Une histoire de la conjecture de Gessel (en combinatoire énumérative)

Kilian Raschel

Travail commun avec A. Bostan & I. Kurkova

Séminaire de combinatoire du LIX
7 mai 2014

Since when do we count paths?

▷ Chevalier de Méré's problem

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Extrait de la lettre de Pascal à Fermat du 29 juillet 1654

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- ## Bertrand's Ballot problem

In combinatorics, Bertrand's ballot problem (solved in 1878) is the question:

In an election where candidate A receives p votes and candidate B receives q votes with $p > q$, what is the probability that A will be strictly ahead of B throughout the count?

The answer is $\dfrac{p-q}{p+q}$.

50's, 60's and 70's

- *Fun with lattice paths*, Howard Grossman (1950)

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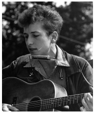

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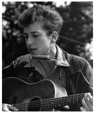

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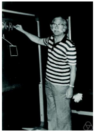

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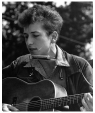

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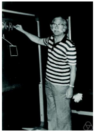

- *An Introduction to Probability Theory and Its Applications*, William Feller (1950–1966)

Context: enumeration of lattice walks

▷ *Nearest-neighbor walks* in the plane $\mathbb{Z}^2$; admissible steps

$$\mathfrak{S} \subseteq \{\swarrow, \leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow\}.$$

▷ $\mathfrak{S}$-walks: walks in $\mathbb{Z}^2$ starting at $(0,0)$ and using steps in $\mathfrak{S}$.

▷ $f_{\mathfrak{S}}(n;i,j)$: number of $\mathfrak{S}$-walks ending at (i,j) and consisting of exactly n steps, possibly *confined to* some subdomain of $\mathbb{Z}^2$ (for us: *the quarter plane*).

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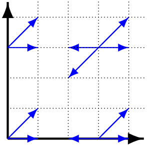
$f_{\mathfrak{S}}(0; 0, 0) = 1$
$f_{\mathfrak{S}}(2n + 1; 0, 0) = 0$
$f_{\mathfrak{S}}(2; 0, 0) = 2$
$f_{\mathfrak{S}}(4; 0, 0) = 11$



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$$Q_{\mathfrak{S}}(t; x, y) = \sum_{n=0}^{\infty} \left(\sum_{i,j=0}^{\infty} f_{\mathfrak{S}}(n; i, j) x^i y^j \right) t^n \in \mathbb{Q}[x, y][[t]].$$

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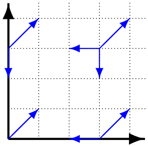
Structure? (algebraic/holonomic) *Explicit form*? *Asymptotics*?

$Q_{\mathfrak{S}}(t;0,0) \leadsto$ counts $\mathfrak{S}$-walks returning to the origin (excursions);
$Q_{\mathfrak{S}}(t;1,1) \leadsto$ counts $\mathfrak{S}$-walks with prescribed length.

First example: Kreweras walk



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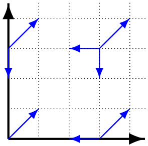

Theorem [Kreweras (1965); 100 pages combinatorial proof!]

$$Q_{\mathfrak{S}}(t;0,0) = {}_3F_2\left(\begin{matrix}1/3 & 2/3 & 1\\ & 3/2 & 2\end{matrix}\,\middle|\, 27\,t^3\right)$$
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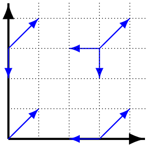

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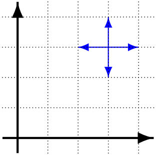
Theorem [Gessel (1986), Bousquet-Mélou (2005), ...]
The trivariate generating function $Q_{\mathfrak{S}}(t;x,y)$ is algebraic.

Second example: the simple random walk (1/2)

SRW in the plane



Pólya (1921):

- Formula for the excursions
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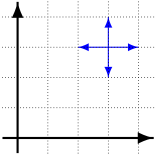

Pólya (1921):

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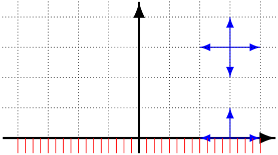
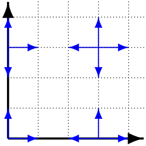
SRW in the half-plane and in the quarter-plane

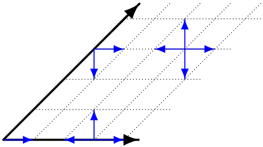


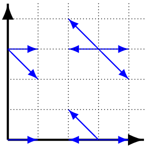


▷ Algebraic (Bousquet-Mélou and Petkovšek, 2003) and holonomic generating functions (Bousquet-Mélou and Mishna, 2010), respectively

Second example: the simple random walk (2/2)

SRW in the angle 45°





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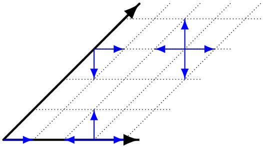

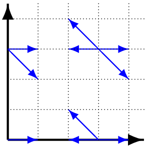

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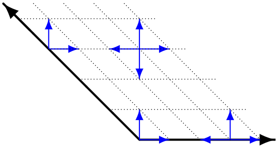
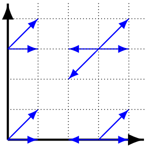
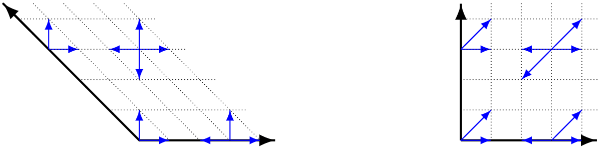
What about the SRW in the angle 135°?





Statement of Gessel's conjecture



Conjecture [Gessel (2001)]

$$Q_{\mathfrak{G}}(t;0,0) = {}_3F_2\left(\begin{matrix} 5/6 & 1/2 & 1 \\ & 5/3 & 2 \end{matrix} \middle| \, 16t^2\right)$$

$$= \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} (4t)^{2n}.$$

Opinion in the combinatorics community
The generating function $Q_{\mathfrak{G}}(t;0,0)$ (thus a fortiori $Q_{\mathfrak{G}}(t;x,y)$) is not algebraic.

The conjecture is shared (and made public)

Gessel sends the conjecture to Bousquet-Mélou (2001)

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Denoting the kernel

$$K_{\mathfrak{S}}(t; x, y) = xyt \left[\sum_{(i,j)\in\mathfrak{S}} x^i y^j - 1/t \right],$$

one has the functional equation

$$K_{\mathfrak{S}} Q_{\mathfrak{S}}(t; x, y) = K_{\mathfrak{S}} Q_{\mathfrak{S}}(t; x, 0) + K_{\mathfrak{S}} Q_{\mathfrak{S}}(t; 0, y) - K_{\mathfrak{S}} Q_{\mathfrak{S}}(t; 0, 0) - xy.$$

Walks with small steps in the quarter plane (1/2)

Systematic study of walks with small steps in the quarter plane (Bousquet-Mélou and Mishna, 2003–2010)

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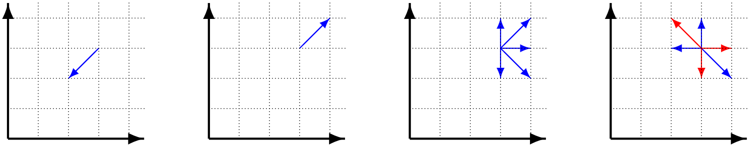
Some of these 2^8 models are:



trivial, simple, half-plane, symmetrical.

Finally, it remains 79 *problems!*

Classifying lattice walks restricted to the quarter plane (Mishna, 2003–2009)

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The polynomial

$$\sum_{(i,j)\in\mathfrak{S}} x^i y^j = \sum_{i=-1}^{1} B_i(y)x^i = \sum_{j=-1}^{1} A_j(x)y^j$$

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is left invariant under

$$\psi(x,y) = \left(x, \frac{A_{-1}(x)}{A_{+1}(x)}\frac{1}{y}\right), \qquad \phi(x,y) = \left(\frac{B_{-1}(y)}{B_{+1}(y)}\frac{1}{x}, y\right),$$

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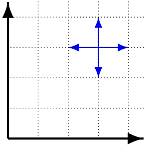
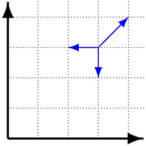
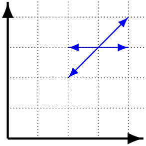
Classification of the $79 = 22 + 1 + 56$ models
(Bousquet-Mélou and Mishna, Bostan and Kauers, 2010)



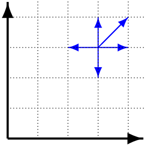
Order 4 *(16)*



Order 6 *(5)*



Order 8 *(2)*



Order ∞ *(56)*

January 28, 2008

Two talks in seminar "Algorithms" (Philippe Flajolet's team) at Inria Paris-Rocquencourt

▷ *Establishing Non-D-finiteness of Combinatorial Generating Functions*, Marni Mishna

▷ *Integration of Algebraic Functions using Gröbner Bases*, Manuel Kauers

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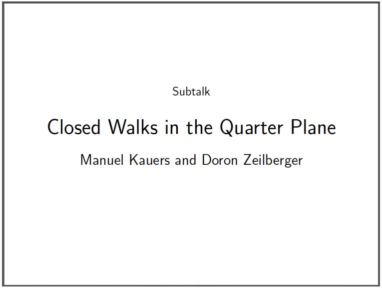

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- ▷ No proof of the conjecture

Summer 2008: proof of Gessel's conjecture

Proof of Ira Gessel's lattice path conjecture
(Kauers, Koutschan and Zeilberger)

Proof of Ira Gessel's lattice path conjecture

Manuel Kauers[a], Christoph Koutschan[a], and Doron Zeilberger[b,1]

[a]Research Institute for Symbolic Computation, Johannes Kepler University, 4040 Linz, Austria; and [b]Mathematics Department, Rutgers University, Piscataway, NJ 08854-0819

Communicated by Richard P. Stanley, Massachusetts Institute of Technology, Cambridge, MA, February 13, 2009 (received for review June 25, 2008)

Summer 2008: proof of Gessel's conjecture

Proof of Ira Gessel's lattice path conjecture
(Kauers, Koutschan and Zeilberger)

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To that end, D.Z. offers a prize of one hundred (100) US dollars for a short, self-contained, human-generated (and computer-free) proof of Gessel's conjecture, not to exceed five standard pages typed in standard font. The longer that prize would remain unclaimed, the more (empirical) evidence we would have that a proof of Gessel's conjecture is indeed beyond the scope of humankind.

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The complete generating function for Gessel walks is algebraic
(Bostan and Kauers)

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AMERICAN MATHEMATICAL SOCIETY
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THE COMPLETE GENERATING FUNCTION FOR GESSEL WALKS IS ALGEBRAIC

ALIN BOSTAN AND MANUEL KAUERS,
WITH AN APPENDIX BY MARK VAN HOEIJ

(Communicated by Jim Haglund)

History of the algebraicity

(waste of time because of the rumor of the non-algebraicity)

▷ January 28: talk of Manuel Kauers

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- February 22: guessed diff. eq. for $Q_{\mathfrak{S}}(t;x,0)$ and $Q_{\mathfrak{S}}(t;0,y)$.
  - The diff. eq. are huge: degree 11 in $\frac{\mathrm{d}}{\mathrm{d}t}$, 96 in t, 78 in x, and integers of 61 decimal digits.
  - In principle, sufficient to prove that $Q_{\mathfrak{S}}(t;x,y)$ is holonomic (closure properties of holonomic functions), but problem of determining 1.5 billon integer coefficients. . .

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▷ August 26: They find vanishing polynomials for the generating functions, and make their result public.

▷ People were surprised. (Why was the algebraicity of $Q_{\mathfrak{G}}(t; 0, 0)$ not discovered earlier?)

Complex analysis methods

Fayolle, Iasnogorodski and Malyshev (70's)

An analytical method in the theory of two-dimensional positive random walks, Malyshev (1972)

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Kurkova and R. (2009)

Explicit expression for the generating function counting Gessel's walks, Kurkova and R. (2009)

▷ Explicit expression

$$Q_{\mathfrak{G}}(t;x,0) = \int_{C_t} \frac{f(t;u)}{u-x} \mathrm{d}u.$$

Other methods

Petkovšek and Wilf (2008)

On a conjecture of Gessel

▷ Expression of Gessel's numbers in terms of determinants

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Fayolle and R. (2009)

On the holonomy or algebraicity of generating functions counting lattice walks in the quarter plane

▷ Algebraic methods (Galois theory)

Summary of our approach

Bostan, Kurkova and R. (2013)

A human[1] proof of Gessel's lattice path conjecture

[1] Of the XIXth century.

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▷ Finding $Q_{\mathfrak{S}}(t; x, y)$ amounts to finding $Q_{\mathfrak{S}}(t; x, 0)$ and $Q_{\mathfrak{S}}(t; 0, y)$.

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▷ With the theory of elliptic functions, we obtain an expression of $Q_{\mathfrak{S}}(t;x,0)$ in terms of ζ-Weierstrass functions.

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▷ A rewriting of the functional equation gives that there is only a finite number of branches and allows to compute the poles of the branches.

▷ With the theory of elliptic functions, we obtain an expression of $Q_{\mathfrak{S}}(t;x,0)$ in terms of ζ-Weierstrass functions.

▷ (Relatively) standard identities from the theory of special functions give $Q_{\mathfrak{S}}(t;0;0)$ as a sum of hypergeometric functions.

[1] Of the XIXth century.

Main result (in terms of ζ-Weierstrass functions)

Bostan, Kurkova and R. (2013)

$$
\begin{aligned}
2z^2Q(0,0) = &+1\zeta\left(\frac{1}{3}\omega_3\right)-3\zeta\left(\frac{2}{3}\omega_3\right)+2\zeta\left(\frac{3}{3}\omega_3\right)\\
&+3\zeta\left(\frac{4}{3}\omega_3\right)-5\zeta\left(\frac{5}{3}\omega_3\right)+2\zeta\left(\frac{6}{3}\omega_3\right),
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with

- $\zeta(\omega) = \frac{1}{\omega} + \sum_{(n_1,n_3)\in\mathbb{Z}^2\setminus(0,0)} \left(\frac{1}{\omega-(n_1\omega_1+4n_3\omega_3)} + \frac{1}{n_1\omega_1+4n_3\omega_3} + \frac{\omega}{(n_1\omega_1+4n_3\omega_3)^2}\right),$

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- $\omega_1 = i\int_{x_1}^{x_2} \frac{\mathrm{d}x}{\sqrt{-d(x)}} \in i\mathbb{R},$
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- $\omega_3 = \frac{3}{4}\int_{x_2}^{1/x_2} \frac{\mathrm{d}x}{\sqrt{d(x)}} \in \mathbb{R},$
- $d(x) = (zx^2 - x + z)^2 - 4z^2x^2,$
- $x_1 = \frac{1+2z-\sqrt{1+4z}}{2z}$, $x_2 = \frac{1-2z-\sqrt{1-4z}}{2z}$.

Other Gessel's conjectures

Dear Colleagues,

Attached is a conjecture about walks in the quarter plane with east, west, northeast, and southwest steps (sometimes called "Gessel walks") that might be of interest. Perhaps it can be derived from the known generating function for these walks. My hope is that it would suggest a simpler approach, though I really don't have any ideas on how to prove it.

Best regards,
Ira Gessel

Other Gessel's conjectures

Objet a conjecture on walks in the quarter plane

De Ira Gessel

À Alin.Bostan@inria.fr, manuel@kauers.de, Koutschan@risc.jku.at, Doron Zeilberger, kourkova@ccr.jussieu.fr, Kilian.Raschel@lmpt.univ-tours.fr, Raimundas Vidūnas, Mireille Bousquet-Mélou, Marni Mishna

Date 2013-03-09 22:11

walk-conjecture.pdf

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A CONJECTURE ON PATHS IN THE QUARTER PLANE 2

I conjecture that there is a similar formula for $q(0, j, 2m)$ for arbitrary j. Let

$$f_j(x) = (-1)^j(2j+1)x^j + 2x^{j+1}\sum_{m=0}^{\infty} q(0,j,2m)x^m.$$

Then we find empirically that

$$f_j\left(x\frac{(1+x)^3}{(1+4x)^3}\right) = (-x)^j \frac{p_j(x)}{(1+4x)^{3j+3/2}}, \qquad (4)$$

where $p_j(x)$ is a polynomial of degree $3j+2$ with positive coefficients. The first few values are

$$p_0(x) = 1 + 8x + 4x^2$$
$$p_1(x) = 3 + 27x + 79x^2 + 101x^3 + 52x^4 + 4x^5$$
$$p_2(x) = 5 + 60x + 285x^2 + 714x^3 + 1035x^4 + 894x^5 + 461x^6 + 128x^7 + 4x^8$$

Open questions

- Combinatorial proof of Gessel's conjecture
Towards a combinatorial understanding of lattice path asymptotics, by Johnson, Mishna and Yeats (2013)

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On lattice walks confined to the positive octant, by Bousquet-Mélou, Bostan, Kauers and Melczer

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▷ Link between the finiteness of the group and the nature of the generating function

Merci pour votre attention !

Important classes of univariate power series

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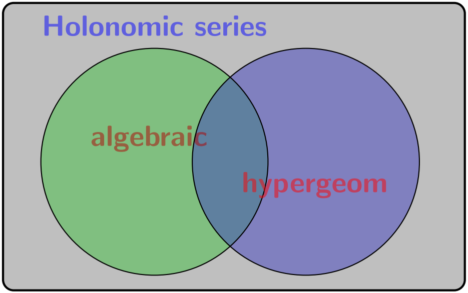
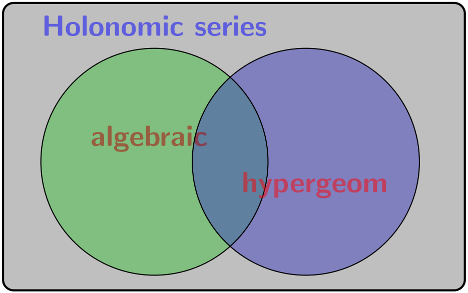


Important classes of univariate power series



Holonomic: $S(t) \in \mathbb{Q}[[t]]$ satisfying a linear differential equation with polynomial coefficients $c_r(t)S^{(r)}(t) + \cdots + c_0(t)S(t) = 0$.

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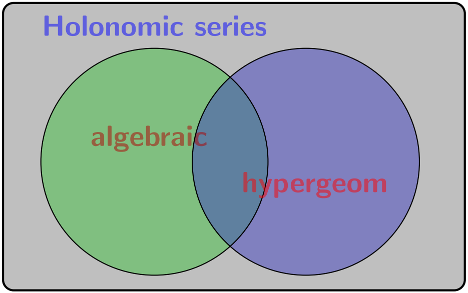

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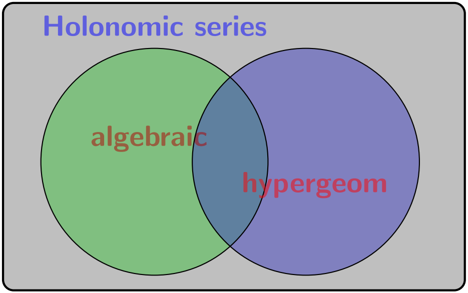

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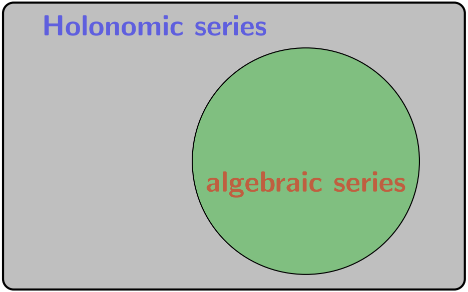
Hypergeometric: $S(t) = \sum_n s_n t^n$ such that $\frac{s_{n+1}}{s_n} \in \mathbb{Q}(n)$. E.g.

$$ {}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix} \,\middle|\, t\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{t^n}{n!}, \quad (a)_n = a(a+1)\cdots(a+n-1). $$

Important classes of multivariate power series



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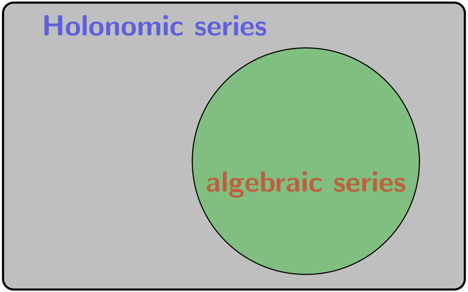

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$S \in \mathbb{Q}[[x, y, t]]$ is *algebraic* if it is the root of a $P \in \mathbb{Q}[x, y, t, T]$.

Main methods for proving holonomy and non-holonomy

Main methods

(1) for proving non-holonomy

  (1a) Infinite number of singularities, or lacunary

  (1b) Asymptotics

(2) for proving holonomy

  (2a) Diagonals, or positive parts, of rational functions

  (2b) Guess'n'Prove

▷ All methods are algorithmic.

“Guess and Prove” approach

Methodology for proving algebraicity

Experimental mathematics—Guess'n'Prove—approach:

(S1) high order expansion of the generating series $F_{\mathfrak{S}}(t; x, y)$;

(S2) guessing candidates for minimal polynomials of $F_{\mathfrak{S}}(t; x, 0)$ and $F_{\mathfrak{S}}(t; 0, y)$, based on Hermite-Padé approximation;

(S3) rigorous certification of the minimal polynomials, based on (exact) polynomial computations.