Compatibility Fan Realizations of Graph Associahedra

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joint work with **Vincent Pilaud** (CNRS, LIX Polytechnique)

January 7th, 2015

Polytope

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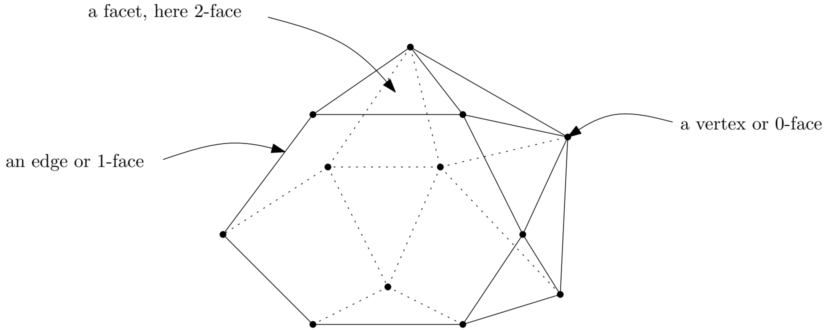

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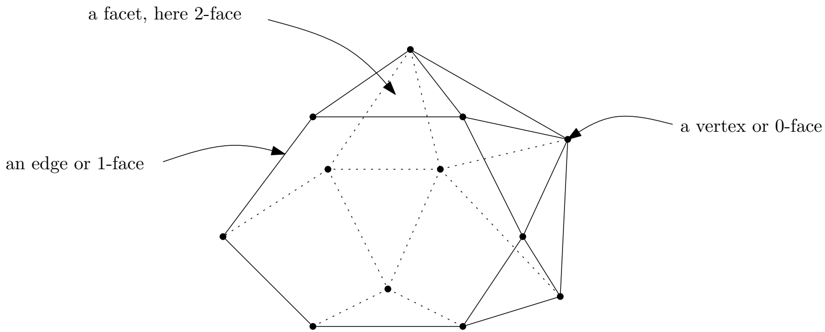
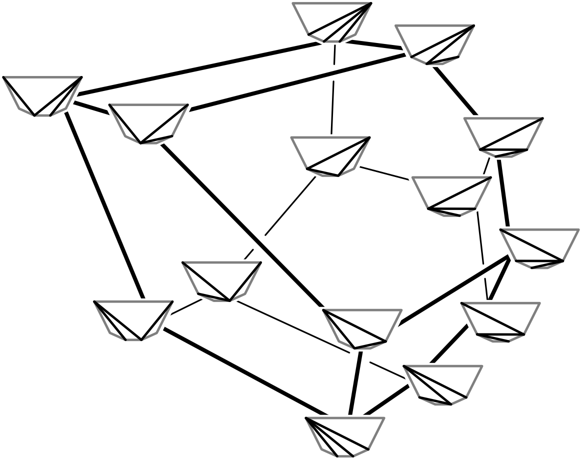


$\rightarrow$ simple: the graph is regular of degree $\dim(P)$.

Definition

An ***associahedron*** is a polytope whose graph is the flip graph of triangulations of a convex polygon.

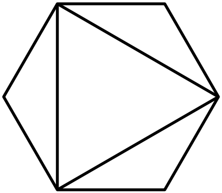


Faces $\leftrightarrow$ dissections of the polygon

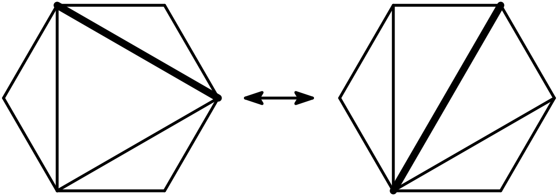
The flip operation

Flip graph on the triangulations of the polygon:

Vertices: *triangulations*



Edges: *flips*



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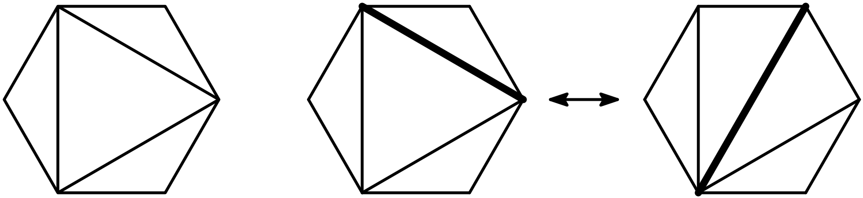
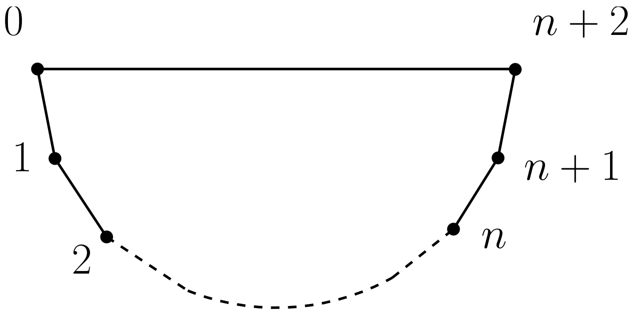


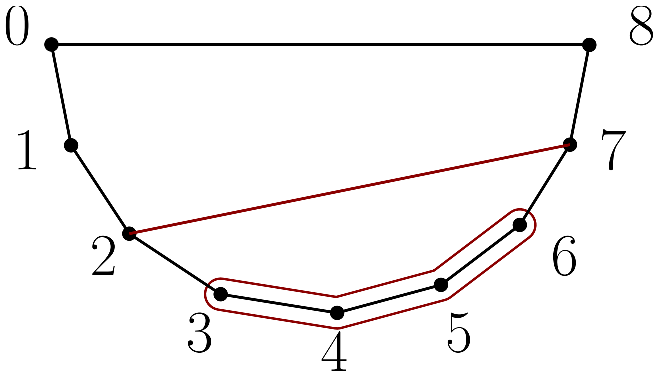
n diagonals $\Rightarrow$ the flip graph is n-regular.

Useful configuration (Loday's)

$G_{n+3} =$ 

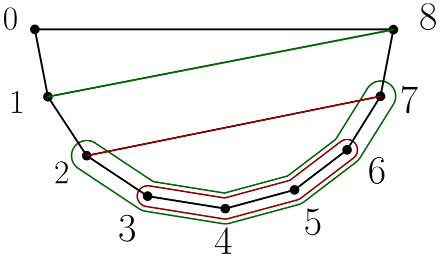
Graph point of view

$$\{\text{diagonals of } G_{n+3}\} \longleftrightarrow \{\text{strict subpaths of the path } [n+1]\}$$

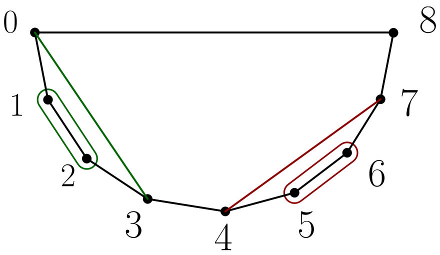


Non-crossing diagonals

Two ways to be non-crossing in Loday's configuration:



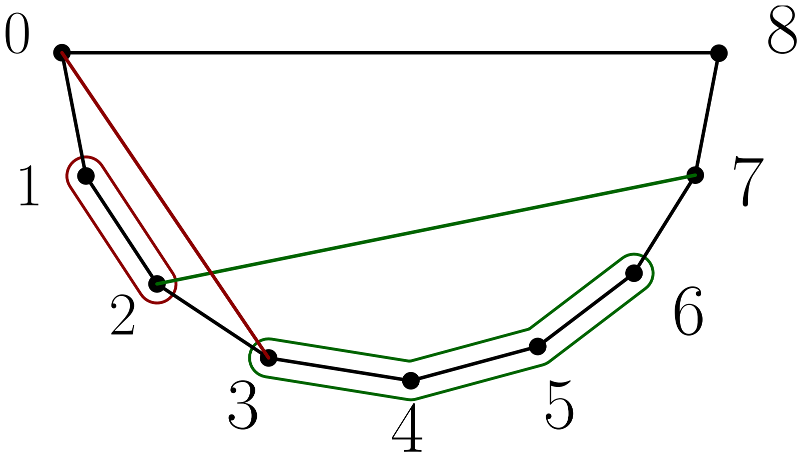
nested subpaths



non-adjacent subpaths

Pay attention to the second case:

The right condition is indeed *non-adjacent*, disjoint is not enough!



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- A ***tube*** of G is a proper subset $t \subseteq V$ inducing a connected subgraph of G;
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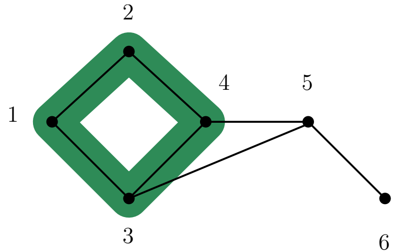
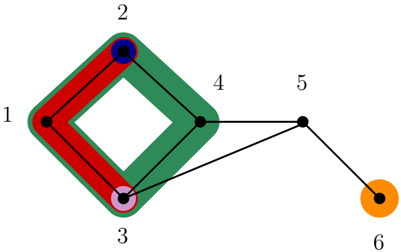
- A ***tubing*** of G is a set of pairwise compatible tubes of G.



A tube

(generalizes a diagonal)



A maximal tubing

(generalizes a triangulation)

Graph associahedra

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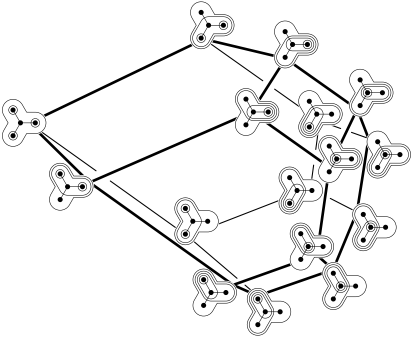
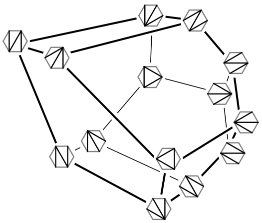
Theorem (Carr-Devadoss '06)

*There exists a polytope called **graph associahedron** of G, denoted $\mathbf{Asso}_G$, whose graph is this flip graph.*

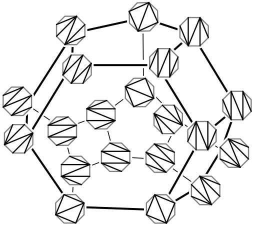


Faces $\leftrightarrow$ tubings of G.

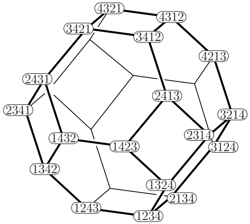
Classical polytopes...



The associahedron

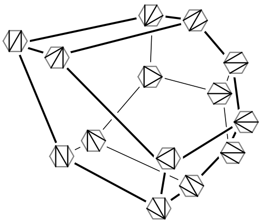


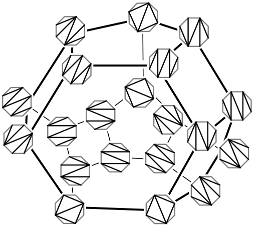
The cyclohedron

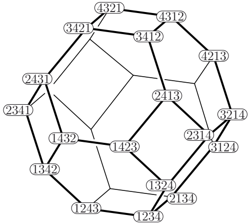


The permutahedron

...can be seen as graph associahedra







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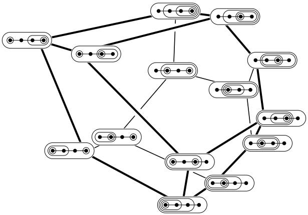

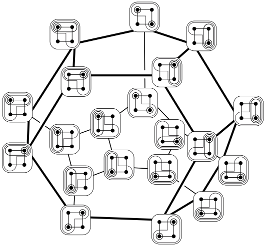

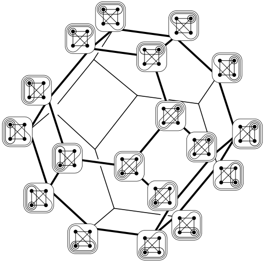

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Complete Fan = fan whose cones cover the whole space.

polytope $\Rightarrow$ normal Fan.

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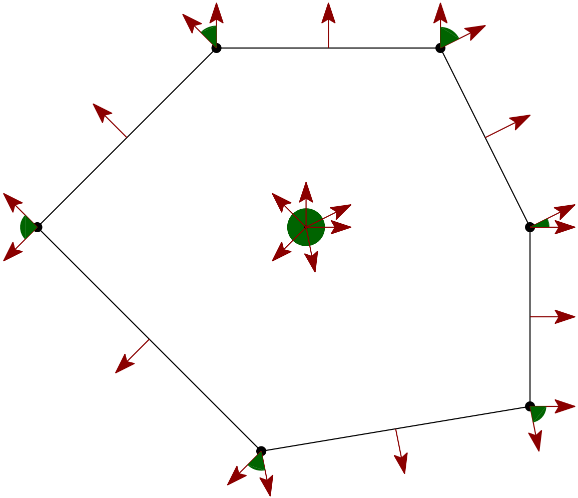
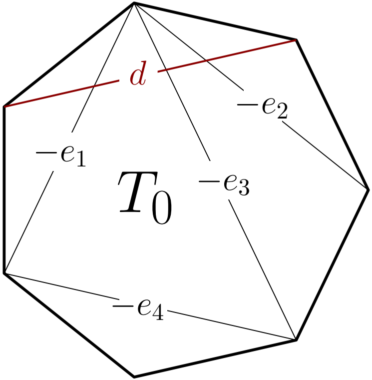
simple polytope $\Rightarrow$ complete simplicial Fan.



F. Santos' construction for the fan

→ choose an initial triangulation T_0 of the polygon.



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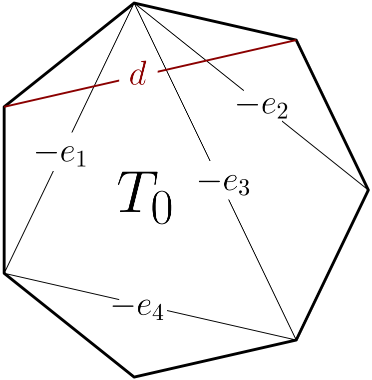

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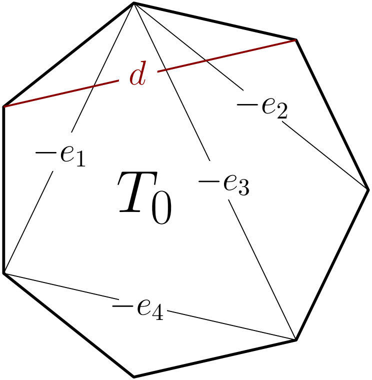

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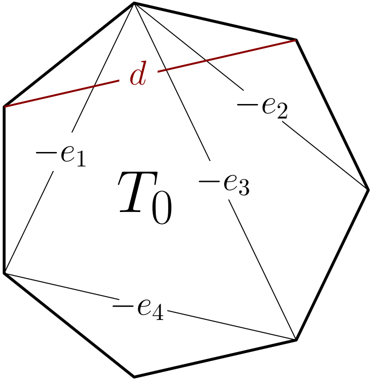

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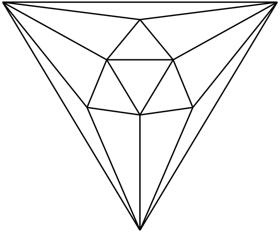
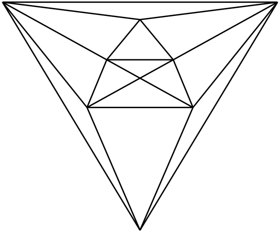
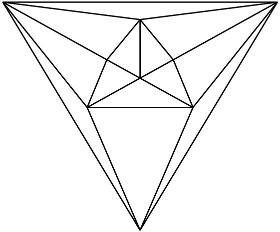

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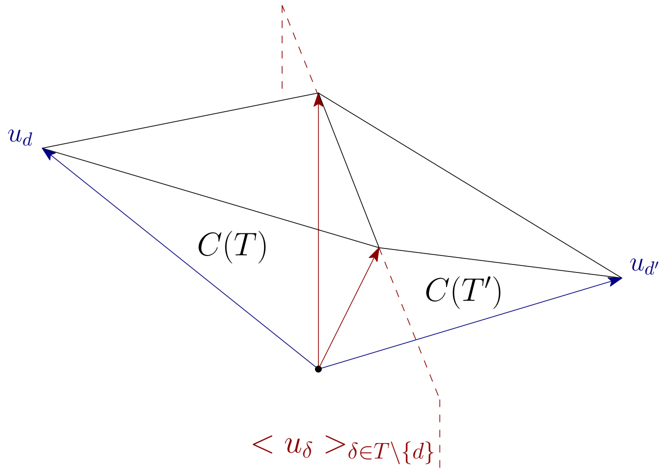

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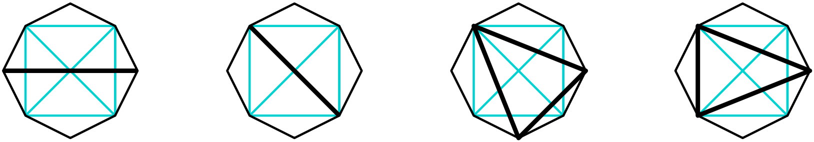

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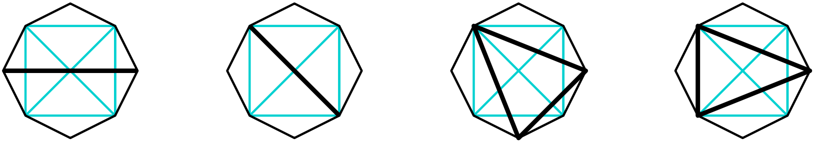
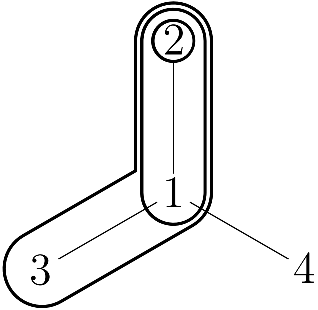
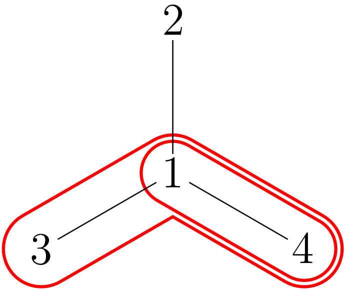


→ Finite number of linear dependences to check explicitly.

For graphs?



T_0



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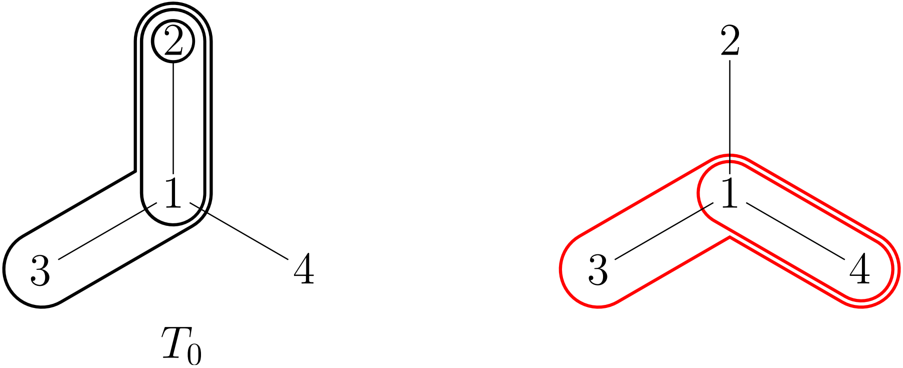

$\rightarrow$ impossible to choose $-1, 0, 1$ coordinates.

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Proposition

- $(t \parallel t') < 0 \Leftrightarrow (t' \parallel t) < 0 \Leftrightarrow t = t'$.
- $(t \parallel t') = 0 \Leftrightarrow (t' \parallel t) = 0 \Leftrightarrow$ *t and t' are compatible.*
- $(t \parallel t') = (t' \parallel t) = 1 \Leftrightarrow$ *t and t' are exchangeable.*

The result!

$\rightarrow$ for $T_0 = \{t_1, \ldots, t_n\}$, define $u_t = ((t \parallel t_1), \ldots, (t \parallel t_n))$

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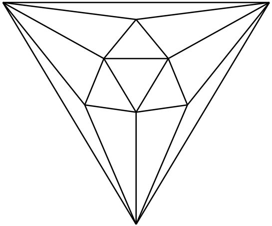

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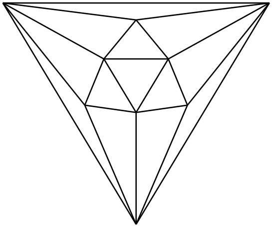

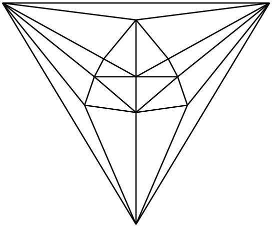

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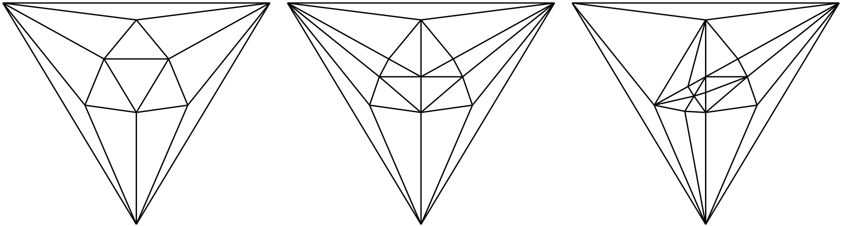

Link with cluster algebras

$\rightarrow$ [CFZ]: compatibility degrees between roots in finite types to construct generalized associahedra.

$$\{\text{Generalized Associahedra}\}\bigcap\{\text{Graph Associahedra}\} = A, B, C.$$

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type	graph
A	path
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roots	tubes
$(\alpha \parallel \alpha')$	$(t \parallel t')$
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$(\alpha \parallel \alpha')$	$(t' \parallel t)$

THANK YOU FOR YOUR PASSIONATED LISTENING!