The half-plactic monoïd

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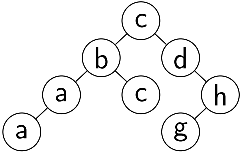
Fevrier 2015

Outline

1. Background: plactic like monoïds
   - Schensted's algorithm and the plactic monoïd
   - Binary search trees and the sylvester monoïd
   - The lattice of combinatorial Hopf algebras
   - Free quasi symmetric functions
2. The half plactic monoïd
   - Orienting the relations ?
   - A rewriting rule on Posets !
   - Binary search posets
   - The insertion algorithm
3. Work in progress: further questions

Binary Search Trees and Young tableaux ...

c	d				
b	b	b	b		
a	a	a	a	a	b



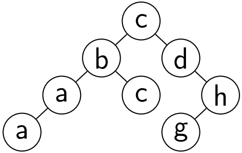
... are **very similar** combinatorial objects:

- Robinson-Schensted correspondence;
- Plactic/Sylvester monoïd;
- Hook formula;
- Hopf algebra;
- Tamari lattice/tableau lattice.

I don't think we fully understand why.

Binary Search Trees and Young tableaux ...

c	*d*				
b	*b*	*b*	*b*		
a	*a*	*a*	*a*	*a*	*b*



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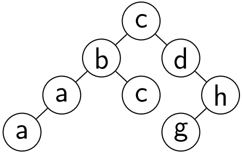

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Plactic like monoïds

Motivation: construct combinatorial Hopf algebras by taking quotient in **FQSym** (Malvenuto-Reutenauer algebra of permutations).

- Plactic monoïd $\mapsto$ **FSym** (Poirier-Reutenauer algebra of tableau). Application: proof of the Littlewood-Richardson rule.
- sylvester monoïd $\mapsto$ **BT** (Loday-Ronco algebra of binary tree). Application: dendriform algebra (shuffle/quasi-shuffle ...).
- hypoplactic monoid $\mapsto$ $(\mathrm{QSym}, \mathbf{NCSF})$. Application: descent of permutations, degenerate Hecke algebra $H_n(0)$.

Schensted's algorithm

Algorithm

Start with an empty row r, insert the letters l of the word one by one from left to right by the following rule:

- *replace the first letter strictly larger that l by l;*
- *append l to r if there is no such letter.*

Insertion of $ababcabbadbab$

$$\emptyset \xrightarrow{a} \begin{array}{|c|}\hline a \\ \hline\end{array} \xrightarrow{b} \begin{array}{|c|c|}\hline a & b \\ \hline\end{array} \xrightarrow{a} \begin{array}{|c|c|}\hline a & a \\ \hline\end{array} \xrightarrow{b} \begin{array}{|c|c|c|}\hline a & a & b \\ \hline\end{array} \xrightarrow{c} \begin{array}{|c|c|c|c|}\hline a & a & b & c \\ \hline\end{array} \xrightarrow{a}$$

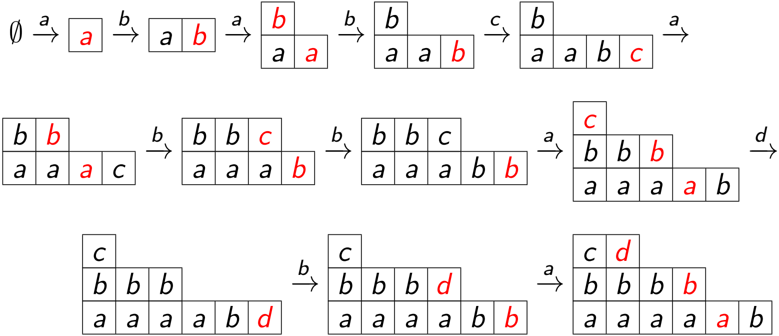
$$\begin{array}{|c|c|c|c|}\hline a & a & a & c \\ \hline\end{array} \xrightarrow{b} \begin{array}{|c|c|c|c|}\hline a & a & a & b \\ \hline\end{array} \xrightarrow{b} \begin{array}{|c|c|c|c|c|}\hline a & a & a & b & b \\ \hline\end{array} \xrightarrow{a}$$

$$\begin{array}{|c|c|c|c|c|}\hline a & a & a & a & b \\ \hline\end{array} \xrightarrow{d} \begin{array}{|c|c|c|c|c|c|}\hline a & a & a & a & b & d \\ \hline\end{array} \xrightarrow{b} \begin{array}{|c|c|c|c|c|c|}\hline a & a & a & a & b & b \\ \hline\end{array} \xrightarrow{a}$$

$$\begin{array}{|c|c|c|c|c|c|}\hline a & a & a & a & a & b \\ \hline\end{array} \xrightarrow{b} \begin{array}{|c|c|c|c|c|c|c|}\hline a & a & a & a & a & b & b \\ \hline\end{array}$$

The Robinson-Schensted's map

Bumping the letters: when a letter is replaced in Schensted algorithm, **insert it in a next row** (placed on top in the drawing).



The Robinson-Schensted's correspondence

Going backward: If we record the order in which the box are added then we can undo the whole process.

$$
ababcabbadba \longleftrightarrow \left(\begin{array}{|c|c|c|c|c|c|} \cline{1-2} c & d \\ \cline{1-4} b & b & b & b \\ \hline a & a & a & a & a & b \\ \hline \end{array} \;,\; \begin{array}{|c|c|c|c|c|c|c|} \cline{1-2} 9 & 13 \\ \cline{1-4} 3 & 6 & 7 & 11 \\ \hline 1 & 2 & 4 & 5 & 8 & 10 & 12 \\ \hline \end{array} \right).
$$

Theorem

This define a bijection between words w and pairs

$$(\textit{semi standard Young tableau}, \textit{standard Young tableau})$$

of the same shape.

The plactic monoïd

Quotient of the free monoid by Knuth's relations:

$$acb \equiv cab \quad \text{if } a \leq b < c$$
$$bac \equiv bca \quad \text{if } a < b \leq c$$

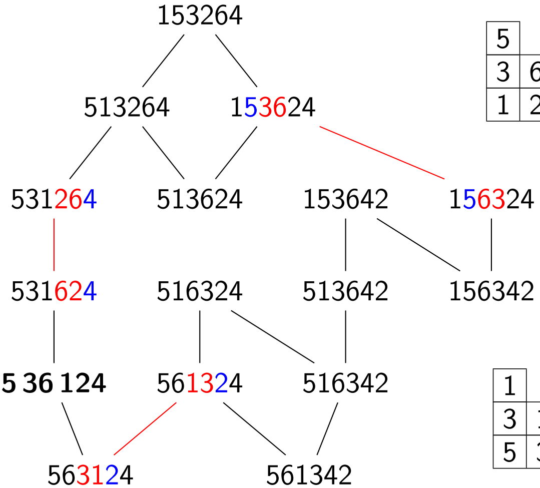
The letter b allowing the exchange is called catalytic.

Theorem

Two words give the same tableau under the RS map if and only if, is it possible to go from one to the other using Knuth's relations.

Proof: rewriting to the row reading of the tableau + green invariants.

A plactic class

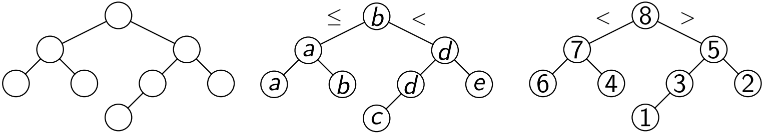


5		
3	6	
1	2	4

1		
3	1	
5	3	1

$$\frac{6!}{3\cdot 3\cdot 5} = 16$$

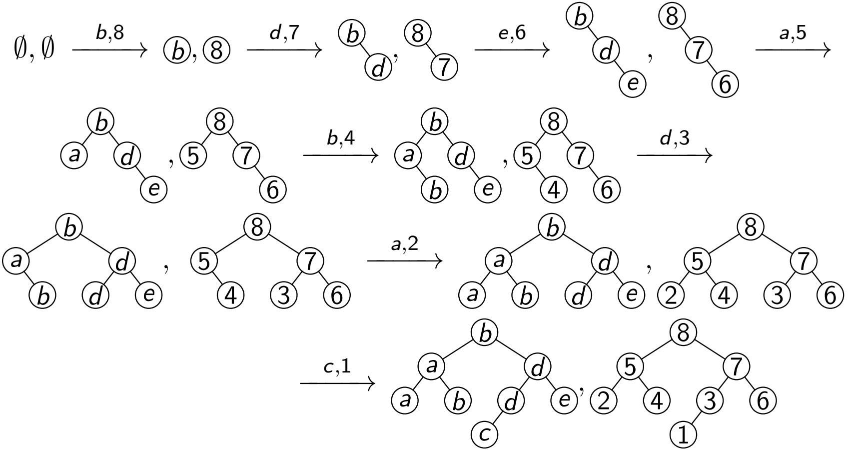
Words and trees !



binary tree (BT) search tree (BST) decreasing tree (DT)

The underlying tree of a BST or a DT is called its shape.

Sylvester RS correspondence for the word *cadbaedb*



The sylvester monoïd

Quotient of the free monoid by the relations:

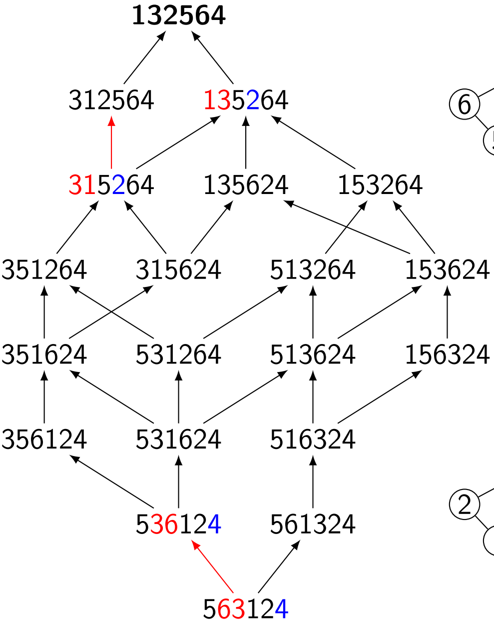
$$ac \cdots b \equiv ca \cdots b \quad \text{if } a \leq b < c$$

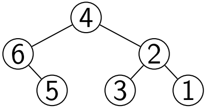
($\cdots$ stands for any number of letters).

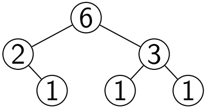
Theorem

Two words give the same binary search tree if and only if, is it possible to go from one to the other using the sylvester relations.

Proof: orienting the relations + permutations avoiding 312.







$$\frac{6!}{6\cdot 3\cdot 2} = 20$$

Standardization of a word

$$\begin{array}{ccc} \text{word of length } n & \longmapsto & \text{permutation of } \mathfrak{S}_n \\ w = l_1 l_2 \dots l_n & \longmapsto & \sigma = \mathrm{Std}(w) \end{array}$$

$$\text{for } i < j \quad \text{then} \qquad \sigma(i) > \sigma(j) \quad \text{iff} \quad l_i > l_j \,.$$

Example: $\mathrm{Std}(abcadbcaa) = 157296834$

$$\begin{array}{ccccccccc} a & b & c & a & d & b & c & a & a \\ a_1 & b_5 & c_7 & a_2 & d_9 & b_6 & c_8 & a_3 & a_4 \\ 1 & 5 & 7 & 2 & 9 & 6 & 8 & 3 & 4 \end{array}$$

Free Quasi symmetric functions

Definition

Sub-algebra of the free algebra generated by

$$\mathbf{F}_\sigma := \sum_{\mathrm{Std}(w)=\sigma^{-1}} w \quad \in \mathbb{Z}\langle A\rangle \tag{1}$$

$$F_{123} = \sum \text{weakly increasing word of length 3}$$

$$F_{321} = \sum \text{strictly decreasing word of length 3}$$

$$F_{2143} = bacb + badc + cadc + cbdc + \ldots$$

Product rule

Proposition

$\alpha \in \mathfrak{S}_m$ and $\beta \in \mathfrak{S}_n$. Then,

$$\mathbf{F}_\alpha \mathbf{F}_\beta = \sum_{\sigma \in \alpha \sqcup\!\sqcup \beta[m]} \mathbf{F}_\sigma \tag{2}$$

$$\begin{aligned} F_{132} F_{21} &= F_{13265} + F_{13625} + F_{13652} + F_{16325} + F_{16352} \\ &+ F_{16532} + F_{61325} + F_{61352} + F_{61532} + F_{65132} \end{aligned}$$

Isomorph to the algebra of permutations of Malvenuto-Reutenauer.

Coproduct on **FQSym**

X and Y : infinite, totally ordered, mutually commuting alphabets.

Definitions

The ordered sum $X \hat{+} Y$ *of* X *and* Y *is the union of* X *and* Y *where the variables of* X *are smaller than the variables of* Y*.*

Coproduct of F_σ *defined by*

$$F_\sigma \longmapsto F_\sigma(X \hat{+} Y) \longmapsto \sum F_\alpha(X) F_\beta(Y) \longmapsto \sum F_\alpha \otimes F_\beta \qquad (3)$$

- By construction, the compatibility relations holds !
- This proves that **FQSym** is a Hopf algebra.

Closed Formula

Proposition

The coproduct in **FQSym** *is given by*

$$\Delta(F_\sigma) = \sum_{k=0}^{n} F_{\mathrm{Std}(w_1 \dots w_k)} \otimes F_{\mathrm{Std}(w_{k+1} \dots w_n)}\,, \tag{4}$$

for all permutation $\sigma = w_1 \dots w_n$.

$$\Delta(F_{312}) = F_{312} \otimes 1 + F_{21} \otimes F_1 + F_1 \otimes F_{12} + 1 \otimes F_{312}$$

Exercice: Prove the compatibility relation using the formulas.

Compatibility with the standardization

Proposition

The sylvester and the plactic congruence are compatible with the standardization:

$$w_1 \equiv w_2 \iff \mathrm{Std}(w_1) \equiv \mathrm{Std}(w_2) \tag{5}$$

$$\textit{and for all letter } x \in A, \quad |w_1|_x = |w_2|_x \tag{6}$$

where $|w|_x$ *denotes the number of occurrences of* x *in* w*.*

$ebcaddab \equiv_{sylv} abeacddb$ and $83516724 \equiv_{sylv} 13825674$

Compatibility with the restriction to intervals

$I = [a_k, \ldots, a_l]$ an interval of the alphabet

u/I : the word obtained by erasing the letters of u that are not in I

Proposition

The sylvester and the plactic congruence are compatible with the restriction to intervals:

$$w_1 \equiv w_2 \quad \Longrightarrow \quad w_1/I \equiv w_2/I$$

For $I = [c, d, e]$:

$$ebcaddab \equiv_{sylv} bcaadedb \quad \text{and} \quad ecdd \equiv_{sylv} cded$$

Constructing Hopf algebras from a congruence

Theorem (H-Nzeutchap 2007)

Let $\equiv$ be a congruence which is compatible with the standardization and the restriction to the intervals. Then

$$P_C := \sum_{\mathrm{class}(\sigma)=C} F_\sigma$$

and

$$Q_C := F_\sigma(A/\equiv) \qquad \textit{for any } \sigma \in C$$

span two dual Hopf algebras.

- The base of these algebras are indexed by standard classes.

The lattice

Proposition (Priez 2013)

The intersection and the union of two standardization / restriction compatible monoids are also compatible.

- Plactic $\cup$ Sylvester $=$ Hypoplactic $\mapsto \mathrm{QSym}$.
- Plactic $\cap$ Sylvester $= \frac{1}{2}$-plactic $\mapsto$???.

The half plactic monoïd

Definition

Quotient of the free monoid by the relations:

$$acb \equiv cab \quad \textit{if } a \leq b < c$$

The catalytic letter must be right after the exchanged letters.

How to describe the classes ?

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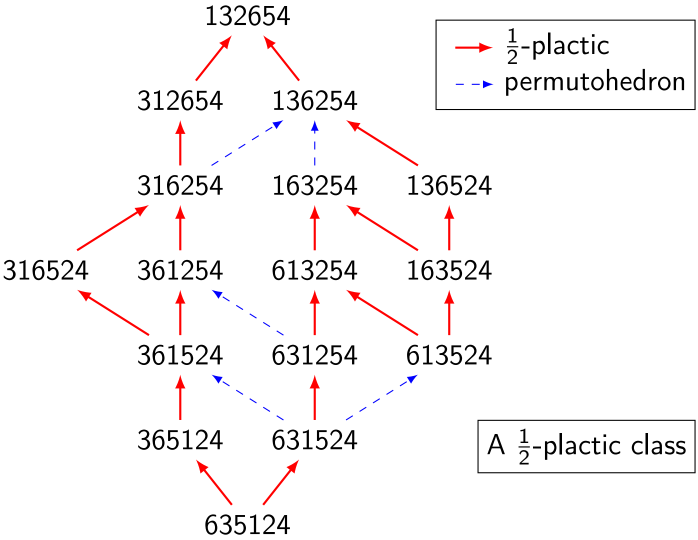


A $\frac{1}{2}$-plactic class

Numerology

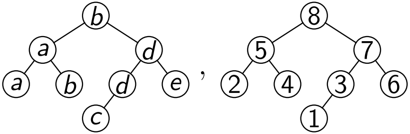
Number of classes:

n	0	1	2	3	4	5	6	7	8	9
#classes	1	1	2	5	16	61	274	1413	8266	54099

Distribution of size of the classes for $n = 7$:

size	1	2	3	4	5	6	7	9	10
#classes	454	320	194	90	132	44	40	54	16
size	12	14	15	16	19	20	21	26	35
#classes	18	16	2	14	2	1	10	2	4

Poset and linear extensions

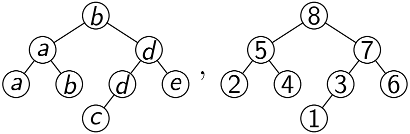
cadbaedb $\longleftrightarrow$ 

$a \downarrow b$: a is after b

153624
513624
153264
513264
531264
531624
536124

Poset and linear extensions

cadbaedb $\longleftrightarrow$



$$\begin{array}{c} a \\ \downarrow \\ b \end{array} : a \text{ is after } b$$

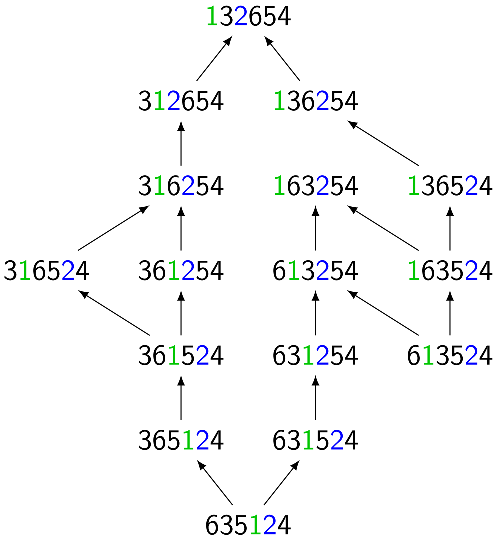
$$\begin{tikzcd} & 4 \arrow[dl] \arrow[dr] & \\ 2 \arrow[d] \arrow[drr] & & 6 \arrow[d] \\ 1 & & 3 \arrow[d] \\ & & 5 \end{tikzcd}$$

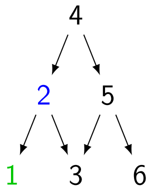
153624
513624
153264
513264
531264
531624
536124

Main Theorem

Theorem

For all $\frac{1}{2}$-plactic class C of size n permutations there exists a partial order on $\{1, \dots, n\}$ whose linear extensions are exactly the elements of C.

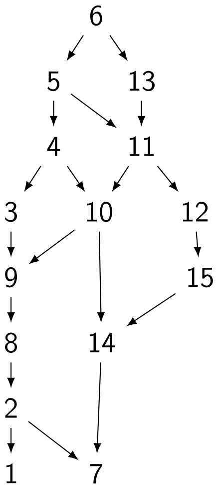

132654
312654
136254
316254
163254
136524
316524
361254
613254
163524
361524
631254
613524
365124
631524
635124


4
2
5
1
3
6

Another $\frac{1}{2}$-plactic class

7 14 1 2 8 9 15 12 3 10 4 11 5 13 6

+ 1926 other linear extensions



Main Theorem (2)

Theorem

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Sketch of the proof:

- a rewriting system on the posets: removable edges;
- rewriting commutes $\Rightarrow$ the system is confluent;
- For each permutation there is a unique minimal non rewritable poset whose linear extensions contains its class;
- induction on minimal elements $\Rightarrow$ actual equality.

Problem

Starting from a permutation σ, how to compute the poset $P(\sigma)$ associated to the class of sigma, without enumerating the class ?

Assuming the theorem is true, we work at minima: which are the pair (i, j) which are not comparable ?

If $\overline{u}cab\overline{v} = \overline{u}acb\overline{v}$ is a legible rewriting, then a and c are not comparable in P.

Idea of the construction

Start with the linear poset associated to σ, and when we find two element that are not comparable, remove the comparison.

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Removable edges

Notations:

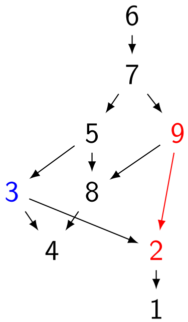
- $<$ for comparison of integers, $\prec$ for the poset
- Poset interval: $]a, b[:= \{x \mid a \prec x \prec b\}$
- Cover: $a \leftarrow b$ means $a \prec b$ and $]a, b[= \emptyset$.

Definition

A cover $a \leftarrow c$ is removable if there exists a b such that

- *either $a < b < c$ or $c < b < a$*
- *$b \not\preceq c$*
- *if $a \prec b$ then $]a, b[\subset \{c\}$*

$a \prec b$ and $]a, b[\subset \{c\}$



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Lemma

A cover $a \leftarrow b$ is removable if and only if there exists a linear extension $\overline{u}acb\overline{v}$ of P such that $\overline{u}acb\overline{v} \equiv \overline{u}cab\overline{v}$.

Note: since $a \prec b$ then $\overline{u}cab\overline{v}$ is not a linear extension of P.

Proposition

If a poset P has no removable edge, then its set of linear extension is closed by $\frac{1}{2}$-plactic rewriting; otherwise said, it is a union of $\frac{1}{2}$-plactic classes.

Problem

Given a permutation σ compute the minimal poset without removable edge which contains σ as linear extension.

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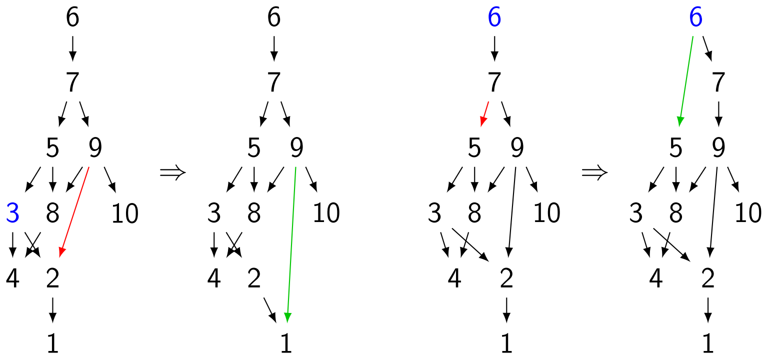
Removing the edges

Warning !!! Removing is on the Poset ($\neq$ Hasse diagram)



Removing commutes

Inclusion on poset $Q \subset P$ means $a \prec_Q b$ implies $a \prec_P b$.

Proposition

Suppose that $Q \subset P$ and $a \leftarrow_P b$ is a removable edge.
If $a \prec_Q b$ then $a \leftarrow_Q b$ is a removable edge.

Corollary

Suppose $(a, b) \neq (c, d)$. If both $a \leftarrow_P b$ and $c \leftarrow_P d$ are removable in P, then $c \leftarrow d$ is removable in $P/a \leftarrow b$.

One can do the removing in any order !

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One can do the removing in any order !

The algorithm

Algorithm

Given a permutation σ compute the poset $P(\sigma)$.

- *set $P := \sigma$ seen as a total order;*
- *while there is a removable edge pick one and remove it from P;*
- *return P.*

Theorem

The $\frac{1}{2}$-plactic class of σ is exactly the linear extensions of $P(\sigma)$.

Proof by induction on the minimums.

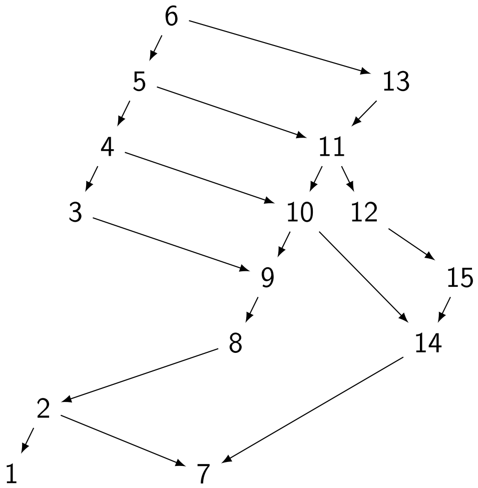
Binary search posets

Proposition

Let P be a poset without a removable edge, then for any x,

- *there is at most one $a < x$ s.t. $a \leftarrow x$*
- *there is at most one $a < x$ s.t. $x \leftarrow a$*
- *there is at most one $a > x$ s.t. $a \leftarrow x$*
- *there is at most one $a > x$ s.t. $x \leftarrow a$*

A binary search posets



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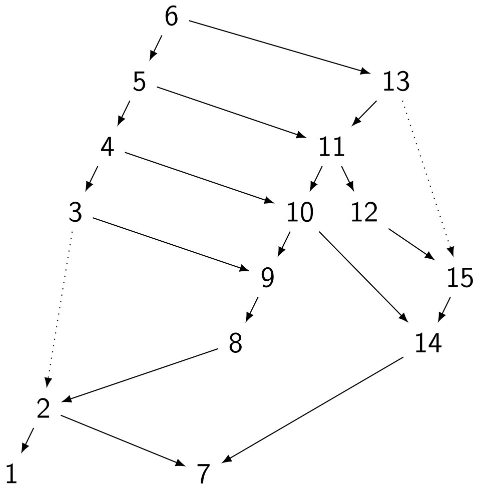
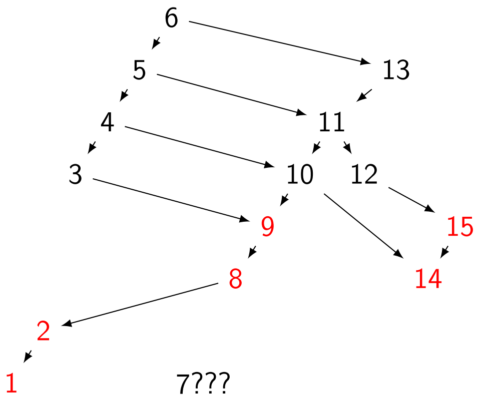


The insertion algorithm

Lemma

For any letter x and word w, $P(xw)/w = P(w)$.



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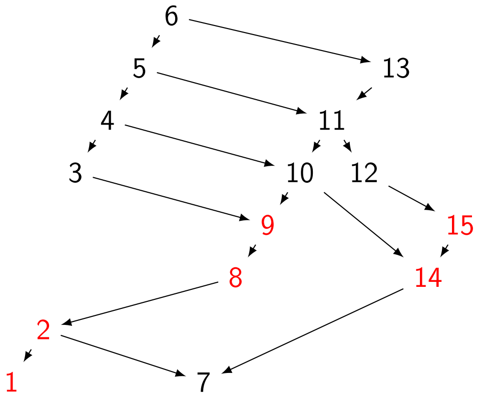

Work in progress: further questions

- Simple characterization of the posets
- Classes are intervals of the permutohedron
- k-sylvester monoïds:

$$ac\overline{w}b \equiv ca\overline{w}b \quad \text{if } a \leq b < c \text{ and } |w| < k$$

The catalytic letter must not be too far

1	1	2	5	16	61	274	1413
1	1	2	5	14	45	162	636
1	1	2	5	14	42	136	472
1	1	2	5	14	42	132	434
1	1	2	5	14	42	132	429

- analogues of the associahedron

Thank You

A characterisation of the posets

Inclusion on poset $Q \subset P$ means $a \prec_Q b$ implies $a \prec_P b$.

Proposition

Suppose that $Q \subset P$ and $a \leftarrow_P b$ is a removable edge.
If $a \prec_Q b$ then $a \leftarrow_Q b$ is a removable edge.

Corollary

The posets associated to the $\frac{1}{2}$-plactic classes are exactly the maximal (for the inclusion) posets among the posets without removable edges.