \leq \varepsilon_k, \varepsilon_l < \min \left\{ \frac{1}{6} \text{wfs}(X) - \varepsilon, \frac{1}{\vartheta_d \rho + 1} (\text{wfs}(X) - \varepsilon) \right\},$$

the oR-ZZ restricted to $\mathcal{R}_{\rho \varepsilon_k}(P_{k+1}) \leftarrow \cdots \leftarrow \mathcal{R}_{\eta \varepsilon_l}(P_l)$ has a persistence barcode made only of full-length intervals and ephemeral (length zero) intervals, the number of full-length intervals being equal to the dimension of $H_*(X^\lambda)$ for any $\lambda \in (0, \text{wfs}(X))$. 


\[ \text{Application 2: Zigzags for homology inference} \]
Application 2: Zigzags for homology inference

2000 data points

(200 \cdot 10^3 \text{ simplices})

[OS13]
Application 2: Zigzags for homology inference

proof strategy:

\[
H\mathcal{R}_{\rho \varepsilon_{i-1}}(P_i) \quad H\mathcal{R}_{\rho \varepsilon_i}(P_{i+1})
\]

\[
H\mathcal{R}_{\eta \varepsilon_{i-1}}(P_{i-1}) \quad H\mathcal{R}_{\eta \varepsilon_i}(P_i) \quad H\mathcal{R}_{\eta \varepsilon_{i+1}}(P_{i+1})
\]
Application 2: Zigzags for homology inference

proof strategy:

- not much control over the topological behavior of Rips complexes
- exploit interleaving with Čech complexes (cf. standard persistence)
- turn oR-ZZ into some Čech-based zigzag while tracking changes in PD
- perform two types of low-level modifications:
  - arrow reversal
  - arrows composition / splitting
Application 2: Zigzags for homology inference

Thm (Arrow Reversal):
"Any arrow in a zigzag module can be reversed while preserving the persistence diagram. The properties of the reverse map also help preserve commutativity."

Thm (Arrow Composition/Splitting):
"Contiguous arrows with same orientation in a zigzag module can be composed, with the same effect on the persistence diagram as in standard persistence."
Application 2: Zigzags for homology inference

**Thm (Arrow Reversal):**
"Any arrow in a zigzag module can be reversed while preserving the persistence diagram. The properties of the reverse map also help preserve commutativity."

![Diagram of Arrow Reversal]

**Thm (Arrow Composition/Splitting):**
"Contiguous arrows with same orientation in a zigzag module can be composed, with the same effect on the persistence diagram as in standard persistence."

![Diagram of Arrow Composition/Splitting]

→ proofs by decomposition (use Gabriel’s theorem)
Application 2: Zigzags for homology inference
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\[ H\mathcal{R}_{\rho \varepsilon_{i-1}}(P_i) \rightarrow H\mathcal{R}_{\rho \varepsilon_i}(P_{i+1}) \]

\[ H\mathcal{R}_{\eta \varepsilon_{i-1}}(P_{i-1}) \rightarrow H\mathcal{R}_{\eta \varepsilon_i}(P_i) \rightarrow H\mathcal{R}_{\eta \varepsilon_{i+1}}(P_{i+1}) \]

\[ \simeq \]

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Application 2: Zigzags for homology inference

Complexity bounds

• Assume the ordering of $P$ is by furthest point sampling.

**Thm (size bound):** Let $m$ be the doubling dimension of $(P, d)$. Then, at any time the number of $k$-simplices in the current complex is:
- $2^{O(kd \log \rho)} |P|$ for the M-ZZ, oR-ZZ and iR-ZZ of parameters $\rho \geq \eta$,
- $2^{O(kd \log \frac{\rho}{\zeta})} |P|$ for the dM-ZZ of parameters $\rho, \zeta$.

**Thm (running time bound):** Let $m$ be the doubling dimension of $(P, d)$. Then, the total number of $k$-simplices inserted in the zigzag is:
- $2^{O(kd \log \rho)} |P|$ for the M-ZZ and iR-ZZ of parameters $\rho \geq \eta$,
- $2^{O(kd \log \frac{\rho}{\zeta})} |P|$ for the dM-ZZ of parameters $\rho, \zeta$,
- $2^{O(kd \log \rho)} |P|^2$ for the oR-ZZ of parameters $\rho \geq \eta$. 

• Assume the ordering of $P$ is by furthest point sampling.
Application 2: Zigzags for homology inference

Concluding remarks on this application:

- Rips-based zigzags with the following properties:
  - controlled size and running time
  - improved signal-to-noise ratio in the barcode
  - Analysis based on arrow reflections (similar but ≠ from reflection functors)

- Perspectives:
  - to be coupled with efficient algorithms of zigzag persistence: cf. application 1
  - zigzag with other complexes: witness complex, graph-induced complex, etc.