Central limit theorem for random Young diagrams with respect to Jack measure
(joint work with Valentin Féray)

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A partition $\lambda$ is a finite non-increasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$. It can be represented by a Young diagram $\lambda$.

A generalized Young diagram is a broken line going from a point $(0, y)$ on the $y$-axis to a point $(x, 0)$ on the $x$-axis such that every piece is either a horizontal segment from left to right or a vertical segment from top to bottom.
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Young diagrams

**Definition**

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![Young diagram example](image)

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Simple operations on generalized Young diagrams

- **Dilation:**
  \[ T_{s,t}(\lambda) - \text{generalized Young diagram obtained by stretching } \lambda \text{ horizontally by a factor } s \text{ and vertically by a factor } t, \text{ where } s, t \in \mathbb{R}_+. \]

\[ \lambda \mapsto T_{3,2}(\lambda) \]

**Examples**

**Special cases:**

- **Blowing of Young diagram:** \( D_s(\lambda) := T_{s,s}(\lambda), \text{ for } s \in \mathbb{R}_+; \)
- **\( \alpha \)-anisotropic Young diagram:** \( \lambda^{(\alpha)} := T_{\sqrt{\alpha},\sqrt{\alpha}^{-1}}(\lambda) \) for \( \alpha \in \mathbb{R}_+; \)
Simple operations on generalized Young diagrams

- **Dilation:**
  - \( T_{s,t}(\lambda) \) - generalized Young diagram obtained by stretching \( \lambda \) horizontally by a factor \( s \) and vertically by a factor \( t \), where \( s, t \in \mathbb{R}_+ \).

**Examples**

- **Blowing of Young diagram:** \( D_s(\lambda) := T_{s,s}(\lambda), \text{ for } s \in \mathbb{R}_+ \);
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Two conventions of drawing Young diagrams

Conventions of drawing Young diagrams:

- **French convention:**

- **Russian convention:**

**Definition**

A **profile** of a generalized Young diagram $\lambda$ is a function $\omega(\lambda) : \mathbb{R} \to \mathbb{R}^+$ such that its graph is a profile of $\lambda$ drawn in Russian convention.
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- **French convention:**
  - Diagram showing the French convention with steps and coordinates.

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Definition

A continuous Young diagram is a function $\omega : \mathbb{R} \rightarrow \mathbb{R}_+$ such that

- $\omega(x) - |x|$ has compact support;
- $|\omega(x_1) - \omega(x_2)| \leq |x_1 - x_2|$ for any $x_1, x_2 \in \mathbb{R}$. 
Problem

Definition

A **continuous Young diagram** is a function \( \omega : \mathbb{R} \rightarrow \mathbb{R}_+ \) such that
- \( \omega(x) - |x| \) has compact support;
- \( |\omega(x_1) - \omega(x_2)| \leq |x_1 - x_2| \) for any \( x_1, x_2 \in \mathbb{R} \).

Problem

- \( \mathbb{Y}_n \) - the set of Young diagrams of size \( n \)
  \((|\lambda| := \lambda_1 + \lambda_2 + \cdots = n)\);
- \( \mathbb{P}_n \) - probability measure defined on the set \( \mathbb{Y}_n \).

Let \( \lambda(n) \) be a sequence of Young diagrams of size \( n \). Does exist some continuous Young diagram \( \omega \) such that, as \( n \rightarrow \infty \), in probability

\[
\| \omega(D_{\sqrt{n}^{-1}}(\lambda(n))) - \omega \| \rightarrow 0?
\]
Vershik-Kerov, Logan-Shepp limit shape

**Theorem (Vershik-Kerov, Logan-Shepp ’77)**

Let $\lambda(n)$ be a random Young diagram of size $n$ distributed with \textit{Plancherel measure} $\mathbb{P}^{(1)}_n$. Then, in probability, as $n \to \infty$

$$\left\| \omega \left( D_{1/\sqrt{n}}(\lambda(n)) \right) - \Omega \right\| \to 0.$$
Plancherel measure

\[ a(\bullet) = \text{number of boxes to the right of the given box} \]
Plancherel measure

\[ a(\bullet) = 4 \]
Plancherel measure

\[ \ell(\bullet) = \text{number of boxes above the given box} \]
Plancherel measure

\[ \ell(\bullet) = 3 \]
Plancherel measure

\[ P_n^{(1)}(\lambda) = \frac{\dim(\lambda)^2}{n!}, \]

where (hook formula:)

\[ \dim(\lambda) = \frac{n!}{\prod_{\square \in \lambda} (a(\square) + \ell(\square) + 1)}. \]
Plancherel measure

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\[ P_n^{(1)}(\lambda) = \frac{n!}{\prod_{\Box \in \lambda} (a(\Box) + \ell(\Box) + 1)^2} . \]
Jack measure

- **Jack measure** is a probability measure on the set $\mathbb{Y}_n$ defined by
  \[
  \mathbb{P}_n^{(\alpha)}(\lambda) := \frac{\alpha^n n!}{\prod_{\square \in \lambda} (\alpha a(\square) + \ell(\square) + 1)(\alpha a(\square)) + \ell(\square) + \alpha),
  \]
  where $\alpha \in \mathbb{R}_+$;
- for $\alpha = 1$ Jack measure $\equiv$ Plancherel measure.

**Theorem (D., Féray)**

Let $\lambda(n)$ be a random Young diagram of size $n$ distributed with Jack measure $\mathbb{P}_n^{(\alpha)}$. Then, in probability, as $n \rightarrow \infty$

\[
\left\| \omega(D_1/\sqrt{n}(\lambda^{(\alpha)}(n))) - \Omega \right\| \rightarrow 0.
\]
Central limit theorem

- \( \Delta(\lambda)(x) := \sqrt{n} \omega \left( \frac{D_1/\sqrt{n}(\lambda)}{\sqrt{n}} \right)(x) - \Omega(x) \);
- \( u_k(x) = U_k(x/2) = \sum_{0 \leq j \leq \lfloor k/2 \rfloor} (-1)^j \binom{k-j}{j} x^{k-2j} \);
- \( u_k(2 \cos(\theta)) = \frac{\sin((k+1)\theta)}{\sin(\theta)} \);
- \( u_k(\lambda) = \int_{\mathbb{R}} u_k(x) \Delta(\lambda)(x) \, dx \).

**Theorem (Kerov, 1993)**

Choose a sequence \( (\Xi_k)_{k=2,3,...} \) of independent standard Gaussian random variables and let \( \lambda_{(n)} \) be a random Young diagram of size \( n \) distributed with Plancherel measure. As \( n \to \infty \), we have:

\[
\left( u_k(\lambda_{(n)}) \right)_{k=1,2,...} \overset{d}{\to} \left( \frac{\Xi_{k+1}}{\sqrt{k + 1}} \right)_{k=1,2,...}.
\]
Theorem (D. Féray)

Choose a sequence \((\Xi_k)_{k=2,3,...}\) of independent standard Gaussian random variables and let \(\lambda(n)\) be a random Young diagram of size \(n\) distributed with Jack measure. As \(n \to \infty\), we have:

\[
\left( u^{(\alpha)}_k(\lambda(n)) \right)_{k=1,2,...} \xrightarrow{d} \left( \frac{\Xi_{k+1}}{\sqrt{k+1}} - \frac{\gamma}{k+1} [k \text{ is odd}] \right)_{k=1,2,...},
\]

where \(u^{(\alpha)}_k(\lambda) = \int_{\mathbb{R}} u_k(x) \Delta(\lambda^{(\alpha)})(x) \, dx\), \(\gamma := \sqrt{\alpha} - \sqrt{\alpha^{-1}}\), and we use the usual notation \([\text{condition}]\) for the indicator function of the corresponding condition.
### Symmetric vs shifted-symmetric functions

**Symmetric functions:**

- \( f = (f_1, f_2, \ldots) \) such that \( f_i \in R[x_1, \ldots, x_i] \);
- \( f_{i+1}(x_1, \ldots, x_i, 0) = f_i(x_1, \ldots, x_i) \);
- \( f_i(x_1, \ldots, x_i) \) is symmetric in \( x_1, \ldots, x_i \);
- \( \left( J_{\mu}^{(\alpha)} \right)_\mu \) - linear basis of Jack symmetric functions

**Shifted symmetric functions:**

- \( f = (f_1, f_2, \ldots) \) such that \( f_i \in R[x_1, \ldots, x_i] \);
- \( f_{i+1}(x_1, \ldots, x_i, 0) = f_i(x_1, \ldots, x_i) \);
- \( f_i(x_1 - 1/\alpha, x_2 - 2/\alpha, \ldots, x_i - i/\alpha) \) is symmetric in \( x_1, \ldots, x_i \);
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### Symmetric vs shifted-symmetric functions

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Reduction for graded algebras

- Proving some properties of the elements \((u_k)_k\), which form a basis of the algebra \(A\) - HARD;
- Proving same properties of the elements \((M_k)_k\), which form a basis of the algebra \(A\) - EASY;
- Define gradation on algebra \(A\) such that
  \[ u_k = M_k + \text{terms of lower degree}; \]
- Deducing required properties of the elements \((u_k)_k\).
Reduction for graded algebras

Example

- $\Lambda_\star^{(\alpha)} \subset \left( \Lambda_\star^{(\alpha)} \right)^{\text{ext}}$ - localisation over the $(\sqrt{\text{Ch}^{(\alpha)}_{(1)}})$;

- $\text{Ch}_{(1)}^{m/2} \text{Ch}_{\mu}^{(\alpha)} := \text{Ch}_{(1)}^{m/2} \prod_{i=1}^{\ell} \text{Ch}_{(\mu_i)}^{(\alpha)}$ - linear basis of $\left( \Lambda_\star^{(\alpha)} \right)^{\text{ext}}$, where $m_1(\mu) = 0$, $m \in \mathbb{Z}$;

- $\deg \left( \text{Ch}_{(1)}^{m/2} \text{Ch}_{\mu}^{(\alpha)} \right) = m + |\mu|$;

- $\left( \Lambda_\star^{(\alpha)} \right)^{\text{ext}} \ni u_k^{(\alpha)} = \frac{\text{Ch}^{(\alpha)}_{(k+1)}}{(k+1) \text{Ch}_{(1)}^{(k+1)/2}} - \frac{\gamma}{k+1} [k \text{ is odd}]$

+ terms of negative degree;
Theorem (D., Féray)

Choose a sequence \((\Xi_k)_{k=2,3,...}\) of independent standard Gaussian random variables. As \(n \to \infty\), we have:

\[
\left( \frac{\text{Ch}_{(k)}^{(\alpha)}(\lambda(n))}{\sqrt{kn^k/2}} \right)_{k=2,3,...} \overset{d}{\to} (\Xi_k)_{k=2,3,...},
\]

where the distribution of \(\lambda(n)\) is Jack measure of size \(n\) and where \(\overset{d}{\to}\) means convergence in distribution of the finite-dimensional law.

\[
\mathbb{E}_{p_n^\alpha}(\text{Ch}_{\mu}^{(\alpha)}) = \begin{cases} n(n-1) \cdots (n-k+1) & \text{if } \mu = 1^k \text{ for some } k \leq n, \\ 0 & \text{otherwise.} \end{cases}
\]
Trick with polynomial interpolation

Theorem (D., Féray)

Let
\[ \text{Ch}^{(\alpha)}_{\mu} \text{Ch}^{(\alpha)}_{\nu} = \sum_{\rho} g^{(\alpha)}_{\mu,\nu;\pi} \text{Ch}^{(\alpha)}_{\pi}. \]

Then, structure constants \( g^{(\alpha)}_{\mu,\nu;\pi} \) are polynomials in \( \gamma := \alpha^{1/2} - \alpha^{-1/2} \) of degree less than
\[
\min_{i=1,2,3} \left( n_i(\mu) + n_i(\nu) - n_i(\pi) \right),
\]
with rational coefficients, where \( n_i(\lambda) \) - natural valued function of \( \lambda \).
Trick with polynomial interpolation

Let $\mu, \nu, \pi \in \mathbb{Y}_n$.

$$c_{\mu,\nu;\pi}^{(\alpha)} = \frac{\alpha^{d(\mu,\nu;\pi)/2}}{z_{\tilde{\mu}}z_{\tilde{\nu}}} \sum_{0 \leq i \leq m_1(\pi)} g_{\tilde{\mu},\tilde{\nu};\tilde{\pi}1i}^{(\alpha)} \cdot z_{\tilde{\pi}} \cdot i! \cdot \binom{n - |\tilde{\pi}|}{i},$$

where

- $\tilde{\mu}$ is created from $\mu$ by removing all parts equal to 1,
- $z_{\mu} = \mu_1\mu_2 \cdots m_1(\mu)!m_2(\mu)! \cdots$,
- $m_i(\mu)$ - the number of parts equal to $i$ in $\mu$,
- $d(\mu, \nu; \pi) = |\mu| - \ell(\mu) + |\nu| - \ell(\nu) - (|\pi| - \ell(\pi))$. 

Trick with polynomial interpolation

Let $\mu, \nu, \pi \in \mathbb{Y}_n$.

\[
c^{(\alpha)}_{\mu, \nu; \pi} = \frac{\alpha^{d(\mu, \nu; \pi)}/2}{z_{\overset{\mu}{\bar{\mu}}} z_{\overset{\nu}{\bar{\nu}}}} \sum_{0 \leq i \leq m_1(\pi)} g^{(\alpha)}_{\mu, \nu; \pi, 1^i} \cdot z_{\overset{\pi}{\bar{\pi}}} \cdot i! \cdot \binom{n - |\overset{\pi}{\bar{\pi}}|}{i},
\]

- LHS and RHS of the equation above are polynomials in $n$;
- knowing $c^{(\alpha)}_{\mu, \nu; \pi}$ one can calculate $g^{(\alpha)}_{\mu, \nu; \pi, 1^i}$;
- $c^{(\alpha)}_{\mu, \nu; \pi}$ have combinatorial interpretation for $\alpha = 1, 2, 1/2$. 
Let $\mathbb{C}[S_n] := \{ f : f : S_n \to \mathbb{C} \}$ be a group algebra of the symmetric group. This is algebra with the multiplication defined by:

$$f \cdot g(\sigma) := \sum_{\sigma_1 \sigma_2 = \sigma} f(\sigma_1)g(\sigma_2).$$

Let

$$Z(\mathbb{C}[S_n]) := \{ f \in \mathbb{C}[S_n] : \forall g \in \mathbb{C}[S_n], fg = gf \}$$

be the center of that algebra. It has a basis $(f_\mu)_{|\mu|=n}$, where

$$f_\mu(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ has cycle type } \mu, \\ 0 & \text{otherwise}. \end{cases}$$
\[ \alpha = 1 - \text{Structure constants of the } \mathbb{Z}(\mathbb{C}[\mathfrak{S}_n]) \]

Let

\[ f_{\mu} f_{\nu} = \sum_{|\rho|=n} c_{\mu,\nu;\rho} f_{\rho}. \]

**Lemma**

_The structure constant \( c_{\mu,\nu;\rho} \) is equal to the number of pairs of permutation \((\sigma_1, \sigma_2)\) such that_  
- \( \sigma_1 \) _has cycle type \( \mu \),  
- \( \sigma_2 \) _has cycle type \( \nu \),  
- \( \sigma_1 \sigma_2 = \sigma \), where \( \sigma \) _is a fixed permutation of the cycle-type \( \rho \)._
$\alpha = 1$ - Structure contants of the $\mathbb{Z}(\mathbb{C}[\mathfrak{S}_n])$

One has a following relation:

$$c^{(1)}_{\mu,\nu;\rho} = c_{\mu,\nu;\rho}.$$

**Remark**

*From the previous theorem and a relation above one can deduce a classical result of Farahat and Higman: $c_{\mu 1^n-|\mu|,\nu 1^n-|\nu|;\rho 1^n-|\rho|}$ is a polynomial in $n$.***
\( \alpha = 2 \) - Structure contants of the Hecke algebra of \((\mathfrak{S}_{2n}, H_n)\)

Let \( \mathfrak{S}_{2n} \) acts on the set \( X_n := \{1, \bar{1}, \ldots, n, \bar{n}\} \) by permutations and let
\[
\mathfrak{S}_{2n} > H_n := \{\sigma \in \mathfrak{S}_{2n} : \forall i \in X_n \sigma(\bar{i}) = \sigma(i)\}
\]
be a hyperoctahedral subgroup.

Hecke algebra \( \mathbb{C}[H_n \backslash \mathfrak{S}_{2n} / H_n] < \mathbb{C}[\mathfrak{S}_{2n}] \) of the pair \((\mathfrak{S}_{2n}, H_n)\) is defined by:
\[
\mathbb{C}[H_n \backslash \mathfrak{S}_{2n} / H_n] := \{x \in \mathbb{C}[\mathfrak{S}_{2n}] : hxh' = x \forall h, h' \in H_n\}.
\]

Double-cosets: equivalence classes for the relation \( x \sim hxh' \) (for \( x \in \mathfrak{S}_{2n} \) and \( h, h' \in H_n \))

- naturally indexed by partitions of size \( n \);
- \( F_\mu = \sum_x \text{ of type } \mu \delta_x \) - linear basis of \( \mathbb{C}[H_n \backslash \mathfrak{S}_{2n} / H_n] \).
\(\alpha = 2 \) - Structure constants of the Hecke algebra of \((\mathfrak{S}_{2n}, H_n)\)

Let

\[
F_\mu F_\nu = \sum_{|\rho|=n} h_{\mu,\nu;\rho} F_\rho.
\]

Then

\[
c_{\mu,\nu;\rho}^{(2)} = \frac{h_{\mu,\nu;\rho}}{2^n n!}.
\]

**Remark**

*From the previous theorem and a relation above one can deduce a result of Tout (2013):*

\[
\frac{h_{\mu 1^{n-|\mu|},\nu 1^{n-|\nu|};\pi 1^{n-|\pi|}}}{n! \ 2^n}
\]

is a polynomial in \(n\).
Introduction

**α = 2 - Structure contants of the Hecke algebra of**

\((\mathfrak{S}_{2n}, H_n)\)

- \(\mathcal{F}_S\) - the set of all (perfect) matchings on a set \(S\);
- \(G(F_1, \ldots, F_k)\) - the multigraph with vertex-set \(S\) whose edges are formed by the pairs in \(F_1, \ldots, F_k \in \mathcal{F}_S\);
- The components of \(G(F_1, F_2)\) are even cycles. Let the list of their lengths in weakly decreasing order be \((2\theta_1, 2\theta_2, \ldots) = 2\theta\), and define \(\Lambda\) by \(\Lambda(F_1, F_2) = \theta\);
- \(\mathcal{F}_n\) - the set of all matchings on the set \(\{1, 2 \ldots, 2n\}\).

**Lemma (Goulden, Jackson 1996)**

Let \(F_1, F_2\) be two fixed matchings in \(\mathcal{F}_n\) such that \(\Lambda(F_1, F_2) = \pi\), where \(|\pi| = n\). Then, for any \(\mu, \nu\) of size \(n\) we have

\[
h_{\mu, \nu; \pi} = 2^n n! |\{F_3 \in \mathcal{F}_n : \Lambda(F_1, F_3) = \mu, \Lambda(F_2, F_3) = \nu\}|.
\]
\[ \alpha = 2 - \text{Structure contants of the Hecke algebra of} \ (S_{2n}, H_n) \]

Example

\[ F_1 = \{ \{1, 7\}, \{2, 3\}, \{4, 6\}, \{5, 11\}, \{8, 9\}, \{10, 12\} \} \]
$\alpha = 2$ - Structure contants of the Hecke algebra of $(\mathfrak{S}_{2n}, H_n)$

**Example**

- $F_1 = \{\{1, 7\}, \{2, 3\}, \{4, 6\}, \{5, 11\}, \{8, 9\}, \{10, 12\}\}$
- $F_2 = \{\{1, 9\}, \{2, 4\}, \{3, 7\}, \{5, 12\}, \{6, 8\}, \{10, 11\}\}$
\( \alpha = 2 \) - Structure constants of the Hecke algebra of \((\mathcal{S}_{2n}, H_n)\)

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- \( \Lambda(F_1, F_2) = (, ) \)
Introduction

$\alpha = 2$ - Structure contants of the Hecke algebra of $(S_{2n}, H_n)$

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$F_1 = \{\{1, 7\}, \{2, 3\}, \{4, 6\}, \{5, 11\}, \{8, 9\}, \{10, 12\}\}$

$F_2 = \{\{1, 9\}, \{2, 4\}, \{3, 7\}, \{5, 12\}, \{6, 8\}, \{10, 11\}\}$

$\Lambda(F_1, F_2) = (4,)$
\( \alpha = 2 \) - Structure constants of the Hecke algebra of \((\mathcal{S}_{2n}, H_n)\)

Example

- \( F_1 = \{\{1, 7\}, \{2, 3\}, \{4, 6\}, \{5, 11\}, \{8, 9\}, \{10, 12\}\} \)
- \( F_2 = \{\{1, 9\}, \{2, 4\}, \{3, 7\}, \{5, 12\}, \{6, 8\}, \{10, 11\}\} \)
- \( \Lambda(F_1, F_2) = (4, 2) \)
Main steps in the proof of the main Theorem

- We want to estimate some mixed moments of $\text{Ch}^{(\alpha)}_{(k)}$;
- $\mathbb{E}_{\mathbb{P}_{n}^{(\alpha)}} \left( \text{Ch}^{(\alpha)}_{(k_1)} \cdots \text{Ch}^{(\alpha)}_{(k_l)} \right)$ is a polynomial in $n$;
- the coefficients of the polynomial above are polynomials in $\gamma$;
- the coefficients of the dominant terms are polynomials in $\gamma$ of small degree;
- the only interesting coefficients have degree bounded by 2;
- it is enough to calculate $g^{(\alpha)}_{\mu,\nu;\rho}$ for some special partitions and $\alpha = 1, 2, 1/2$;
- it is possible because of the combinatorial interpretation.
Thank you for your attention.

Any questions?