Autour des éléments pleinement commutatifs dans les groupes de Coxeter finis et affines

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Collaboration avec Riccardo Biagioli et Philippe Nadeau

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Coxeter groups

(W, S) Coxeter group W given by Coxeter matrix $(m_{st})_{s,t \in S}$

Relations: $\begin{cases} s^2 = 1 \\ \underbrace{sts\cdots}_{m_{st}} = \underbrace{tst\cdots}_{m_{st}} \end{cases}$ → Braid relations

$m_{st} = 2$: commutation relation

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Matsumoto property (1964): Given two reduced decompositions of w, there is a sequence of braid relations which can be applied to transform one into the other

FC elements

Full commutativity is a strenghtening of Matsumoto's property

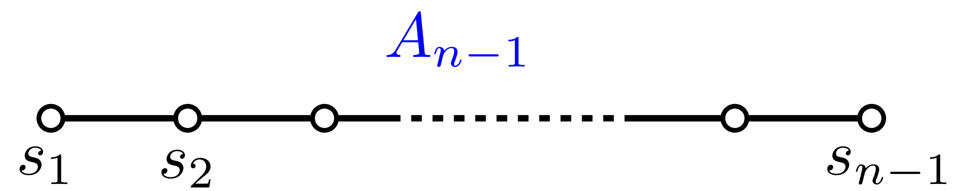
An element w is **fully commutative** if given two reduced decompositions of w, there is a sequence of commutation relations which can be applied to transform one into the other

Equivalently, w is fully commutative if its reduced decompositions form only one commutation class

Type A_{n-1} $\rightarrow$ The symmetric group S_n

Consider $S = \{s_1, \ldots, s_{n-1}\}$, with relations $s_i^2 = 1$ and

$$\begin{cases} s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \\ s_i s_j = s_j s_i, \quad |j-i| > 1 \end{cases}$$



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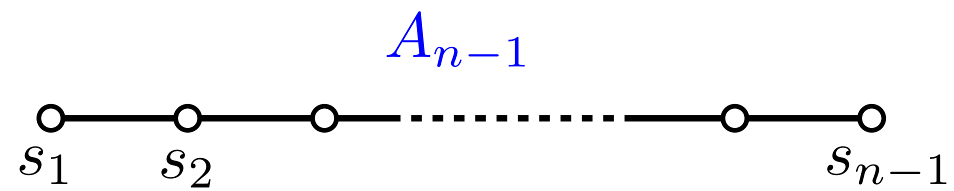

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Theorem [Billey-Jockush-Stanley (1993)]

w is fully commutative $\Leftrightarrow$ $\vartheta(w)$ is 321-avoiding

One can use this to show that FC elements in type A_{n-1} are counted by Catalan numbers, i.e., $|S_n^{FC}| = \dfrac{1}{n+1}\dbinom{2n}{n}$

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- [Green–Losonczy (2001), Shi (2003), ...] connect FC elements to Kazhdan-Lusztig polynomials
- [Barcucci et al (2001)] enumerate in type A with respect to the Coxeter length using pattern-avoidance
- [Hanusa–Jones (2010)] enumerate in type $\widetilde{A}$ with respect to the Coxeter length, using affine permutations

Outline

We enumerate FC elements and involutions according to the Coxeter length for any finite or affine Coxeter group W

$$W^{FC}(t) := \sum_{w \text{ is FC}} t^{\ell(w)} \text{ and } \bar{W}^{FC}(t) := \sum_{w \text{ is FC involution}} t^{\ell(w)}$$

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[B-J-N (2012-14)] We compute $W^{FC}(t)$ and $\bar{W}^{FC}(t)$ for any finite or affine W. When W is affine, the coefficients of the series form ultimately periodic sequences

I will focus on types A and $\widetilde{A}$, corresponding to the finite and affine symmetric groups. The idea is to encode the FC elements in these cases by certain lattice paths

Characterization of FC elements

Proposition[Stembridge] A reduced word represents a FC element if and only if no element of its commutation class contains a factor $\underbrace{sts\cdots}_{m_{st}}$ for a $m_{st} \geq 3$

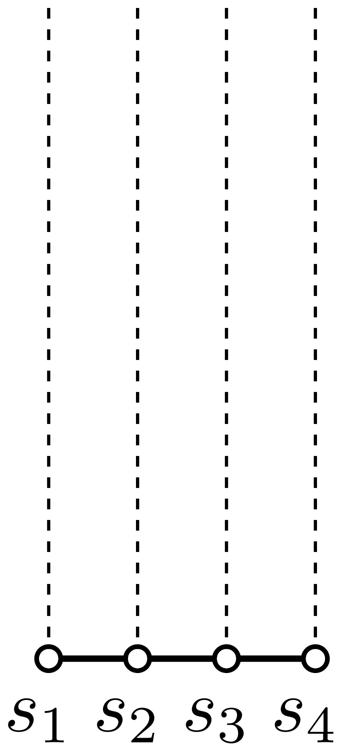
How to see if a commutation class verifies the above property ?
$\Rightarrow$ use the theory of heaps, which are posets encoding commutation classes

Example of heaps in $A_4 (= S_5)$

Heap of a word $=$ poset H labeled by generators s_i of W

Linear extensions of $H \Leftrightarrow$ words of the commutation class

$s_1 s_3 s_4 s_1 s_2 s_3$



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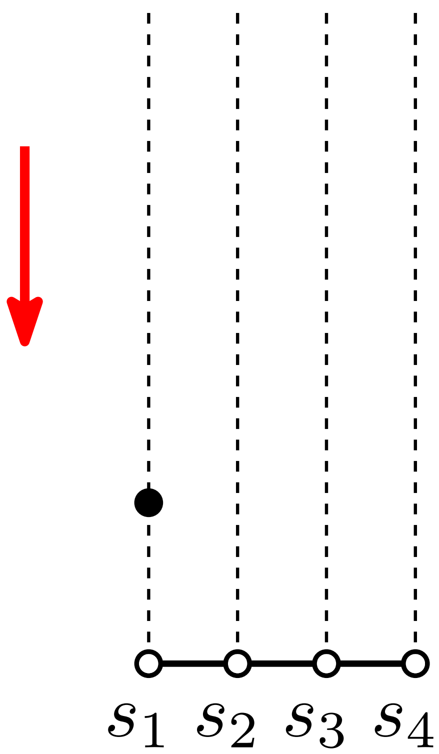

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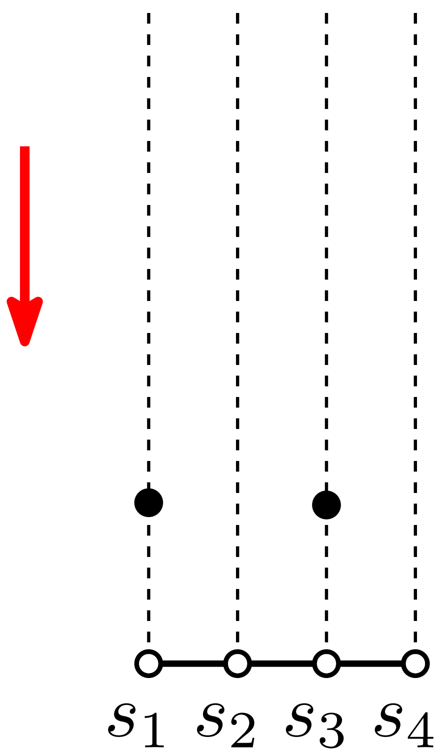

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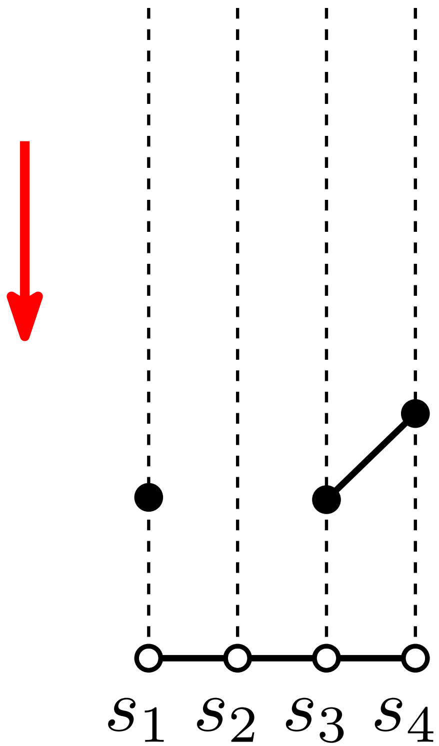

Vertex stays above if corresponding generators do not commute.

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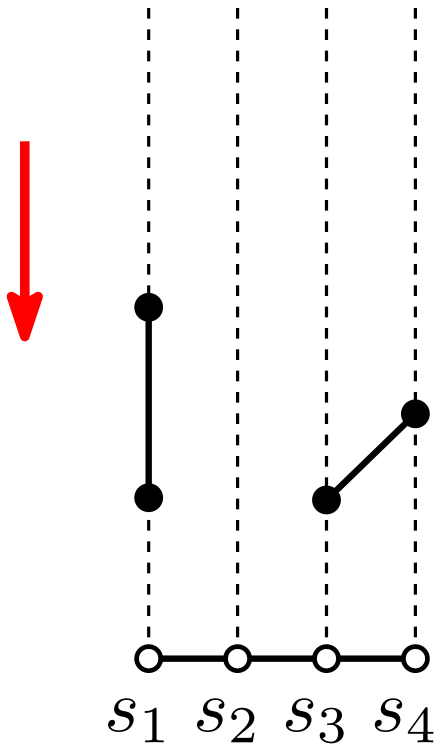

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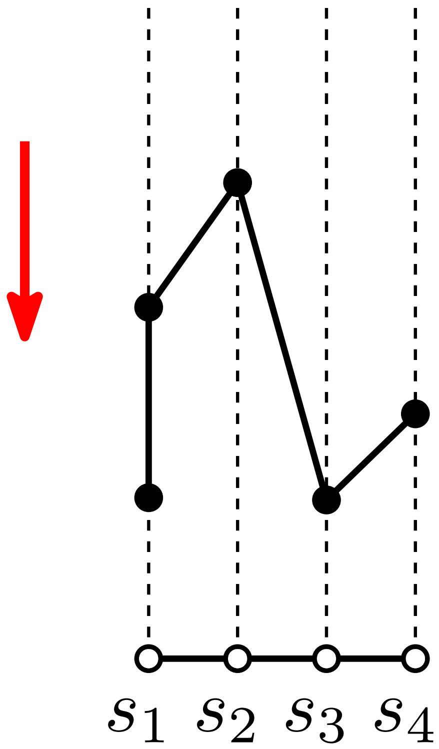

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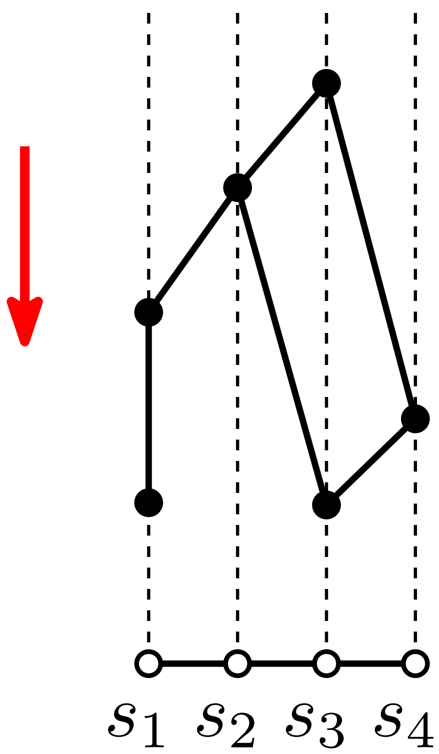

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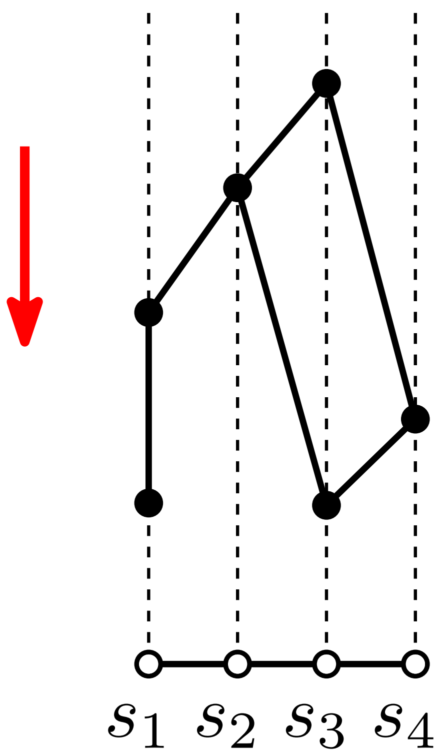

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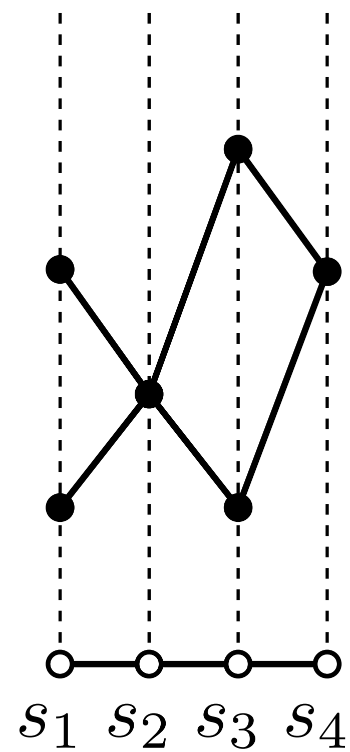

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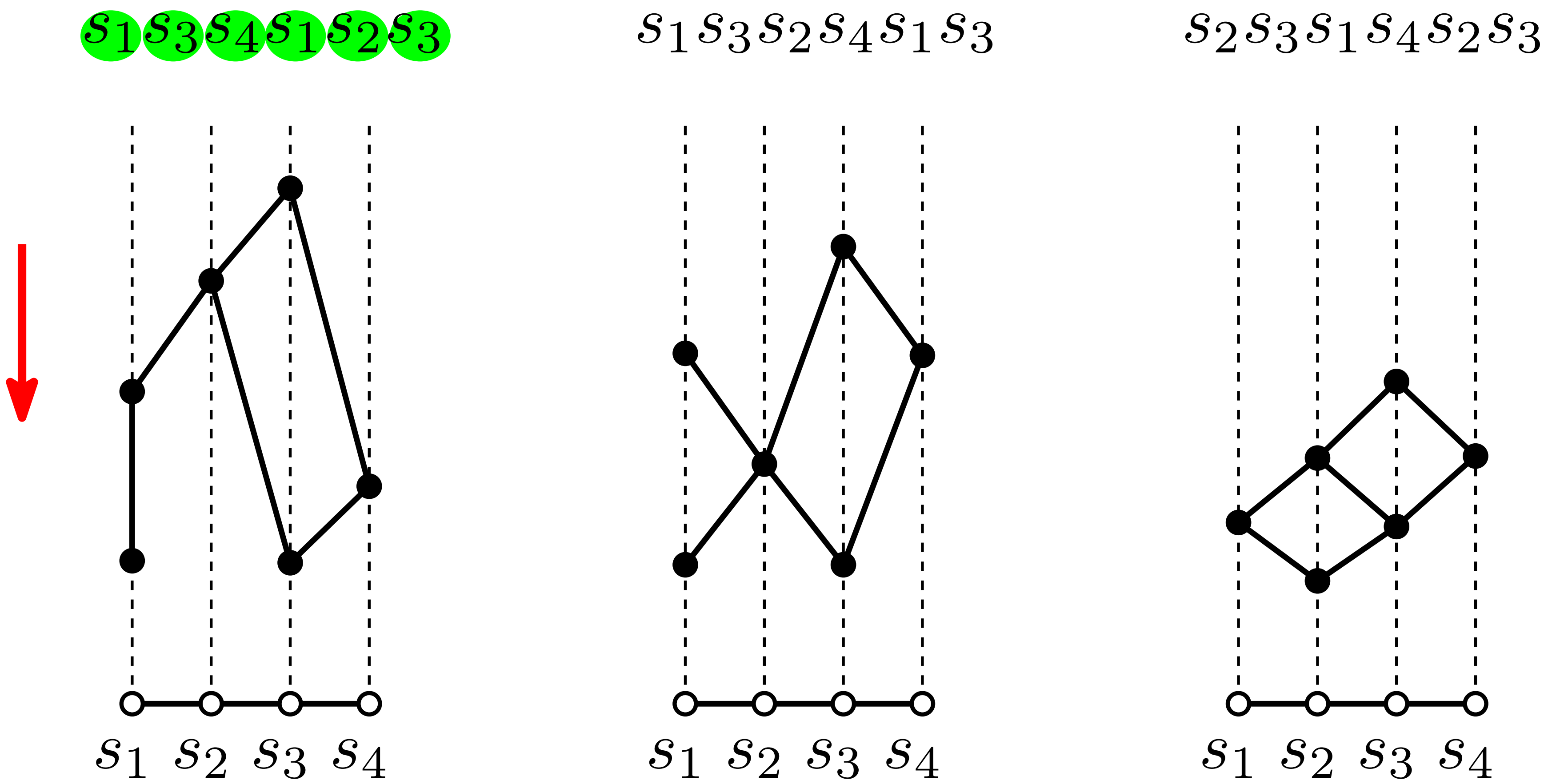

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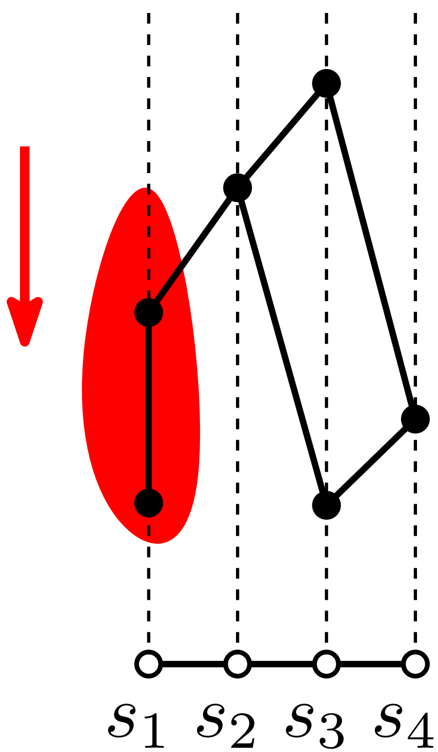
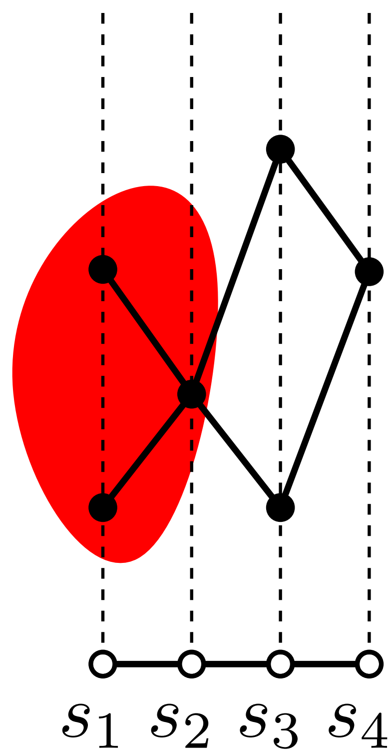


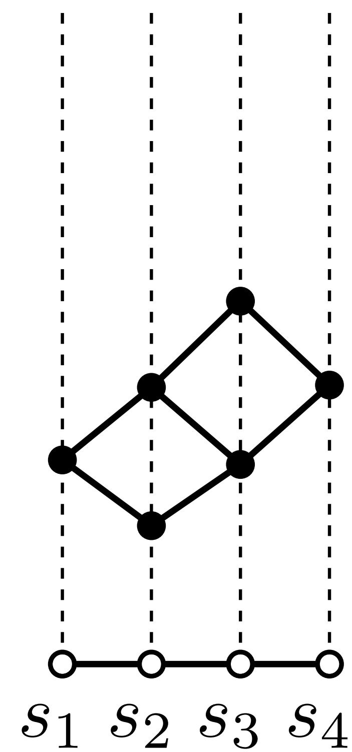
NOT REDUCED

$s_1 s_3 s_2 s_4 s_1 s_3$



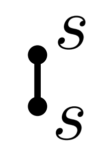
NOT FC

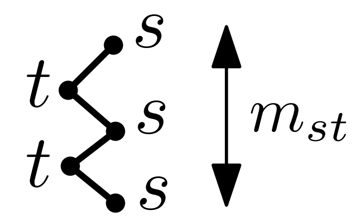
$s_2 s_3 s_1 s_4 s_2 s_3$



FC

Characterization of FC heaps

FC element w $\longleftrightarrow$ Heap H avoiding 

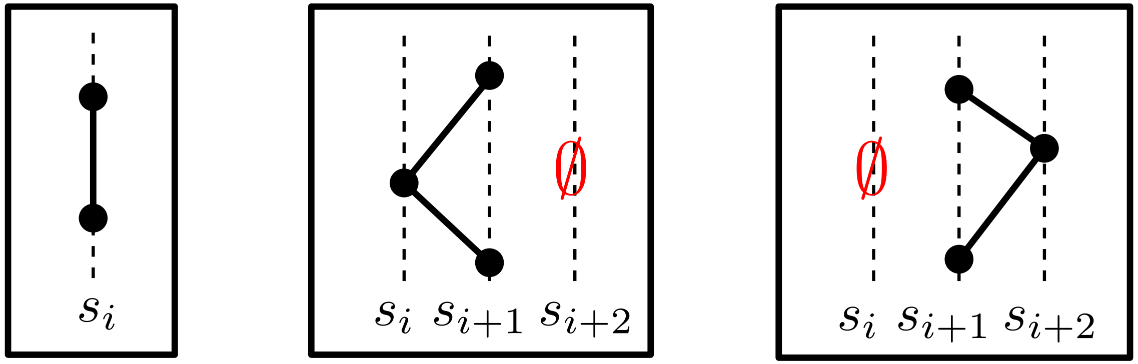
and 

Length $\ell(w)$ $\longleftrightarrow$ Number of elements $|H|$

In type A and $\widetilde{A}$: FC heaps above are particularly simple

Type A

FC heaps avoid precisely



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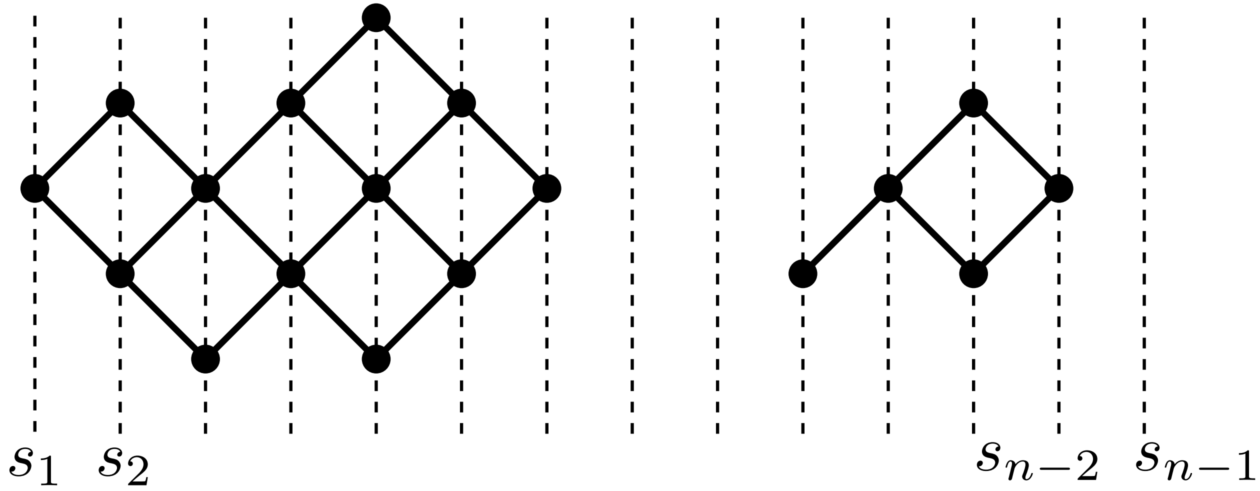

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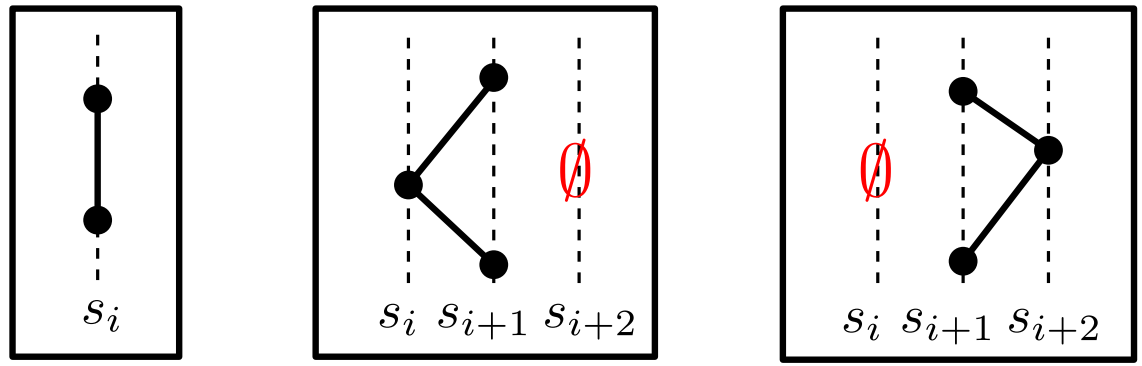

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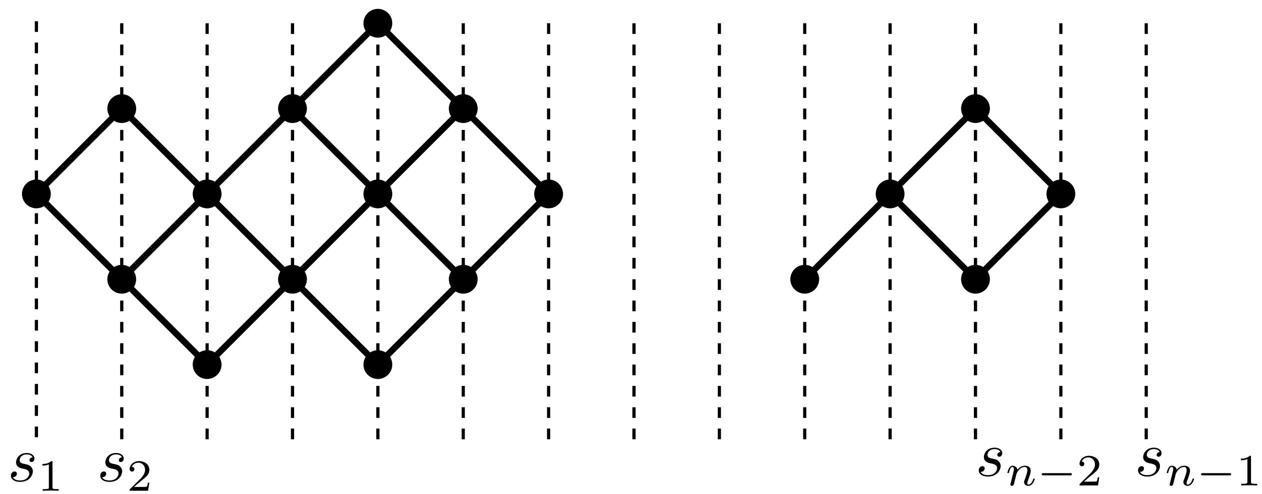
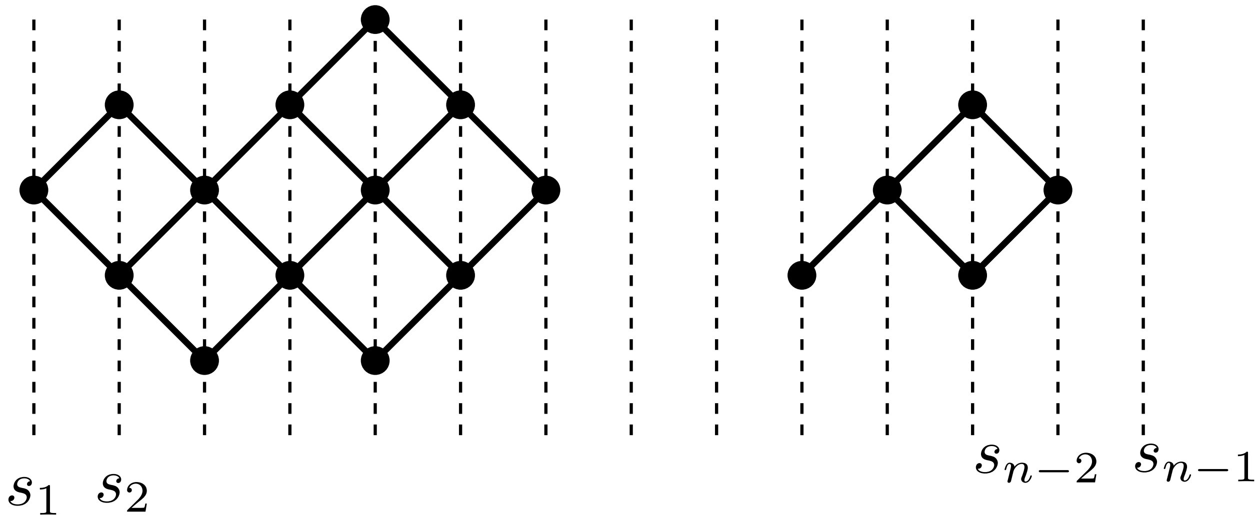


Proposition FC Heaps of type A are characterized by:
(a) At most one occurrence of s_1 (*resp.* s_{n-1})
(b) $\forall i$, elements with labels s_i, s_{i+1} form an alternating chain

Type A: bijection with Motzkin-type paths

FC Heap



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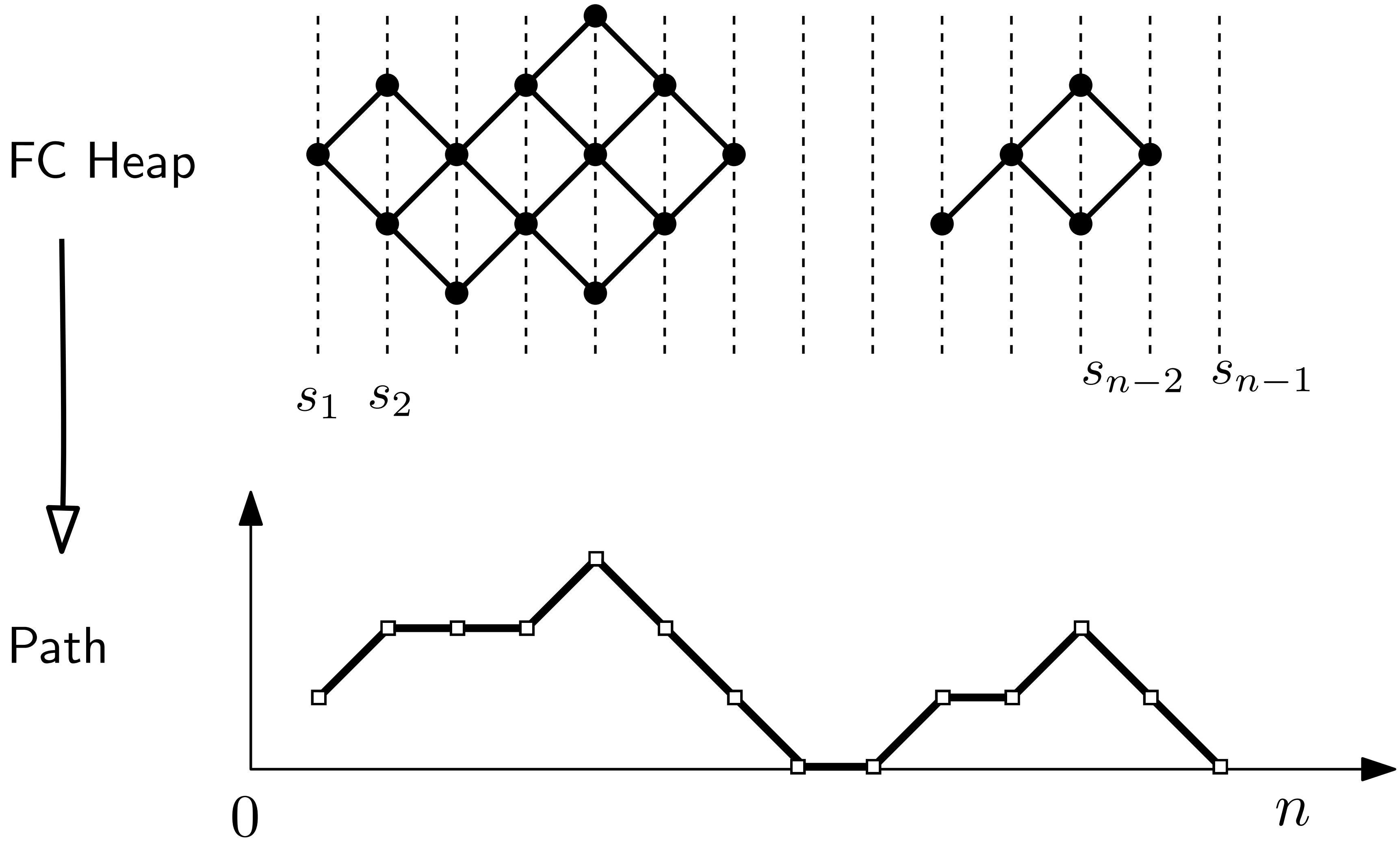

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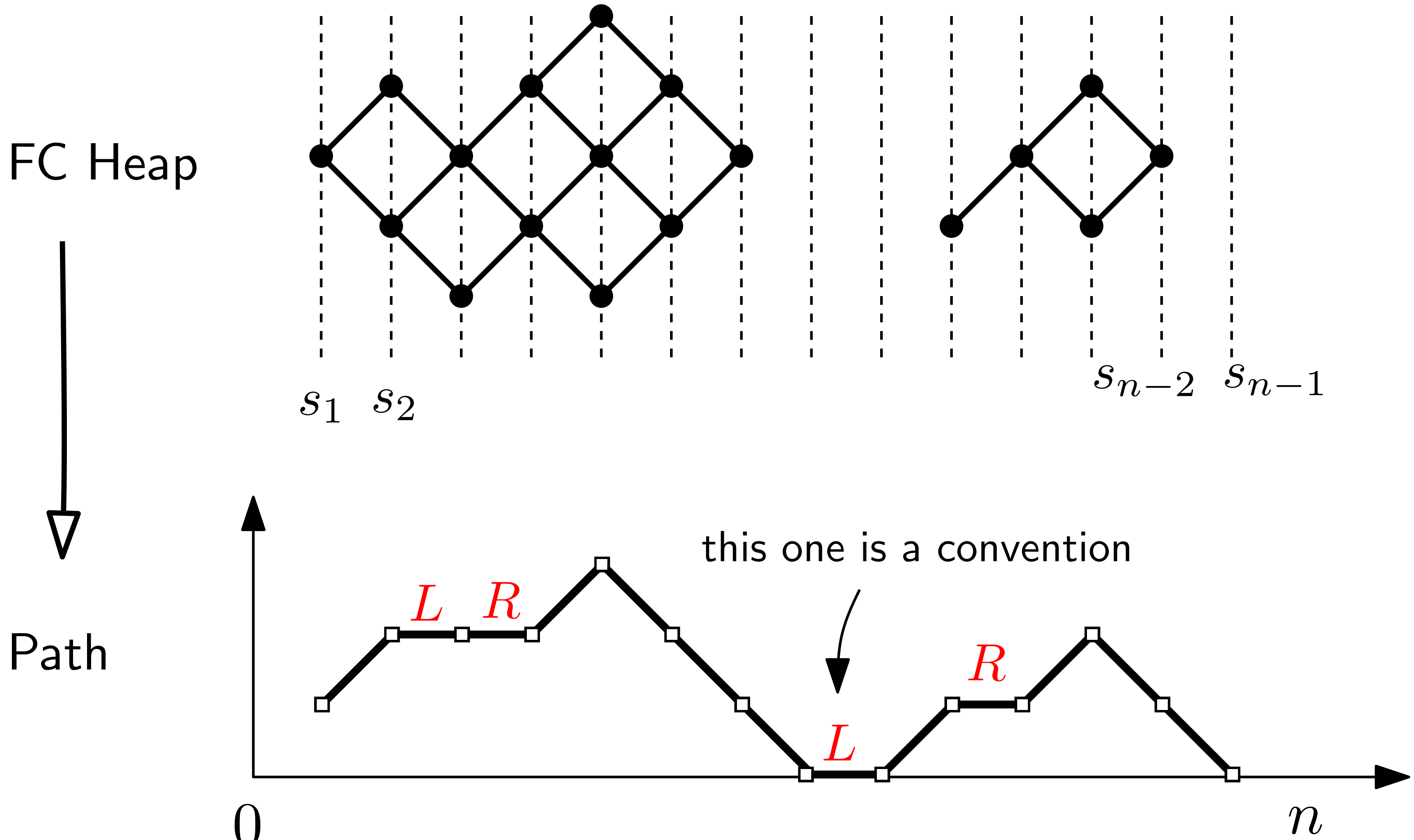

An extra information is needed

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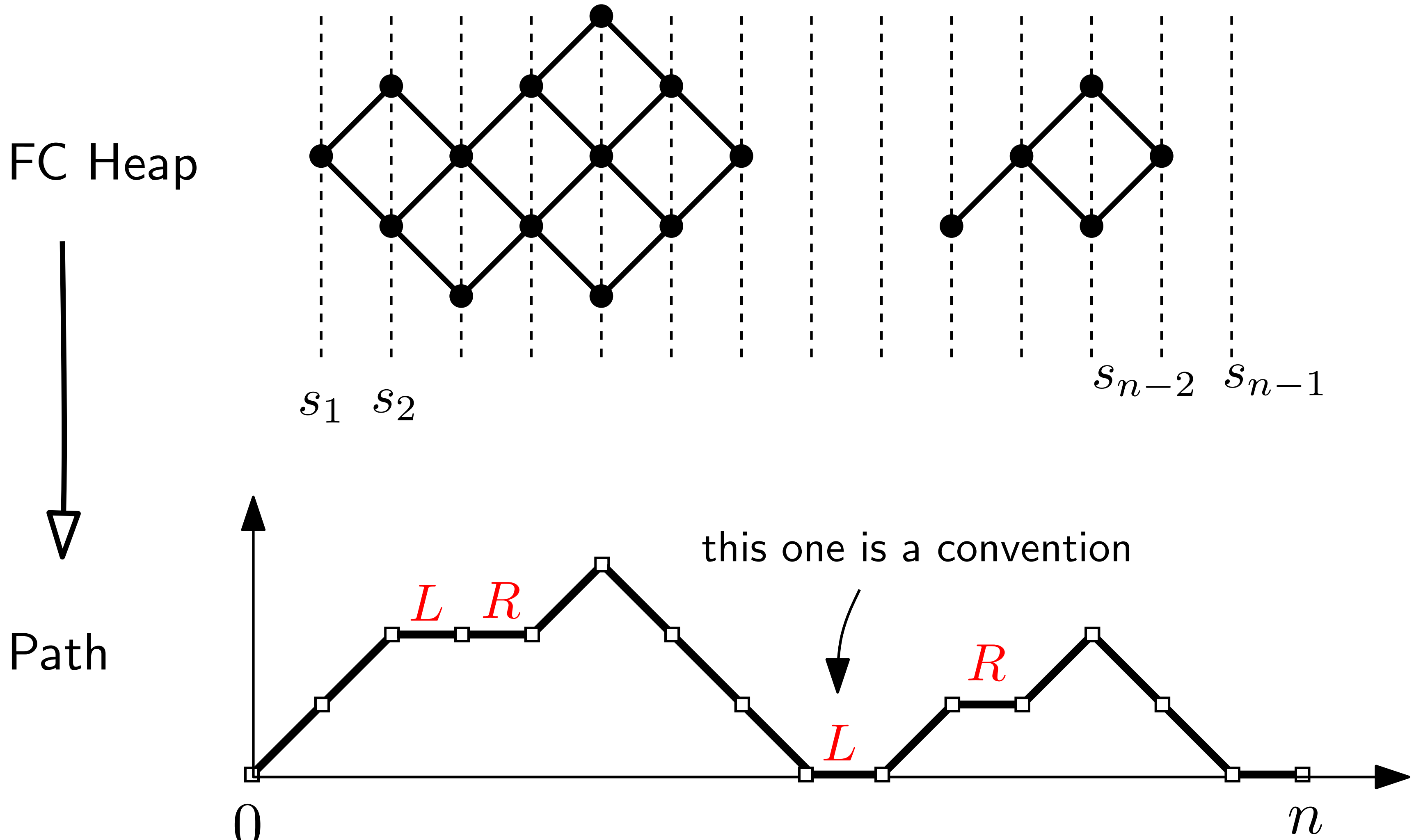

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To finish, add initial and final steps to the path

Type A: bijection with Motzkin paths

Theorem [BJN (2012)]: this is a bijection between FC heaps of type A_{n-1} and Motzkin paths of length n with horizontal steps at height $h > 0$ (*resp.* $h = 0$) labeled L or R (*resp.* labeled L)

Size of the heap $\Leftrightarrow$ **Area** of the path

(Sum of the heights of all vertices)

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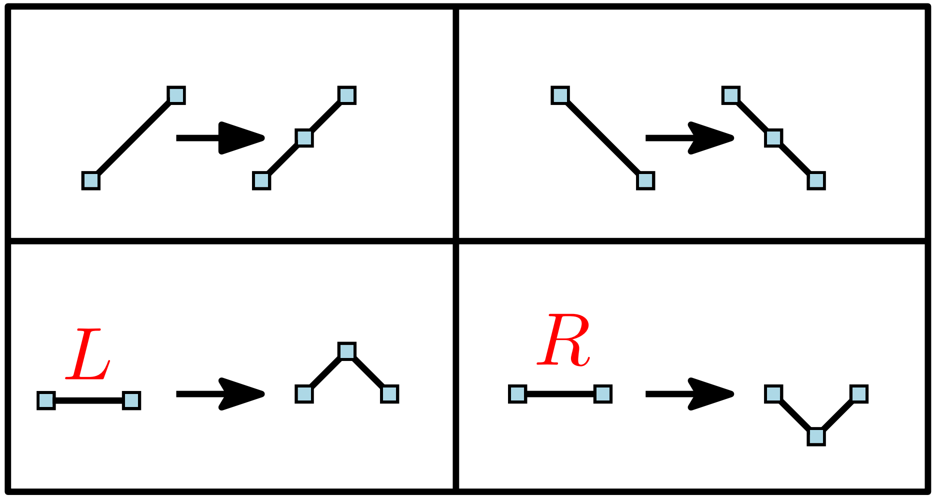
Remark



transforms these paths into Dyck paths $\Rightarrow$ Catalan numbers

Generating functions

We have: $A^{FC}(x) := \sum_{n \geq 1} A^{FC}_{n-1}(t) x^n = M^*(x) - 1$

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We have to count our labeled Motzkin paths with respect to their area $\rightarrow$ use recursive decompositions

Corollary [Barcucci et al. (2001)] We have:

$$A^{FC}(x) = x + xA^{FC}(x) + txA^{FC}(x)(A^{FC}(tx) + 1)$$

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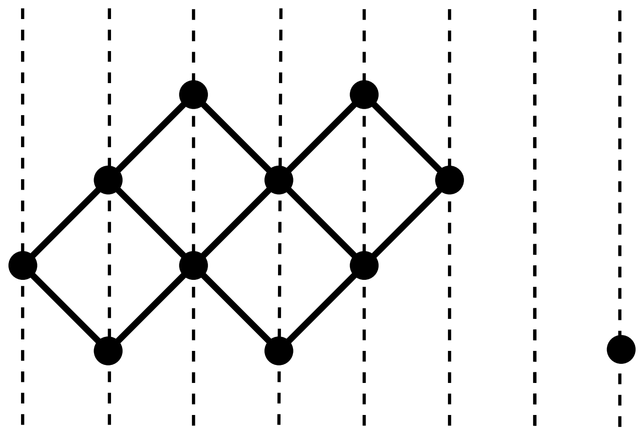
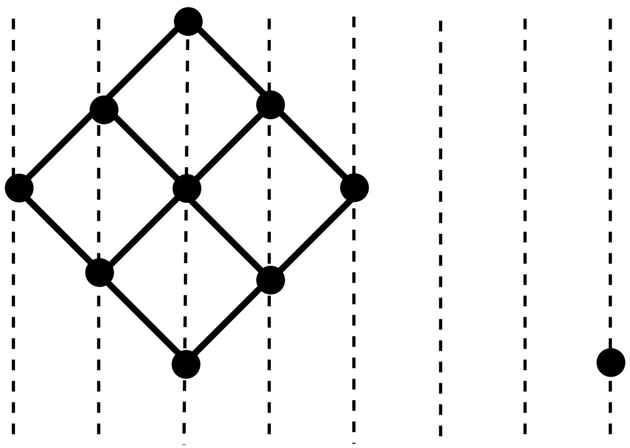


$s_2 s_4 s_8 s_1 s_3 s_5 s_2 s_4 s_6 s_3 s_5$

Not palindromic



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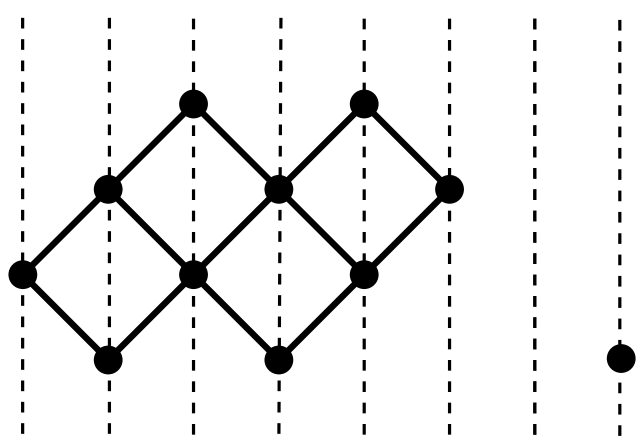
FC involutions in $\bar{W}$ are FC elements whose commutation class is palindromic: it includes the mirror images of its members



$s_2s_4s_8s_1s_3s_5s_2s_4s_6s_3s_5$
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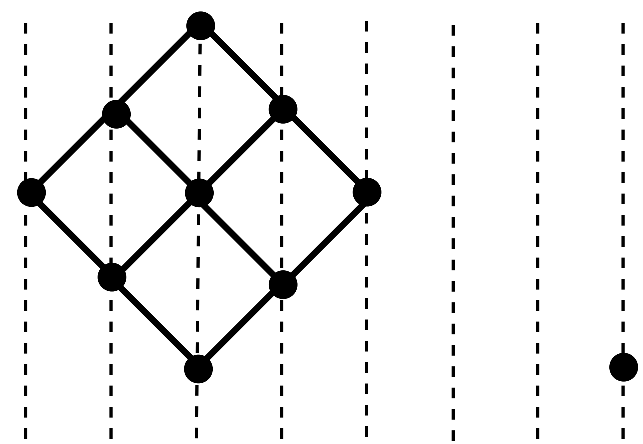

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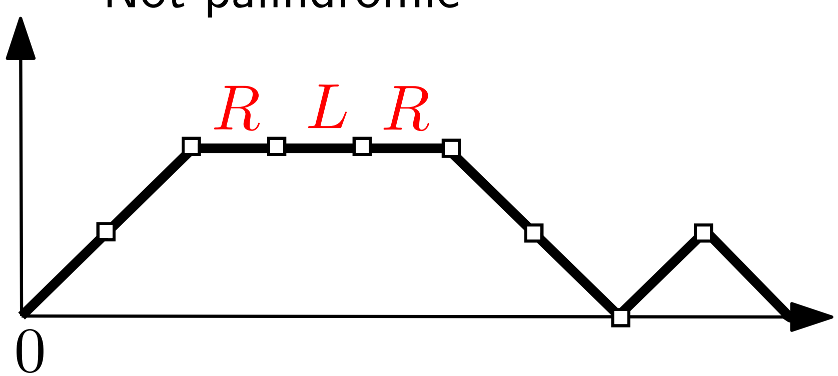
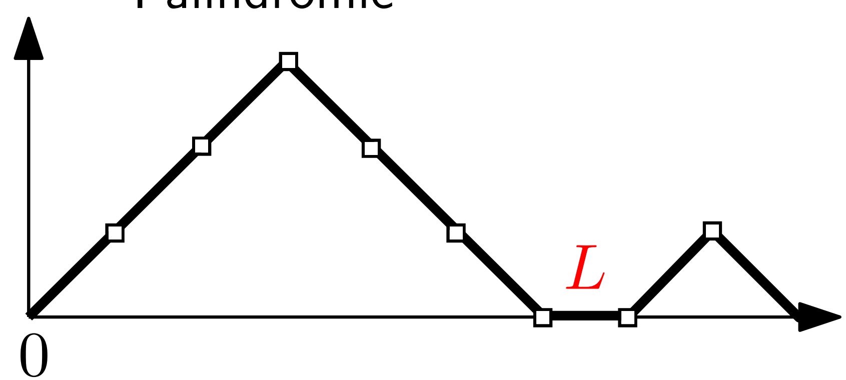


Motzkin path
$M^*(x) - 1$



Dyck paths and h steps
$\frac{Cat(x)}{1-xCat(x)} - 1$

Major index

Descent set of $w \in W$ is $\mathrm{Des}(w) = \{s \in S \mid \ell(ws) < \ell(w)\}$

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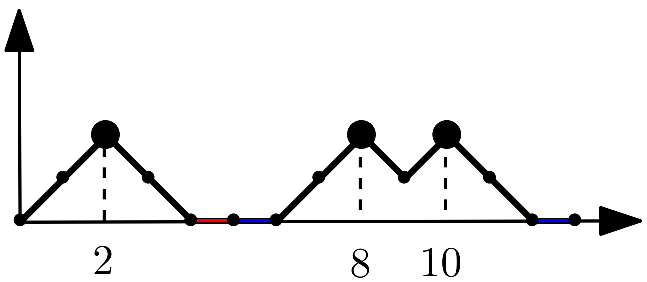

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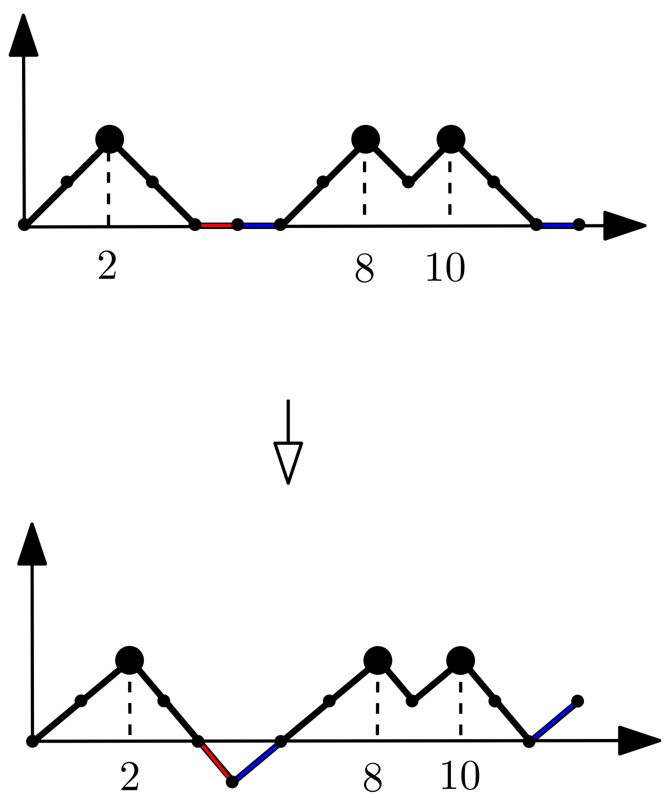

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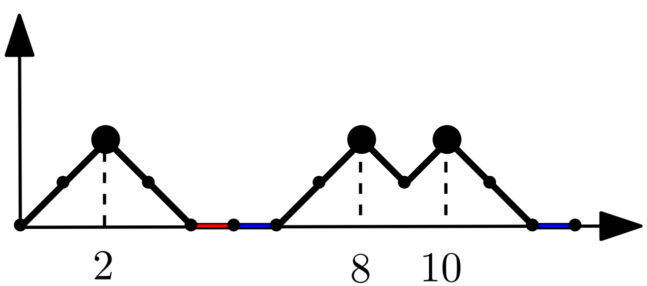

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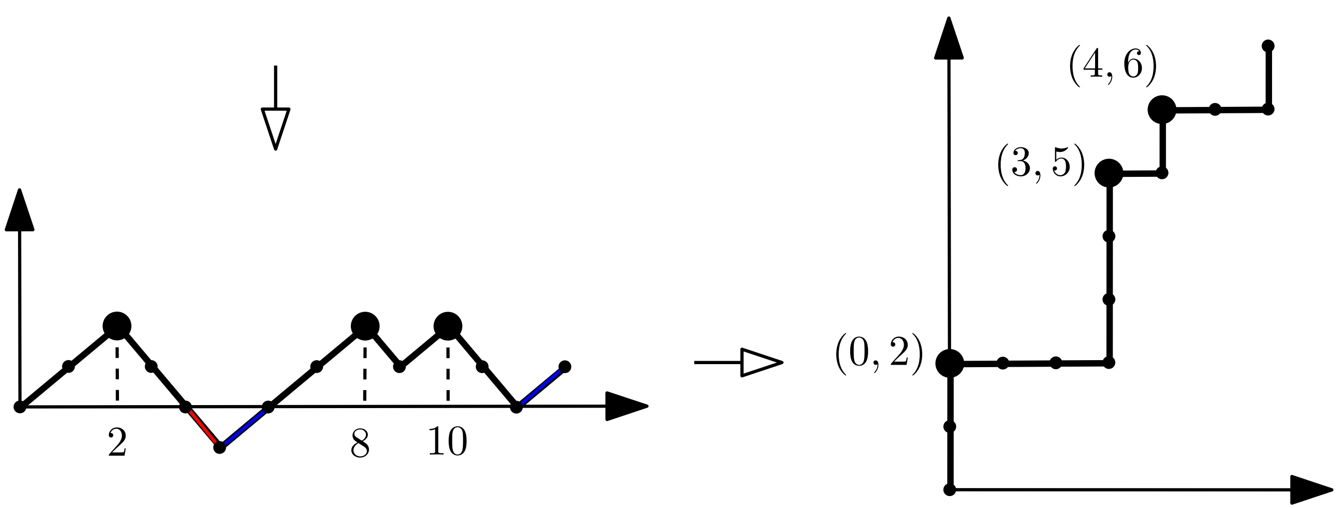

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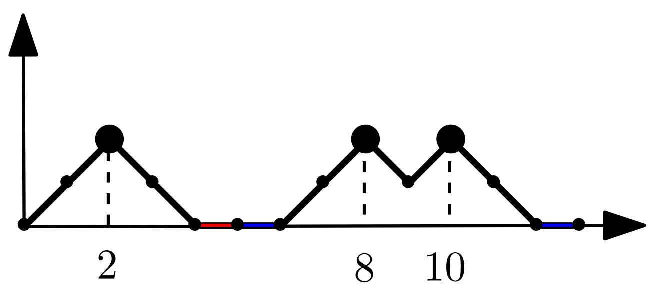

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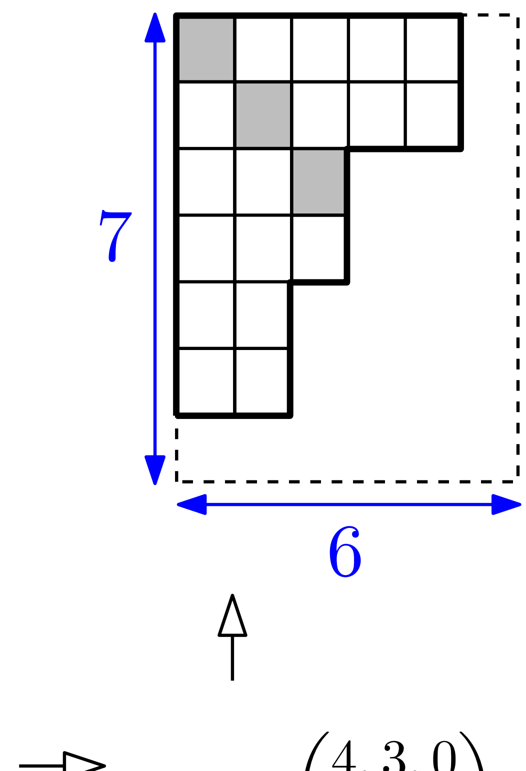

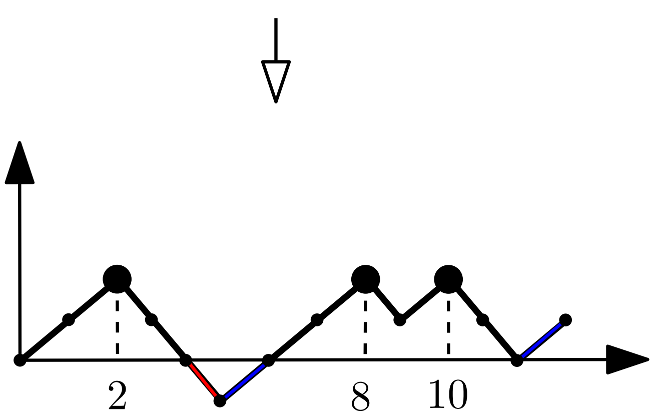

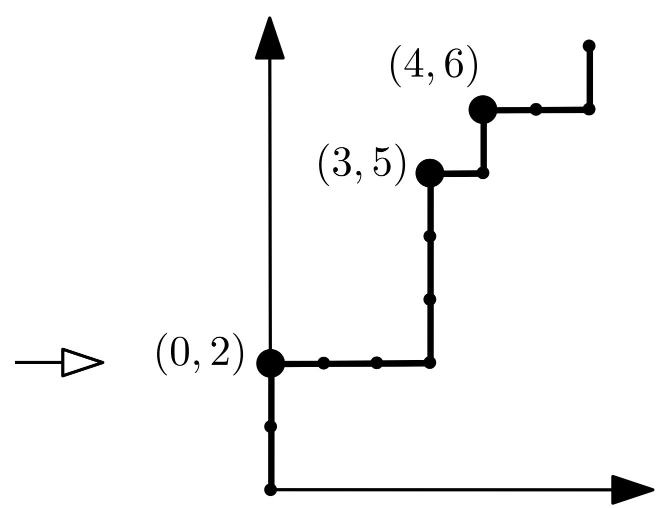

$$\lambda = \begin{pmatrix} 4, 3, 0 \\ 5, 4, 1 \end{pmatrix}$$

Major index generating functions of FC involutions

$$\lambda = \begin{pmatrix} a_1 \cdots a_j \\ b_1 \cdots b_j \end{pmatrix} \text{ with } a_1 < \lfloor \tfrac{n}{2} \rfloor, b_1 < \lceil \tfrac{n}{2} \rceil \Leftrightarrow \lambda \subset \lfloor \tfrac{n}{2} \rfloor \times \lceil \tfrac{n}{2} \rceil$$

Proposition [Barnabei et al, BJN (2014)]

$$\sum_{w \in \bar{A}^{FC}_{n-1}} q^{\mathrm{maj(w)}} = \begin{bmatrix} n \\ \lfloor n/2 \rfloor \end{bmatrix}_q$$

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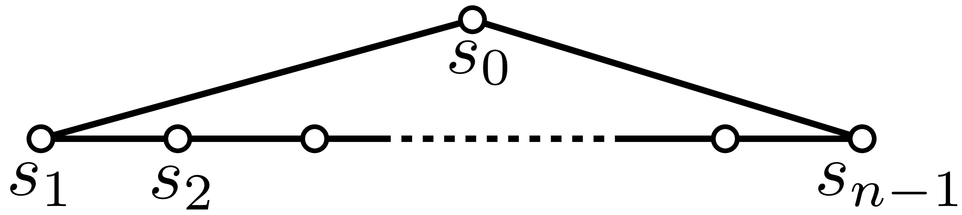
Our approach generalizes to types B and D

Proposition [BJN (2014)]

$$\sum_{w \in \bar{B}^{FC}_{n}} q^{\mathrm{maj(w)}} = \sum_{h=1}^{n} q^h \sum_{i=0}^{h-1} \begin{bmatrix} h-1 \\ i \end{bmatrix}_q + \begin{bmatrix} n \\ \lfloor n/2 \rfloor \end{bmatrix}_q$$

Affine types

$\widetilde{A}_{n-1}$



Theorem [Green (2001)] FC elements of type $\widetilde{A}_{n-1}$ correspond to 321-avoiding affine permutations

Affine types

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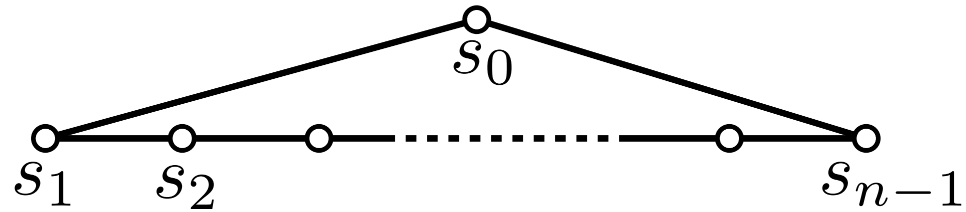


Theorem [Green (2001)] FC elements of type $\widetilde{A}_{n-1}$ correspond to 321-avoiding affine permutations

Hanusa–Jones used this to compute $\widetilde{A}_{n-1}^{FC}(t)$ and derived a complicated expression for this infinite series

Theorem [Hanusa-Jones (2010)] The coefficients of $\widetilde{A}_{n-1}^{FC}(t)$ are ultimately periodic of period dividing n

Generating functions

They computed the generating functions $f_n(t) = \widetilde{A}^{FC}_{n-1}(t)$; here are the first ones

$$f_3(t) = 1 + 3t + \mathbf{6t^2} + \mathbf{6t^3} + \mathbf{6t^4} + \cdots$$

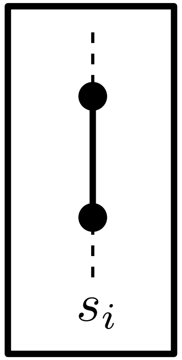
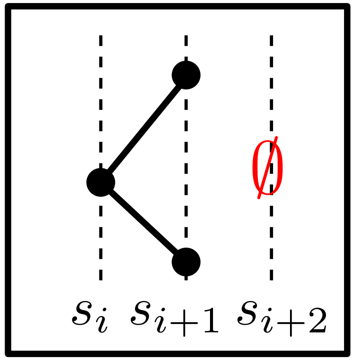
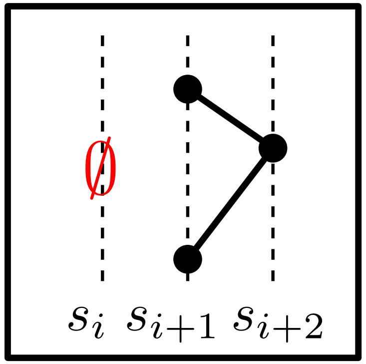
$$f_4(t) = 1 + 4t + 10t^2 + \mathbf{16t^3} + \mathbf{18t^4} + \mathbf{16t^5} + \mathbf{18t^6} + \cdots$$

$$\begin{aligned} f_5(t) = {} & 1 + 5t + 15t^2 + 30t^3 + 45t^4 \\ & + \mathbf{50t^5} + \mathbf{50t^6} + \mathbf{50t^7} + \mathbf{50t^8} + \mathbf{50t^9} + \cdots \end{aligned}$$

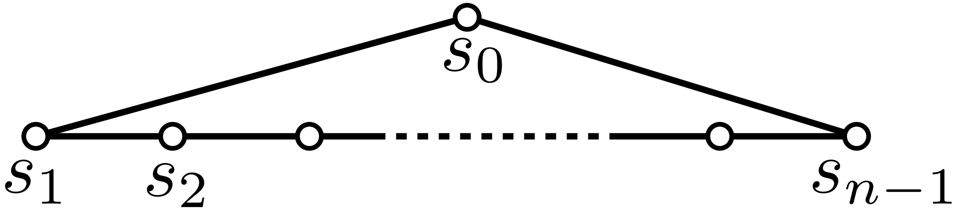
$$\begin{aligned} f_6(t) = {} & 1 + 6t + 21t^2 + 50t^3 + 90t^4 + 126t^5 + 146t^6 \\ & + \mathbf{150t^7} + \mathbf{156t^8} + \mathbf{152t^9} + \mathbf{156t^{10}} + \mathbf{150t^{11}} + \mathbf{158t^{12}} \\ & + \mathbf{150t^{13}} + \mathbf{156t^{14}} + \mathbf{152t^{15}} + \mathbf{156t^{16}} + \mathbf{150t^{17}} + \mathbf{158t^{18}} \\ & + \cdots \end{aligned}$$

FC elements in type $\widetilde{A}$

FC heaps satisfy the same local conditions as in finite type A

$\rightarrow$ The heaps must avoid   

Difference: the cyclic shape of the Coxeter diagram



$\rightarrow$ The labels above must be taken with index modulo n; the heaps must be thought of as "drawn on a cylinder"

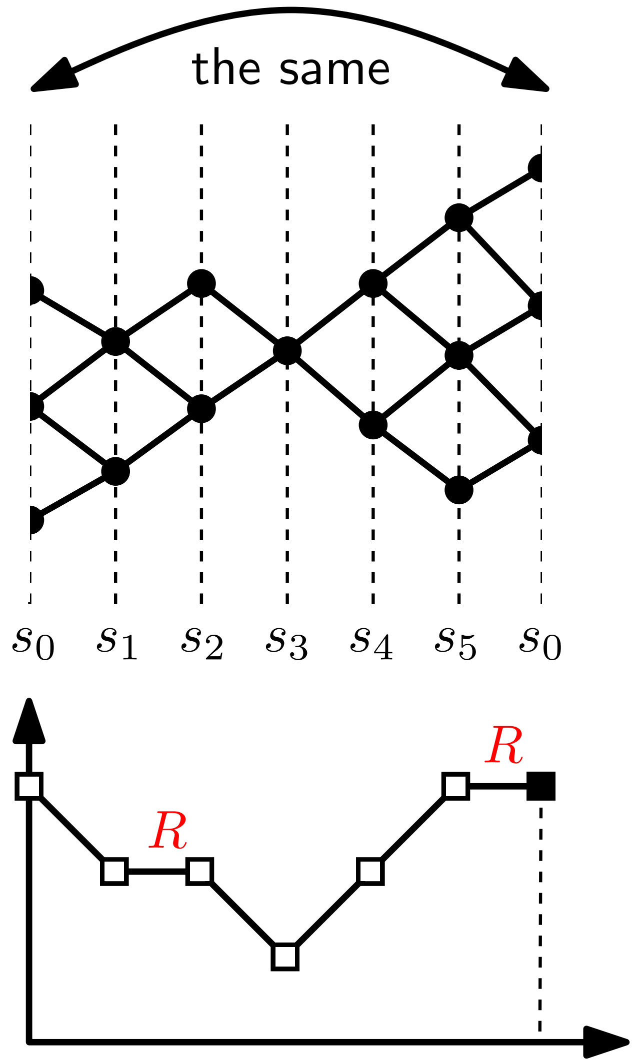
Heaps become Motzkin-type paths

We can form a path as before from a heap: because of the cyclic diagram, our paths will start and end at the same height

Example:

Heap

Path



The area does not take into account the final height

Bijection

Starting from a FC element in $\widetilde{A}_{n-1}$, we thus obtain a path in $\mathcal{O}_n^*$, the set of length n paths with starting and ending point at the same height

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Theorem[BJN (2012)] This is a bijection between

1. FC elements in $\widetilde{A}_{n-1}$ and
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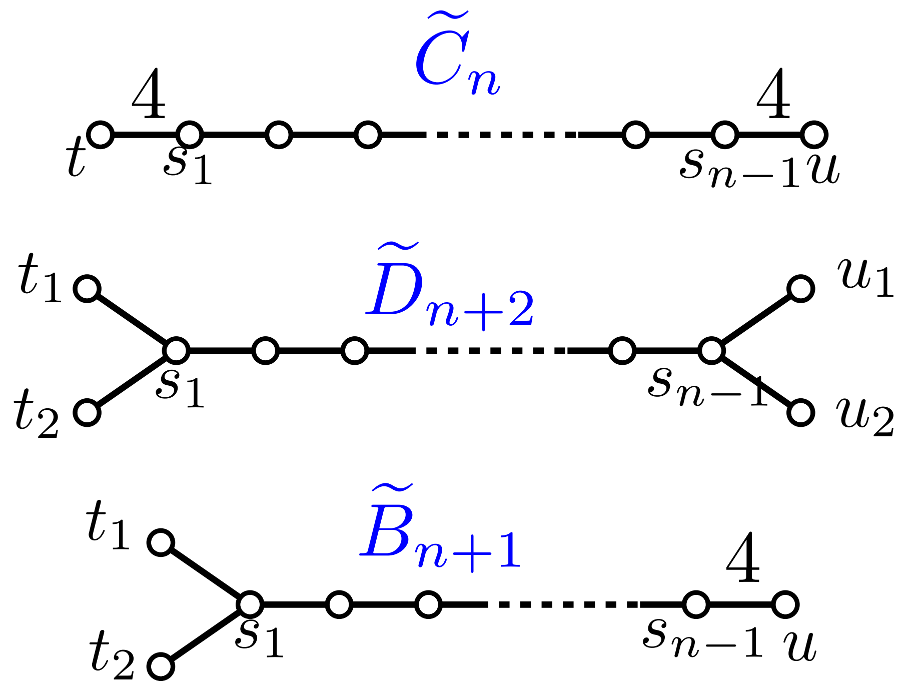
Corollary $\widetilde{A}_{n-1}^{FC}(t) = \mathcal{O}_n^*(t) - \dfrac{2t^n}{1-t^n} = t^n \dfrac{\check{\mathcal{O}}_n(t) - 2}{1-t^n} + \check{\mathcal{O}}_n^*(t)$

Other affine types

There are 3 classical types



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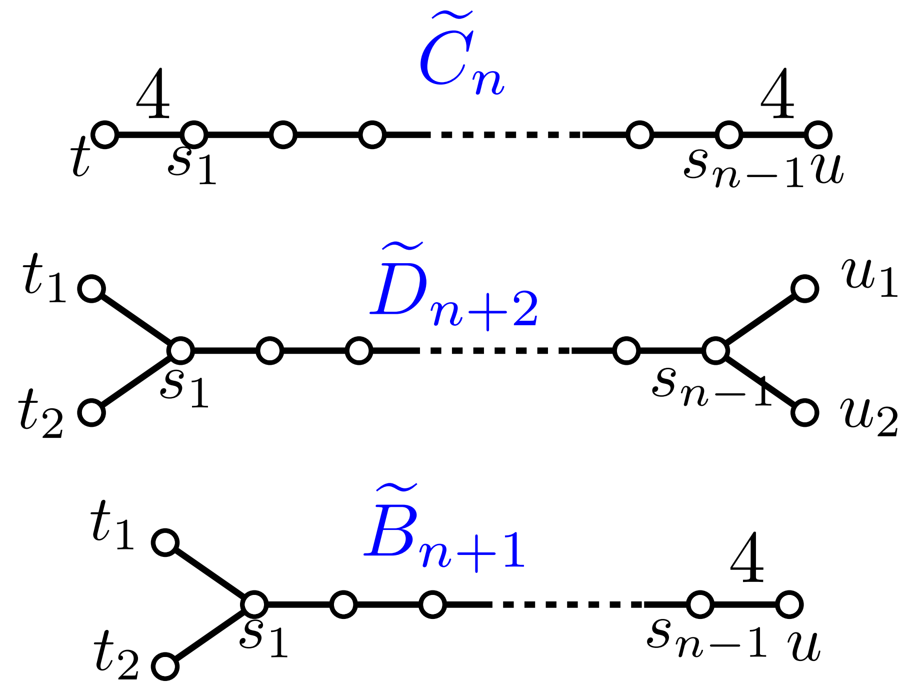

Theorem [BJN (2012)]: for each irreducible affine group W, the sequence of coefficients of $W^{FC}(t)$ is ultimately periodic, with period dividing the following values:

Affine Type	$\widetilde{A}_{n-1}$	$\widetilde{C}_n$	$\widetilde{B}_{n+1}$	$\widetilde{D}_{n+2}$	$\widetilde{E}_6$	$\widetilde{E}_7$	$\widetilde{G}_2$	$\widetilde{F}_4, \widetilde{E}_8$
Periodicity	n	$n+1$	$(n+1)(2n+1)$	$n+1$	4	9	5	1

Moreover, we have the same kind of table for FC involutions

What are the exact periods?

$$\widetilde{A}^{FC}_{n-1}(t) \equiv \frac{1}{1-t^n} \sum_{k=1}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_t^2 \equiv \frac{1}{1-t^n} \left(\check{O}_n(t) - 2\right)$$

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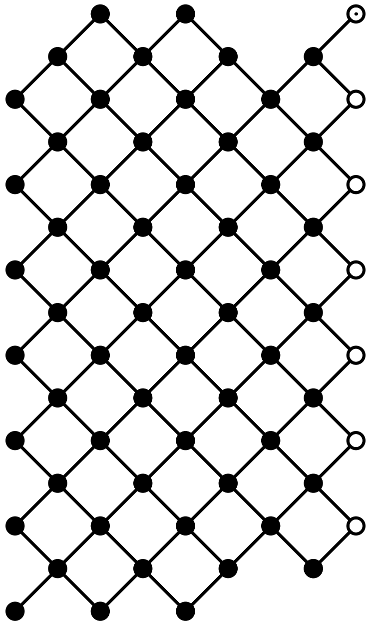

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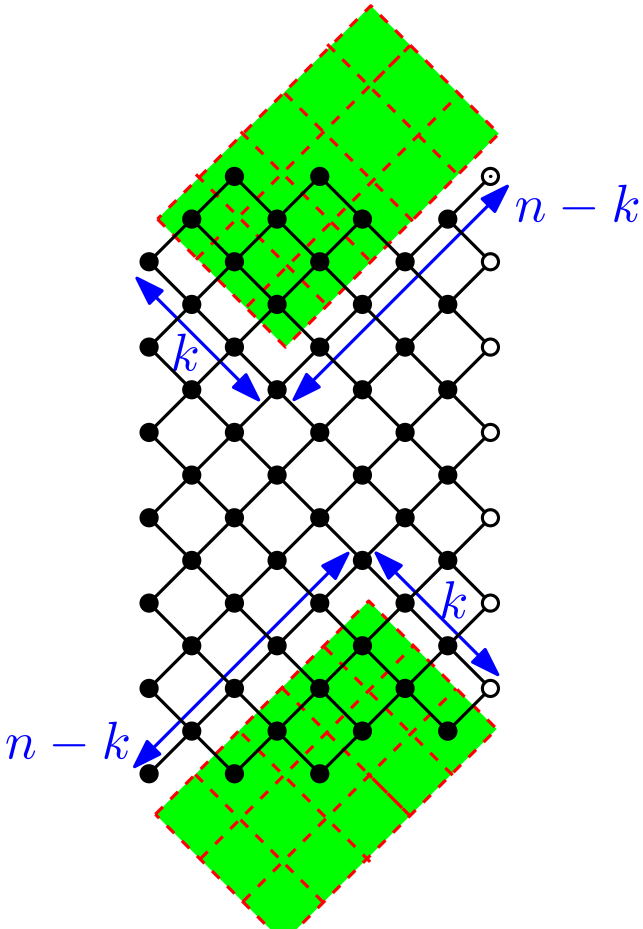

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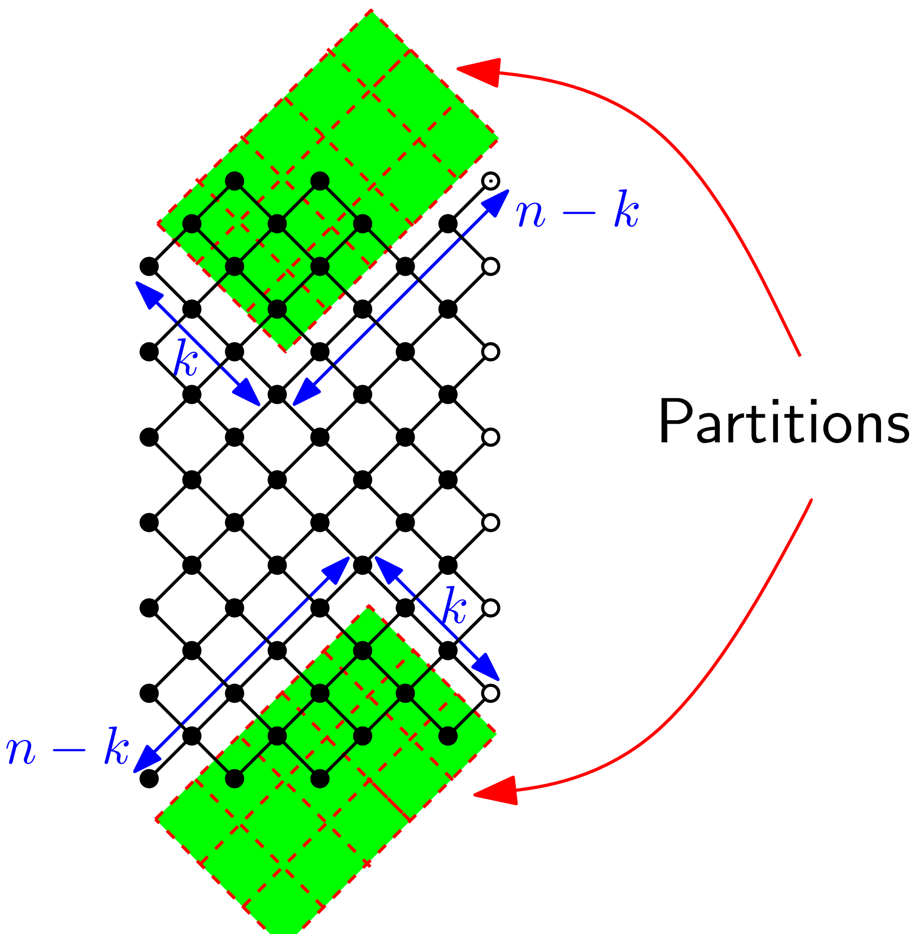

$$\widetilde{A}^{FC}_{n-1}(t) \equiv \frac{1}{1-t^n}\sum_{k=1}^{n-1}\begin{bmatrix} n \\ k \end{bmatrix}_t^2 t^{k^2}$$

$$\equiv \frac{1}{1-t^n}\left(\begin{bmatrix} 2n \\ n \end{bmatrix}_t - 2\right)$$

Exact period for type $\widetilde{A}_{n-1}$

Lemma 1: $\dfrac{P(t)}{1-t^n} \equiv \dfrac{1}{n}\displaystyle\sum_{j=0}^{n-1} \dfrac{P(\xi_n^{-j})}{1-t\xi_n^j}$ with $\xi_n := \mathrm{e}^{\frac{2i\pi}{n}}$, $P \in \mathbb{C}[t]$

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Therefore $\widetilde{A}_{n-1}^{FC}(t) \equiv \dfrac{1}{n}\displaystyle\sum_{j=0}^{n-1} \dfrac{\binom{2d}{d}-2}{1-t\xi_n^j}$

The minimal period is the least common multiple of all the integers in $\{\text{order}(\xi_n^j) \mid d > 1\}$: it is the least common multiple of the numbers n/d for $j = 0, 1, \ldots, n-1$ with $d > 1$

Exact periods in classical affine types

Theorem[JN (2013)]: in type $\widetilde{A}_{n-1}$, the minimal period is $p^{\alpha-1}$ if $n = p^{\alpha}$, and n otherwise.
In type $\widetilde{C}_n$ (*resp.* $\widetilde{B}_{n+1}$, *resp.* $\widetilde{D}_{n+2}$), the minimal period is given by $2m+1$ (*resp.* $(2m+1)(2n+1)$, *resp.* $n+1$) where $2m+1$ is the largest odd divisor of $n+1$

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Moreover, using Ramanujan sums, the number of elements of large enough length ℓ in $\widetilde{A}_{n-1}^{FC}$ is equal to

$$\frac{\binom{2n}{n}}{n}(1 + \mathrm{O}(n\,2^{-n})), \quad n \to +\infty$$

We deduce that for n and ℓ large enough, it is close to the mean value over a period $\dfrac{\binom{2n}{n} - 2}{n}$

A cyclic sieving phenomenon

Let X be a finite set endowed with the action of a finite cyclic group $C = \langle c \rangle$ of order n.
Set $P \in \mathbb{N}[q]$ and $X^g := \{$elements of X fixed by $g \in C\}$

Definition[Reiner–Stanton–White (2004)]: (X, C, P) exhibits the cyclic sieving phenomenon if $P(\xi_n^j) = |X^{c^j}|$ for any $j \in \{0, \ldots, n-1\}$

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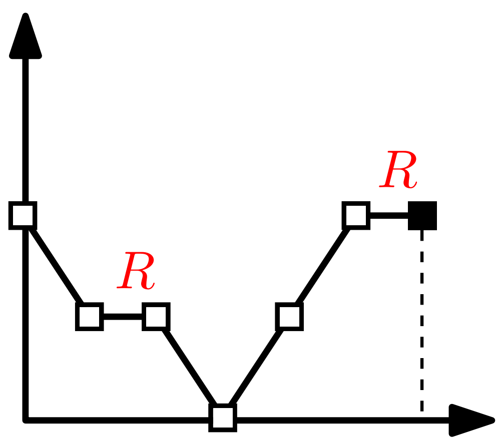

Choose the set $X := \check{\mathcal{O}}_n$
Cyclic action: generated by the rotation $\mathbf{r}$ of paths one unit to the right

Polynomial: $\check{O}_n(t)$

Proposition[JN (2013)]: The triple $(\check{\mathcal{O}}_n, \langle \mathbf{r} \rangle, \check{O}_n(t))$ exhibits the cyclic sieving phenomenon.