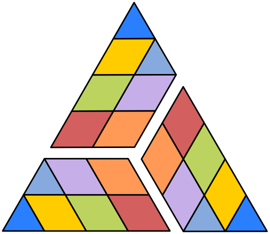


Dyck path triangulations of products of simplices and extendability

Arnau Padrol
(j.w.w. **César Ceballos** & **Camilo Sarmiento**)
Freie Universität Berlin

Séminaire Équipe Modèles Combinatoires — 17/12/2014

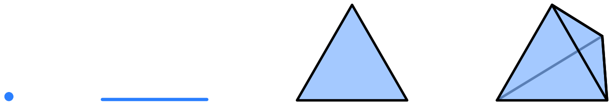
Triangulations of products of simplices

Simplices

A *d-simplex*, is the convex hull of $d+1$ affinely independent points in $\mathbb{R}^d$.

The *standard simplex* is

$$\Delta_{n-1} := \operatorname{conv}\left\{\mathbf{e}_i \ : \ \mathbf{e}_i \in \mathbb{R}^n\right\}.$$



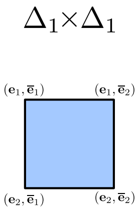
Δ_0 Δ_1 Δ_2 Δ_3

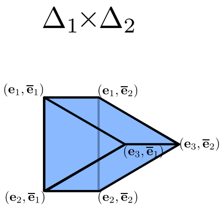
Products of simplicies

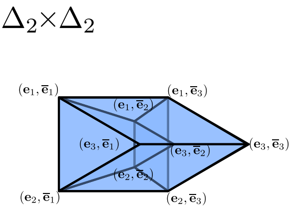
The (cartesian) *product of two (standard) simplices* is the polytope

$$\Delta_{m-1} \times \Delta_{n-1} := \operatorname{conv}\left\{(\mathbf{e}_i, \overline{\mathbf{e}}_j) : \mathbf{e}_i \in \mathbb{R}^m,\ \overline{\mathbf{e}}_j \in \mathbb{R}^n\right\} \subset \mathbb{R}^{m+n}$$

with $m \cdot n$ vertices and of dimension $m+n-2$.







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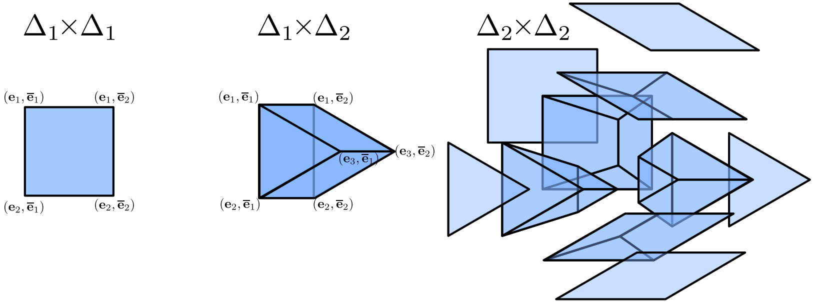

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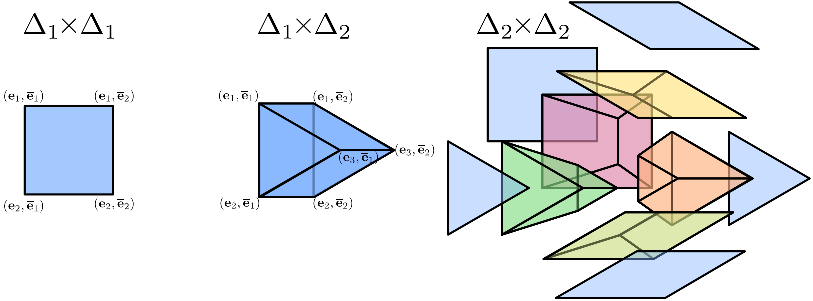
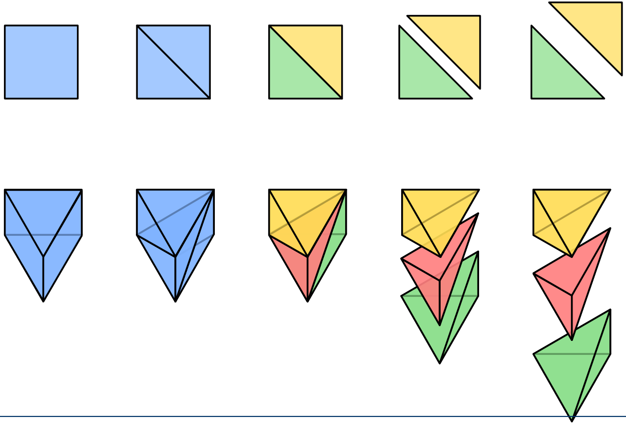


A *triangulation* of P is a subdivision of P into simplices $\{T_1, \ldots, T_n\}$ spanned by the vertices that "intersect well":

- $V(T_i) \subseteq V(P)$,
- $\bigcup T_i = P$,
- $T_i \cap T_j$ is a common face of T_i and T_j.



Triangulations of products of simplices

- Among the simplest polytopes:
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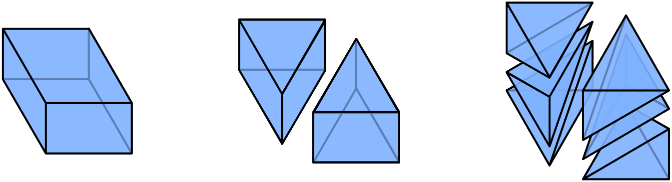

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Triangulations of products of simplices

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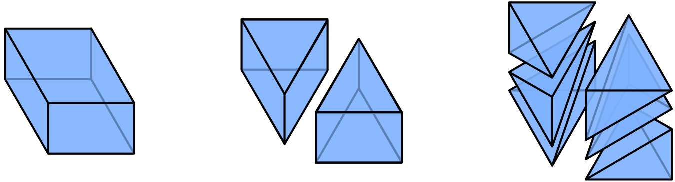
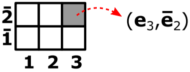


Open: What is the minimal # simplicies in a triangulation of $[0,1]^d$?

- Many combinatorial & algebraic interpretations:
  - Newton polytopes of products of all minors of a matrix **[Babson-Billera '98]**
  - Matroid polytope subdivisions of hypersimplices **[Kapranov '92, Speyer '08, Herrmann-Joswig-Speyer '12]**
  - Matroid of lines of arrangements of complete flags **[Ardila-Billey '07, Ardila-Ceballos '11]**
  - Arrangements of tropical hyperplanes **[Sturmfels-Develin '04, Ardila-Develin '09 et al.]**

Grid representation

Vertices of $\Delta_{m-1} \times \Delta_{n-1}$ can be represented in a $n \times m$ rectangular grid:



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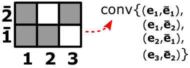

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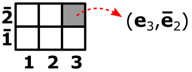

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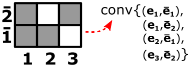
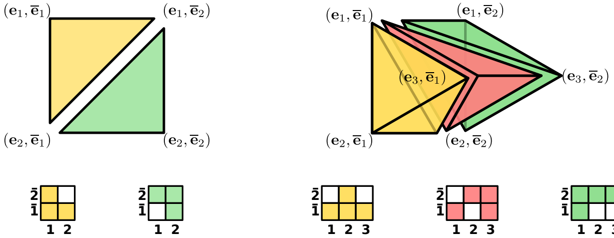
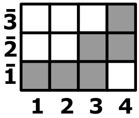


So a triangulation of $\Delta_{m-1} \times \Delta_{n-1}$ can look like:



Staircase triangulation

- Consider a *staircase* in a $n \times m$ grid.
- The corresponding vertices of $\Delta_{m-1} \times \Delta_{n-1}$ span a $(n+m-2)$-simplex.



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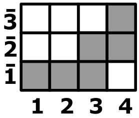
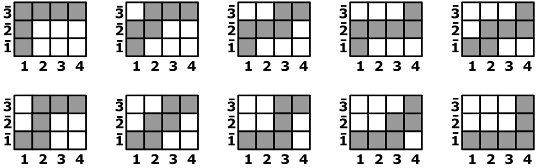
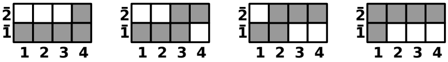


  - The $\binom{n+m-2}{n-1}$ simplices obtained from all staircases cover $\Delta_{m-1} \times \Delta_{n-1}$ and intersect well:



  - They form the *staircase triangulation of $\Delta_{m-1} \times \Delta_{n-1}$*.

When $m = 2$...



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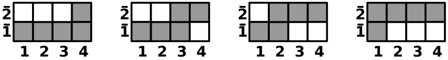

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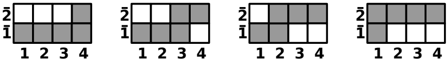

- When $m = 2$ there are $n!$ staircase triangulations
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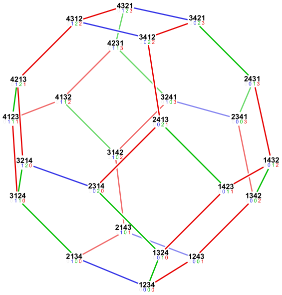
- The secondary polytope of $\Delta_1 \times \Delta_{n-1}$ is a Permutahedron.



The Permutahedron
(figure from Wikipedia)

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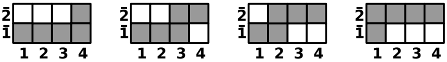

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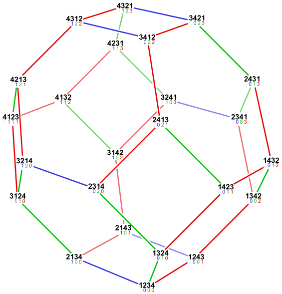

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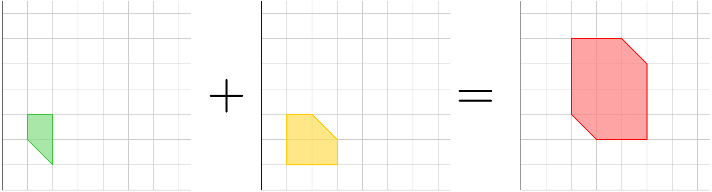
It is an open problem due to Gel'fand, Kapranov and Zelevinsky to find an explicit description of all triangulations of $\Delta_{m-1} \times \Delta_{n-1}$.
[Sturmfels '91]

Minkowski sums

The *Minkowski sum*: $A + B = \{a + b \,:\, a \in A, b \in B\}$



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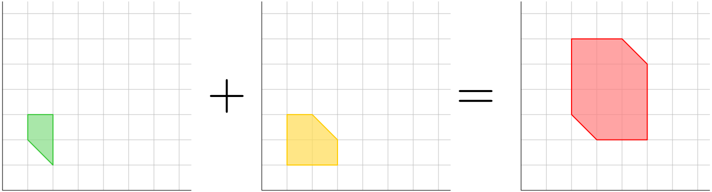
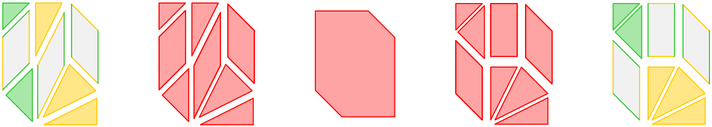


Mixed subdivisions of $P+Q$ with cells $F+G$ where F,G are faces of subdivisions of P and Q.



The Cayley trick

Theorem (The Cayley trick [Huber-Rambau-Santos ’00])

$$\left\{\begin{matrix} \textit{mixed subdivisions} \\ \textit{of } P+Q \end{matrix}\right\} \xleftrightarrow{\textbf{Cayley trick}} \left\{\begin{matrix} \textit{subdivisions} \\ \textit{of } \mathrm{Cay}(P,Q) \end{matrix}\right\}$$

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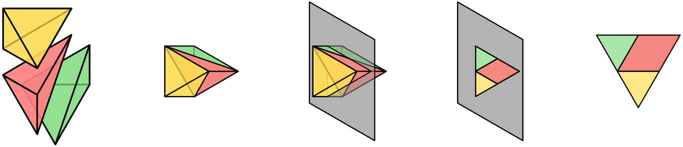

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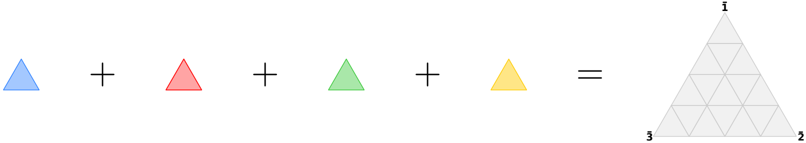

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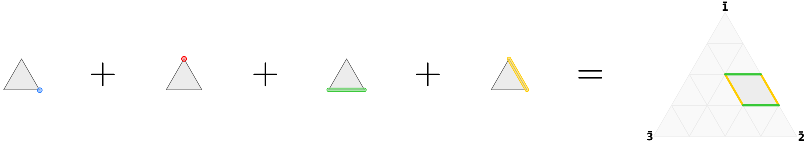

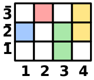

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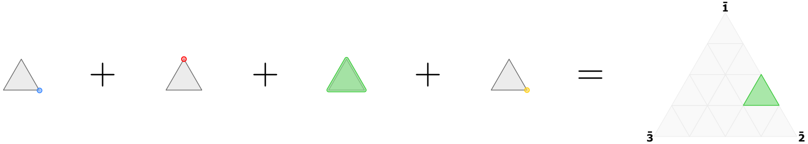

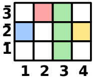

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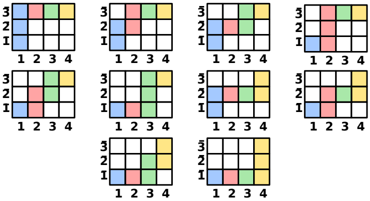
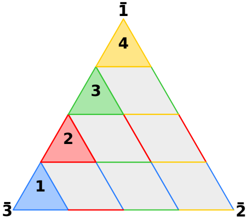
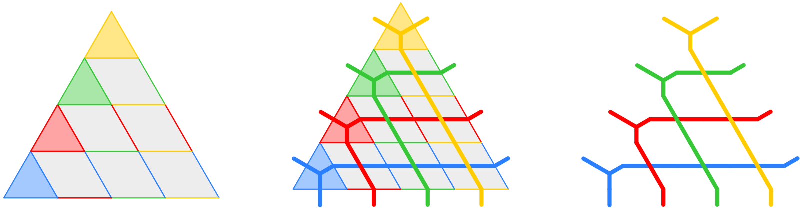




Tropical arrangements



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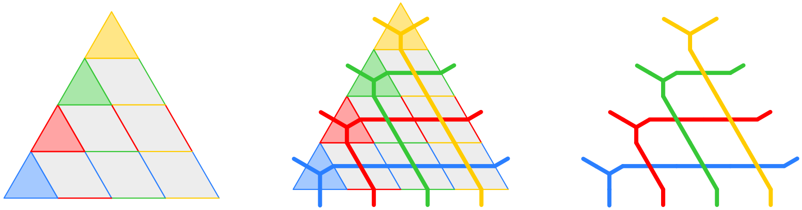

Tropical semiring

$(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$

where $x \oplus y := \min(x, y)$ and $x \odot y = x + y$

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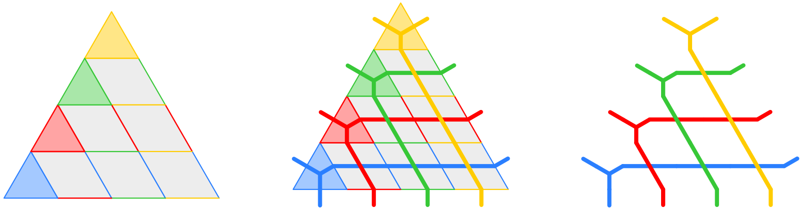

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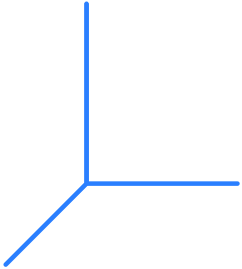
$\rightsquigarrow$

Tropical Geometry

Tropical polynomials, tropical curves, tropical hyperplanes, tropical lines . . .



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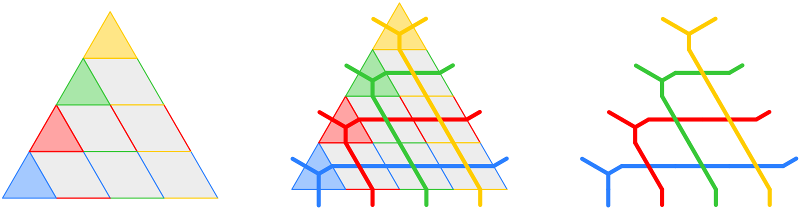

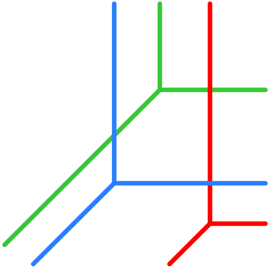

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⇣

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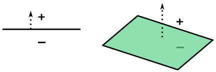
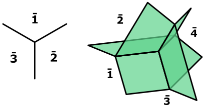
Tropical hyperplane arrangements

Tropical Oriented Matroids

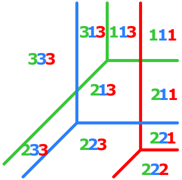
classical hyperplane



tropical hyperplane



[codim-1 skeleton of polar fan of (n-1)-simplex]



Tropical Oriented Matroids

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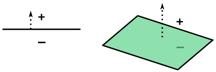

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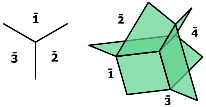

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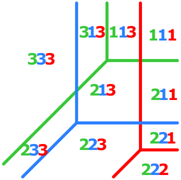
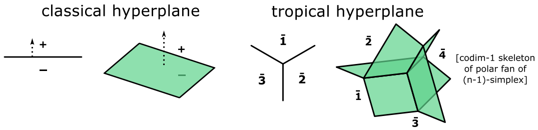
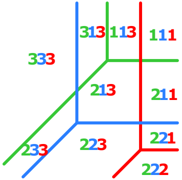


Theorem (Develin-Sturmfels '04)

$$\left\{\begin{matrix} \textit{regular triangulations} \\ \textit{of } \Delta_{m-1} \times \Delta_{n-1} \end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} \textit{(combintorial types of) generic arrangements of} \\ m \textit{ tropical hyperplanes in } \mathbb{TP}^{n-1} \end{matrix}\right\}$$

Tropical Oriented Matroids





Theorem (Develin-Sturmfels '04, Santos '04, Ardila-Develin '09, Oh-Yoo '12, Horn '12)

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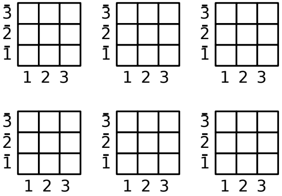
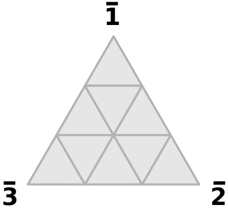
The Dyck path triangulations

Dyck path triangulation of $\Delta_{n-1} \times \Delta_{n-1}$

Consider the *Dyck paths* in an $n \times n$ grid





with their orbits under $(i, \bar{j}) \mapsto (i+1 \bmod n,\ \bar{j}+1 \bmod n)$

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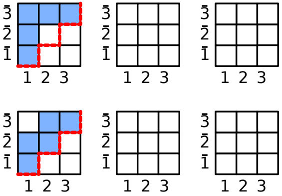

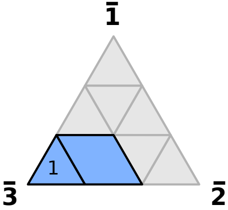

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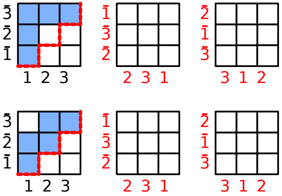

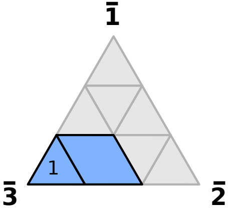

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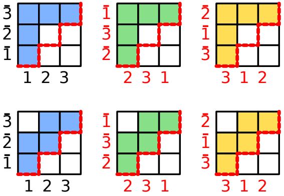

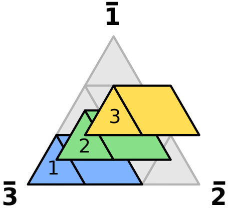

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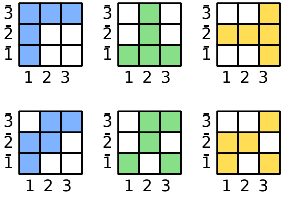

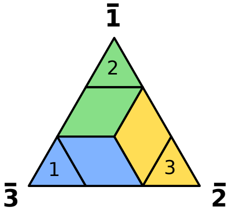

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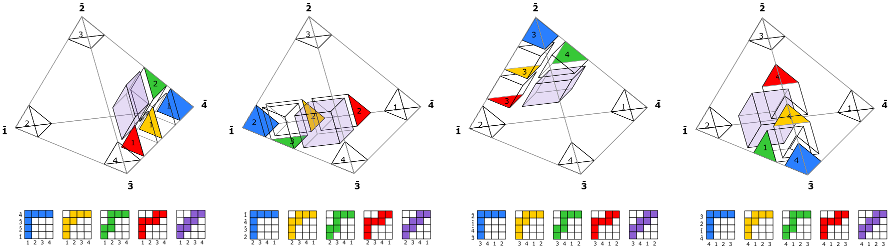

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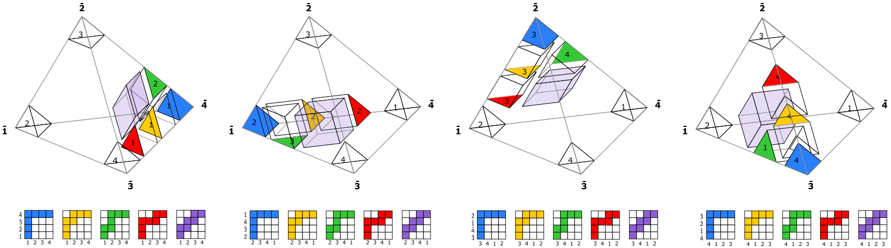


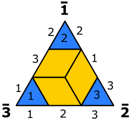
Theorem (CPS '14)

The resulting $n \cdot \frac{1}{n}\binom{2(n-1)}{n-1}$ simplices form a regular *triangulation of $\Delta_{n-1} \times \Delta_{n-1}$: the* Dyck path triangulation*.*

Some relatives

Theorem (CPS '14)
The following are all (regular) triangulations.

Flipped Dyck path triangulation:



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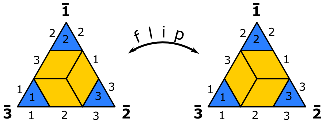

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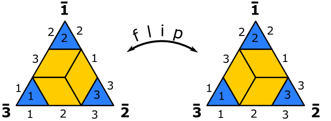
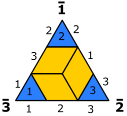


Extended Dyck path triangulation:



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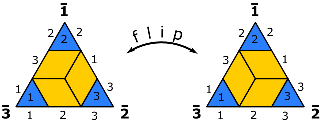

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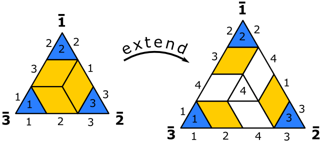

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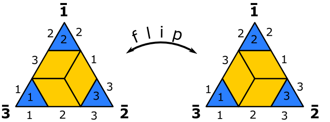

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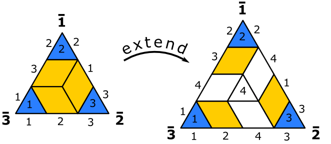
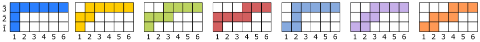
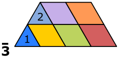


“Rational” Dyck path triangulation





Some relatives

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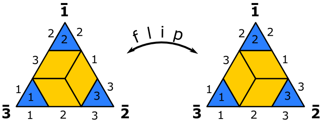

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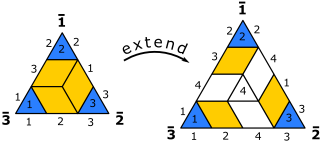

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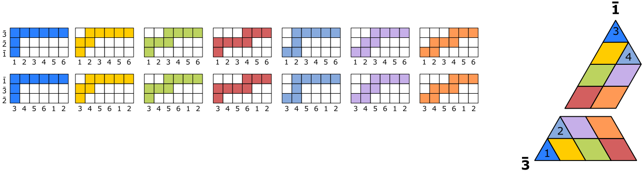

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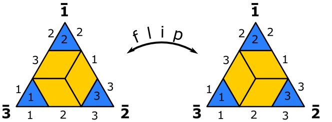

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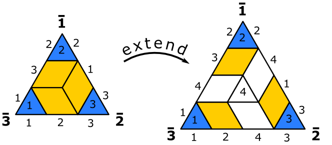

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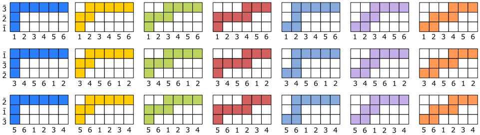
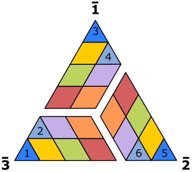
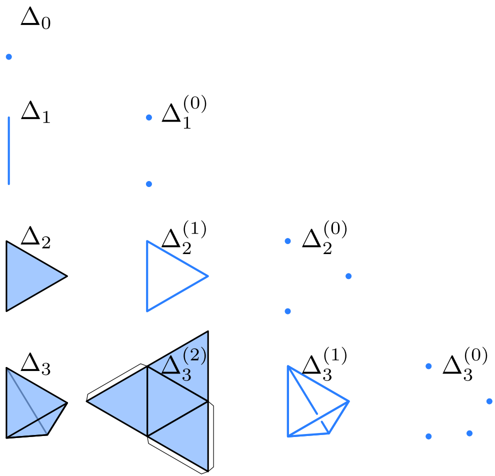




Extendability

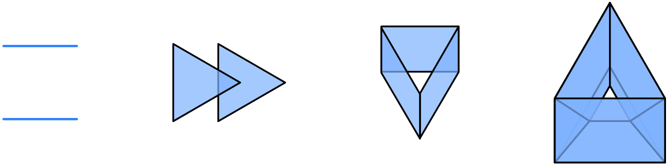
Skeletons

The *k-skeleton* of P, $P^{(k)}$, is the complex of $(\leq k)$-faces of P.



Semiskeletons

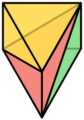
$$\color{red}{\Delta_{m-1}^{(k-1)} \times \Delta_{n-1}} \subset (\Delta_{m-1} \times \Delta_{n-1})^{(k+n-2)}$$



$\Delta_1^{(0)} \times \Delta_1$ $\Delta_1^{(0)} \times \Delta_2$ $\Delta_2^{(1)} \times \Delta_1$ $\Delta_2^{(1)} \times \Delta_2$

Partial triangulations

Triangulations of $\Delta_{m-1} \times \Delta_{n-1}$ $\rightsquigarrow$ triangulations of $\Delta_{m-1}^{(k-1)} \times \Delta_{n-1}$.



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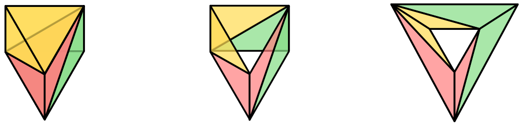

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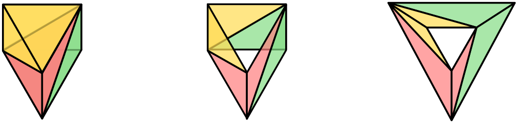

Question

Which triangulations of $\Delta_{m-1}^{(k-1)} \times \Delta_{n-1}$ can be extended to $\Delta_{m-1} \times \Delta_{n-1}$?.

Partial triangulations

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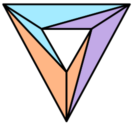

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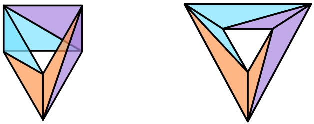

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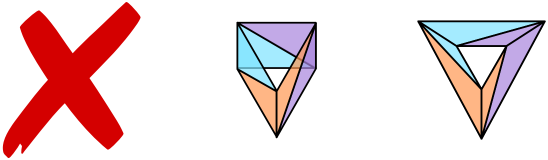

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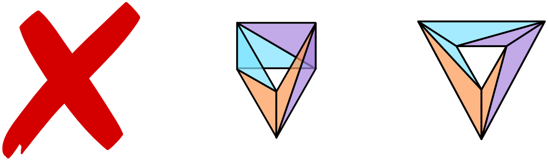


Question

Which triangulations of $\Delta_{m-1}^{(k-1)} \times \Delta_{n-1}$ can be extended to $\Delta_{m-1} \times \Delta_{n-1}$?.

- $k = 2$, $\min\{m, n\} \leq 3$: one obstruction, complete characterization **[Ardila-Ceballos '11]**
- $k = 2$, $\min\{m, n\} > 3$: more obstructions, open **[Santos '11, CPS '14]**
- **Conjecture:** for $k = 2$, general m, n, there are ∞-many obstructions
- $m \geq n > k$: open. **Conjecture:** there are ∞-many obstructions
- $m \geq k \geq n$: solved **[CPS '14]**

Extendability result

Theorem (CPS '14)

Let $m \geq \boxed{k \geq n} \in \mathbb{N}$. If a triangulation of $\Delta_{m-1}^{(k-1)} \times \Delta_{n-1}$ extends, then the extension is unique triangulation of $\Delta_{m-1} \times \Delta_{n-1}$.

Extendability result

Theorem (CPS ’14)

Let $m \geq \boxed{k \geq n} \in \mathbb{N}$. *If a triangulation of* $\Delta_{m-1}^{(k-1)} \times \Delta_{n-1}$ *extends, then the extension is unique triangulation of* $\Delta_{m-1} \times \Delta_{n-1}$.

Theorem (CPS ’14)

Let $m \geq \boxed{k > n} \in \mathbb{N}$. *Every triangulation of* $\Delta_{m-1}^{(k-1)} \times \Delta_{n-1}$ *extends to a unique triangulation of* $\Delta_{m-1} \times \Delta_{n-1}$.

Extendability result

Theorem (CPS '14)

Let $m \geq \boxed{k \geq n} \in \mathbb{N}$. If a triangulation of $\Delta_{m-1}^{(k-1)} \times \Delta_{n-1}$ extends, then the extension is unique triangulation of $\Delta_{m-1} \times \Delta_{n-1}$.

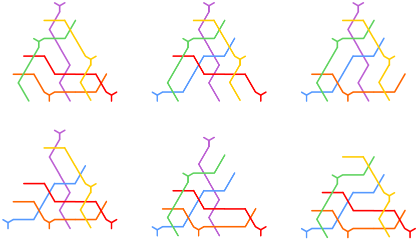
Theorem (CPS '14)

Let $m \geq \boxed{k > n} \in \mathbb{N}$. Every triangulation of $\Delta_{m-1}^{(k-1)} \times \Delta_{n-1}$ extends to a unique triangulation of $\Delta_{m-1} \times \Delta_{n-1}$.

Some alternative interpretations

- "as m increases, triangulations of $\Delta_{m-1} \times \Delta_{n-1}$ don't get much more complicated than triangulations of $\Delta_n \times \Delta_{n-1}$"
- "when $m \gg n$, compatibly piecing together triangulations of $\Delta_n \times \Delta_{n-1}$ we can always build any triangulation of $\Delta_{m-1} \times \Delta_{n-1}$"

Extendability result "tropically"



Extendability result "tropically"

Tropically, when $m \geq n$:

- a generic arrangement of m **tropical** pseudohyperplanes in $\mathbb{TP}^{n-1}$ is completely determined by its $\binom{m}{n}$ generic subarrangements of n **tropical** pseudohyperplanes in $\mathbb{TP}^{n-1}$.

when $m > n$:

- a generic arrangement of m **tropical** pseudohyperplanes in $\mathbb{TP}^{n-1}$ gives rise to a "compatible collection" of $\binom{m}{n+1}$ generic subarrangements of $n+1$ **tropical** pseudohyperplanes in $\mathbb{TP}^{n-1}$.
- conversely, every "compatible" collection of $\binom{d}{n+1}$ generic subarrangements of $n+1$ **tropical** pseudohyperplanes in $\mathbb{TP}^{n-1}$ equals the collection of restrictions of a unique generic arrangement of m **tropical** pseudohyperplanes in $\mathbb{TP}^{n-1}$

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Extendability result “tropically”

Tropically, when $m \geq n$:

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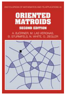
when $m > n$:

- a generic arrangement of m **tropical** pseudohyperplanes in $\mathbb{TP}^{n-1}$ gives rise to a “compatible collection” of $\binom{m}{n+1}$ generic subarrangements of $n+1$ **tropical** pseudohyperplanes in $\mathbb{TP}^{n-1}$.
- conversely, every “compatible” collection of $\binom{d}{n+1}$ generic subarrangements of $n+1$ **tropical** pseudohyperplanes in $\mathbb{TP}^{n-1}$ equals the collection of restrictions of a unique generic arrangement of m **tropical** pseudohyperplanes in $\mathbb{TP}^{n-1}$

without adjective “tropical”, this is in 

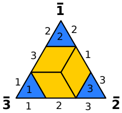
Non-extendability result

The bound is optimal: $\boxed{k > n}$ is necessary for existence, i.e.,

Theorem (CPS '14)

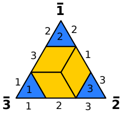
For every natural number $n \geq 2$ there is a non-extendable triangulation of $\Delta_n^{(n-1)} \times \Delta_{n-1}$.

Sketch of proof



Dyck path triangulation
of $\Delta_{n-1} \times \Delta_{n-1}$

Sketch of proof



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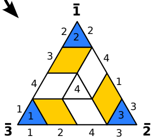
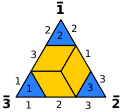


Extended Dyck path triangulation
of $\Delta_n \times \Delta_{n-1}$

Sketch of proof



Dyck path triangulation
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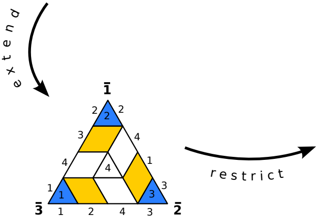

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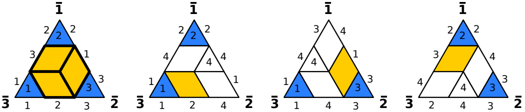
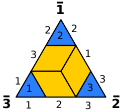


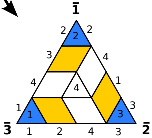
Partial triangulation of $\Delta_n^{(n-1)} \times \Delta_{n-1}$

Sketch of proof



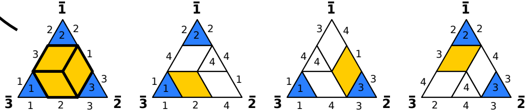
Dyck path triangulation of $\Delta_{n-1} \times \Delta_{n-1}$

extend



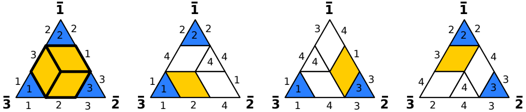
Extended Dyck path triangulation of $\Delta_n \times \Delta_{n-1}$

restrict



Partial triangulation of $\Delta_n^{(n-1)} \times \Delta_{n-1}$

flip



Theorem (CPS '14): The resulting triangulation of $\Delta_n^{(n-1)} \times \Delta_{n-1}$ is non extendable

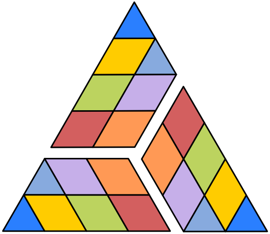
More rational Dyck path triangulations?

Dyck path triangulations exploit the identity

$$n \cdot C_{n-1} = \binom{2n-2}{n-1},$$

Rational Dyck path triangulations use that

$$n \cdot C(n, rn-1) = \binom{(r+1)n-2}{n-1},$$



Question

What about these?

$$a \cdot C(a,b) = \binom{a+b-1}{a-1} \qquad \text{or} \qquad (a+b)\cdot C(a,b) = \binom{a+b}{a}$$

where $C(a,b) = \frac{1}{a+b}\binom{a+b}{a}$ (for a and b relatively prime) are the *rational Catalan numbers*.

Moltes Gràcies! Merci Beaucoup!