Factorisations d'un élément de Coxeter dans les groupes de réflexions (complexes)

Guillaume Chapuy (CNRS – Université Paris 7)

travail commun avec
Christian Stump (Hannover)

LIX, Juin 2013.

Part 1: the objects

Minimal factorizations of a full cycle – Cayley's formula

- In the symmetric group $\mathbb{S}_n$ we consider factorizations of the full cycle $(1, 2, \ldots, n)$ into a product of $(n-1)$ transpositions

- **Theorem** [Cayley's formula] The number of such factorizations is

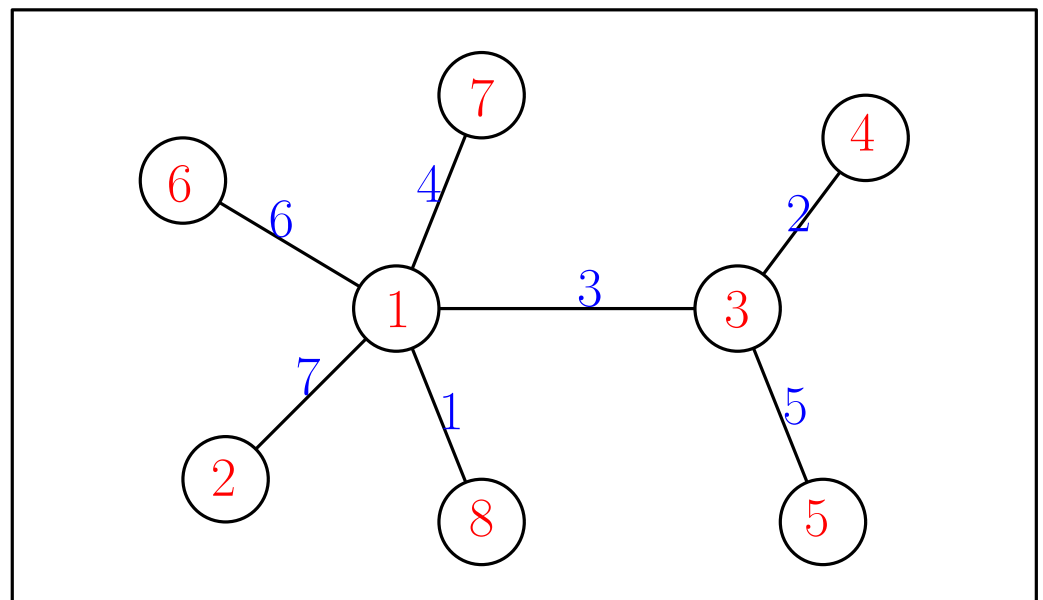
$$\#\big\{\tau_1\tau_2\ldots\tau_{n-1} = (1, 2, \ldots, n)\big\} = n^{n-2}$$

Minimal factorizations of a full cycle – Cayley's formula

- In the symmetric group $\mathbb{S}_n$ we consider factorizations of the full cycle $(1, 2, \ldots, n)$ into a product of $(n-1)$ transpositions

- **Theorem** [Cayley's formula] The number of such factorizations is

$$\#\{\tau_1\tau_2\ldots\tau_{n-1} = (1, 2, \ldots, n)\} = n^{n-2}$$



a Cayley tree with labelled edges :
there are $(n-1)!n^{n-2}$ of them

Minimal factorizations of a full cycle – Cayley's formula

- In the symmetric group $\mathbb{S}_n$ we consider factorizations of the full cycle $(1, 2, \ldots, n)$ into a product of $(n-1)$ transpositions

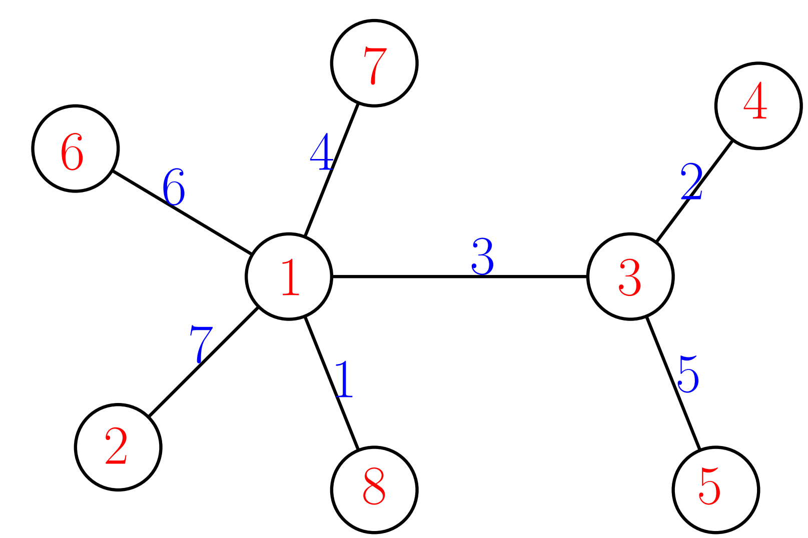
- **Theorem** [Cayley's formula] The number of such factorizations is

$$\#\{\tau_1 \tau_2 \ldots \tau_{n-1} = (1, 2, \ldots, n)\} = n^{n-2}$$

$$\begin{array}{ccccccc} \tau_7 & \tau_6 & \tau_5 & \tau_4 & \tau_3 & \tau_2 & \tau_1 \\ (1\ 2) & (1\ 6) & (3\ 5) & (1\ 7) & (1\ 3) & (3\ 4) & (1\ 8) \end{array}$$

$= (1\ 8\ 5\ 3\ 4\ 7\ 6\ 2)$

a factorization of an arbitrary full cycle



a Cayley tree with labelled edges : there are $(n-1)!n^{n-2}$ of them

Minimal factorizations of a full cycle – Cayley's formula

- In the symmetric group $\mathbb{S}_n$ we consider factorizations of the full cycle $(1, 2, \dots, n)$ into a product of $(n-1)$ transpositions

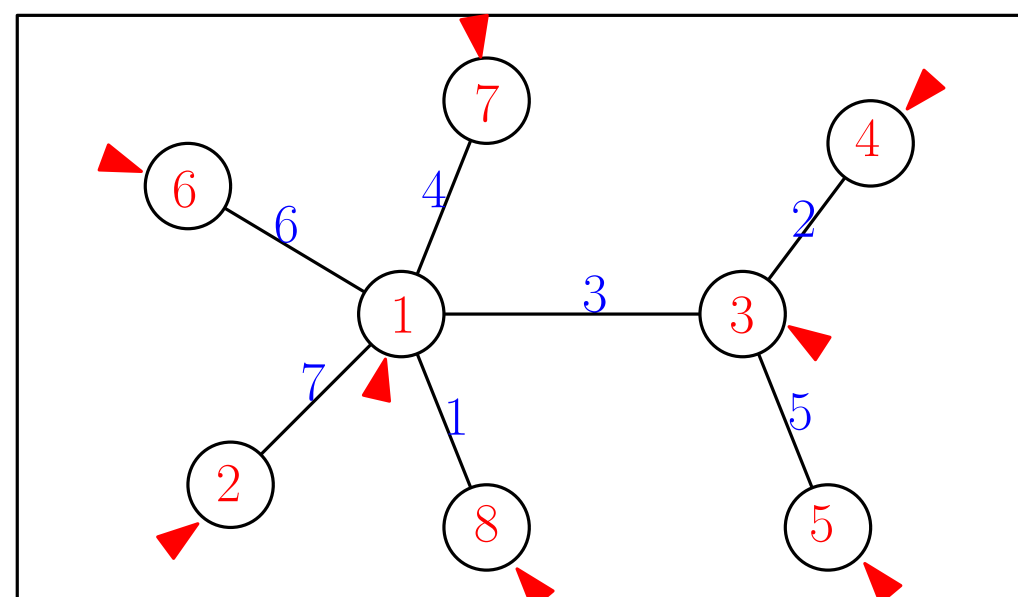
- **Theorem** [Cayley's formula] The number of such factorizations is

$$\#\{\tau_1\tau_2\dots\tau_{n-1} = (1,2,\dots,n)\} = n^{n-2}$$

$$\begin{array}{ccccccc} \tau_7 & \tau_6 & \tau_5 & \tau_4 & \tau_3 & \tau_2 & \tau_1 \\ (1\ 2) & (1\ 6) & (3\ 5) & (1\ 7) & (1\ 3) & (3\ 4) & (1\ 8) \end{array}$$

$$= (1\ 8\ 5\ 3\ 4\ 7\ 6\ 2)$$

a factorization of an arbitrary full cycle



a Cayley tree with labelled edges : there are $(n-1)!n^{n-2}$ of them

Minimal factorizations of a full cycle – Cayley's formula

- In the symmetric group $\mathbb{S}_n$ we consider factorizations of the full cycle $(1, 2, \dots, n)$ into a product of $(n-1)$ transpositions

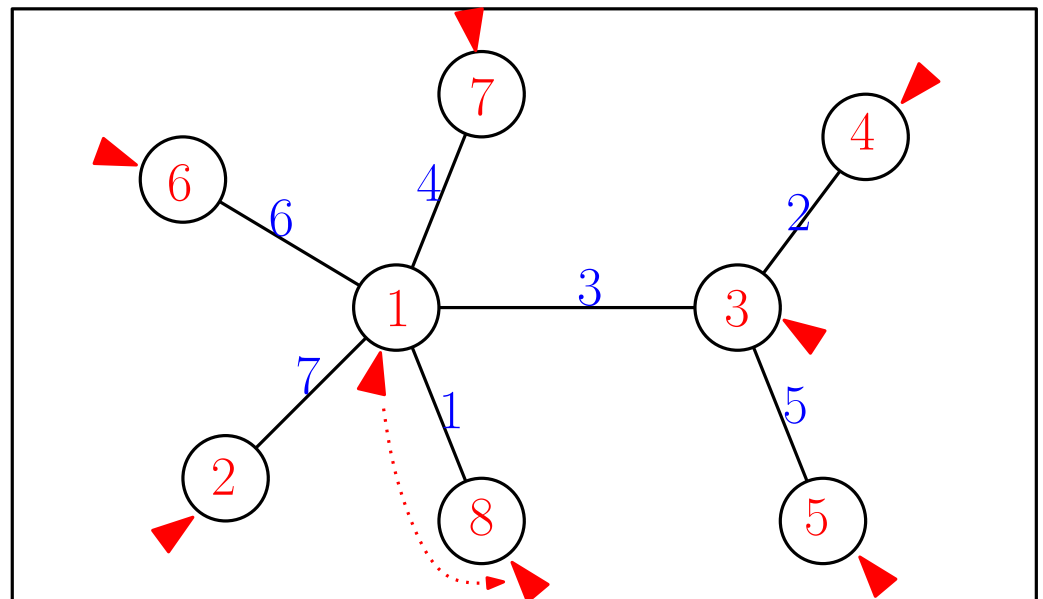
- **Theorem** [Cayley's formula] The number of such factorizations is

$$\#\{\tau_1\tau_2\dots\tau_{n-1} = (1,2,\dots,n)\} = n^{n-2}$$

$$\begin{array}{ccccccc} \tau_7 & \tau_6 & \tau_5 & \tau_4 & \tau_3 & \tau_2 & \tau_1 \\ (1\ 2) & (1\ 6) & (3\ 5) & (1\ 7) & (1\ 3) & (3\ 4) & (1\ 8) \end{array}$$

$$=(1\ 8\ 5\ 3\ 4\ 7\ 6\ 2)$$

a factorization of an arbitrary full cycle



a Cayley tree with labelled edges : there are $(n-1)!n^{n-2}$ of them

Minimal factorizations of a full cycle – Cayley's formula

- In the symmetric group $\mathbb{S}_n$ we consider factorizations of the full cycle $(1, 2, \ldots, n)$ into a product of $(n-1)$ transpositions

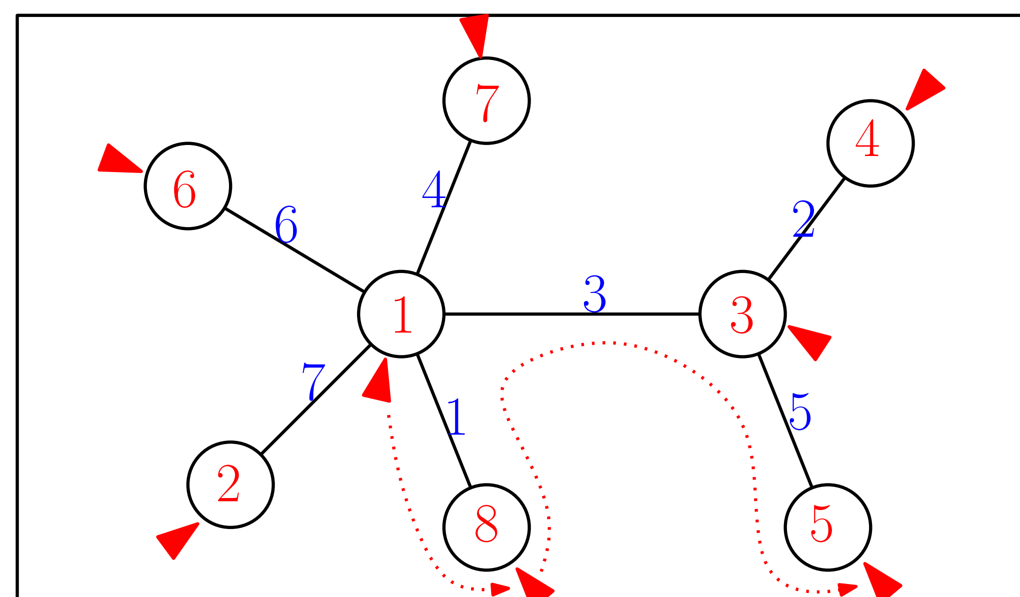
- **Theorem** [Cayley's formula] The number of such factorizations is

$$\#\{\tau_1\tau_2\ldots\tau_{n-1} = (1, 2, \ldots, n)\} = n^{n-2}$$

$$\underset{\tau_7}{(1\ 2)}\ \underset{\tau_6}{(1\ 6)}\ \underset{\tau_5}{(3\ 5)}\underset{\tau_4}{(1\ 7)}\ \underset{\tau_3}{(1\ 3)}\ \underset{\tau_2}{(3\ 4)}\ \underset{\tau_1}{(1\ 8)}$$

$$=(1\ 8\ 5\ 3\ 4\ 7\ 6\ 2)$$

a factorization of an arbitrary full cycle



a Cayley tree with labelled edges : there are $(n-1)!n^{n-2}$ of them

Minimal factorizations of a full cycle – Cayley's formula

- In the symmetric group $\mathbb{S}_n$ we consider factorizations of the full cycle $(1, 2, \ldots, n)$ into a product of $(n-1)$ transpositions

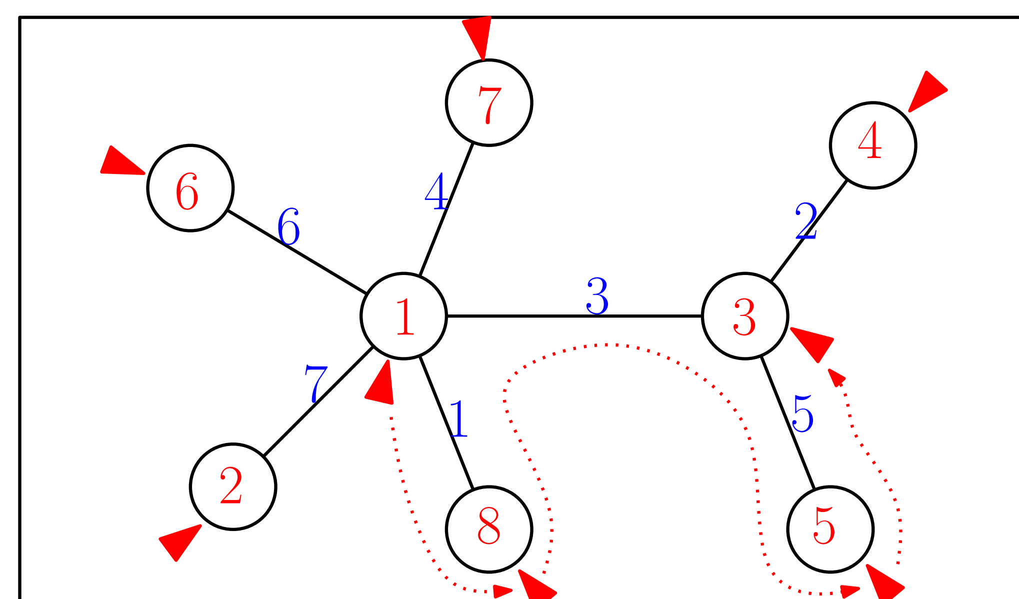
- **Theorem** [Cayley's formula] The number of such factorizations is

$$\#\{\tau_1\tau_2\ldots\tau_{n-1} = (1, 2, \ldots, n)\} = n^{n-2}$$

$$\overset{\tau_7}{(1\ 2)}\ \overset{\tau_6}{(1\ 6)}\ \overset{\tau_5}{(3\ 5)}\overset{\tau_4}{(1\ 7)}\ \overset{\tau_3}{(1\ 3)}\ \overset{\tau_2}{(3\ 4)}\ \overset{\tau_1}{(1\ 8)}$$

$$=(1\ 8\ 5\ 3\ 4\ 7\ 6\ 2)$$

a factorization of an arbitrary full cycle



a Cayley tree with labelled edges : there are $(n-1)!n^{n-2}$ of them

Minimal factorizations of a full cycle – Cayley's formula

- In the symmetric group $\mathbb{S}_n$ we consider factorizations of the full cycle $(1, 2, \ldots, n)$ into a product of $(n-1)$ transpositions

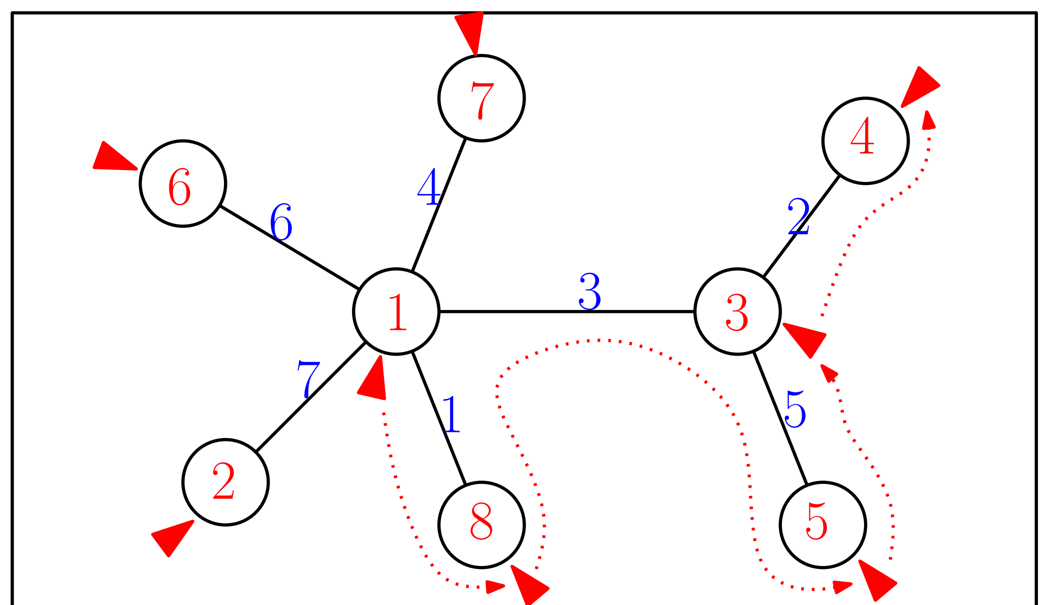
- **Theorem** [Cayley's formula] The number of such factorizations is

$$\#\{\tau_1 \tau_2 \ldots \tau_{n-1} = (1, 2, \ldots, n)\} = n^{n-2}$$

$$\overset{\tau_7}{(1\ 2)}\ \overset{\tau_6}{(1\ 6)}\ \overset{\tau_5}{(3\ 5)}\overset{\tau_4}{(1\ 7)}\ \overset{\tau_3}{(1\ 3)}\ \overset{\tau_2}{(3\ 4)}\ \overset{\tau_1}{(1\ 8)}$$

$$=(1\ 8\ 5\ 3\ 4\ 7\ 6\ 2)$$

a factorization of an arbitrary full cycle



a Cayley tree with labelled edges : there are $(n-1)!n^{n-2}$ of them

Minimal factorizations of a full cycle – Cayley's formula

- In the symmetric group $\mathbb{S}_n$ we consider factorizations of the full cycle $(1, 2, \ldots, n)$ into a product of $(n-1)$ transpositions

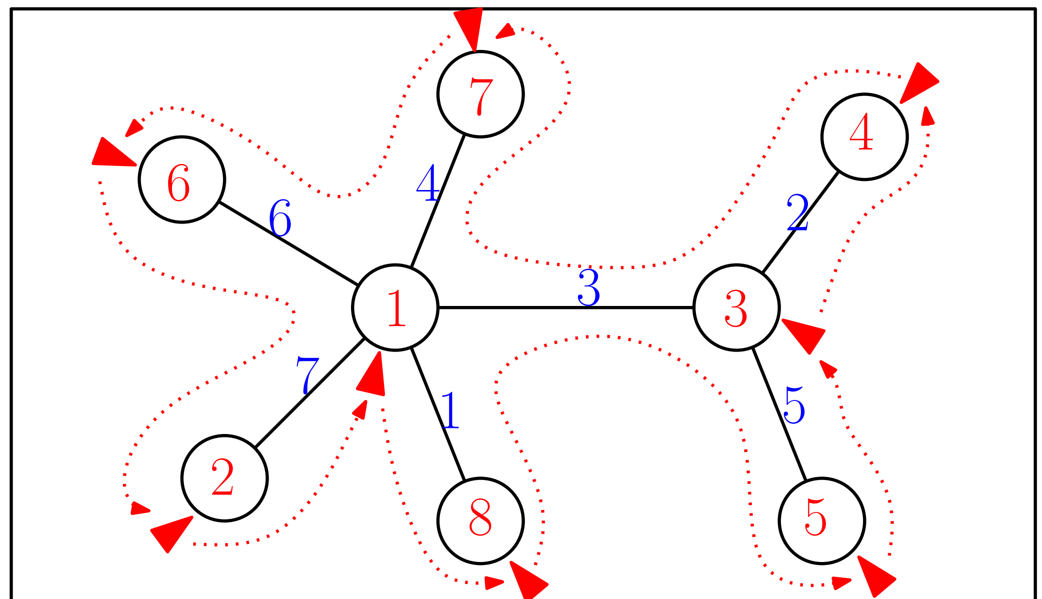
- **Theorem** [Cayley's formula] The number of such factorizations is

$$\#\{\tau_1\tau_2\ldots\tau_{n-1} = (1, 2, \ldots, n)\} = n^{n-2}$$

$$\underset{\tau_7}{(1\ 2)}\ \underset{\tau_6}{(1\ 6)}\ \underset{\tau_5}{(3\ 5)}\underset{\tau_4}{(1\ 7)}\ \underset{\tau_3}{(1\ 3)}\ \underset{\tau_2}{(3\ 4)}\ \underset{\tau_1}{(1\ 8)}$$

$$=(1\ 8\ 5\ 3\ 4\ 7\ 6\ 2)$$

a factorization of an arbitrary full cycle



a Cayley tree with labelled edges : there are $(n-1)!n^{n-2}$ of them

Minimal factorizations of a full cycle – Cayley's formula

- In the symmetric group $\mathbb{S}_n$ we consider factorizations of the full cycle $(1, 2, \ldots, n)$ into a product of $(n-1)$ transpositions

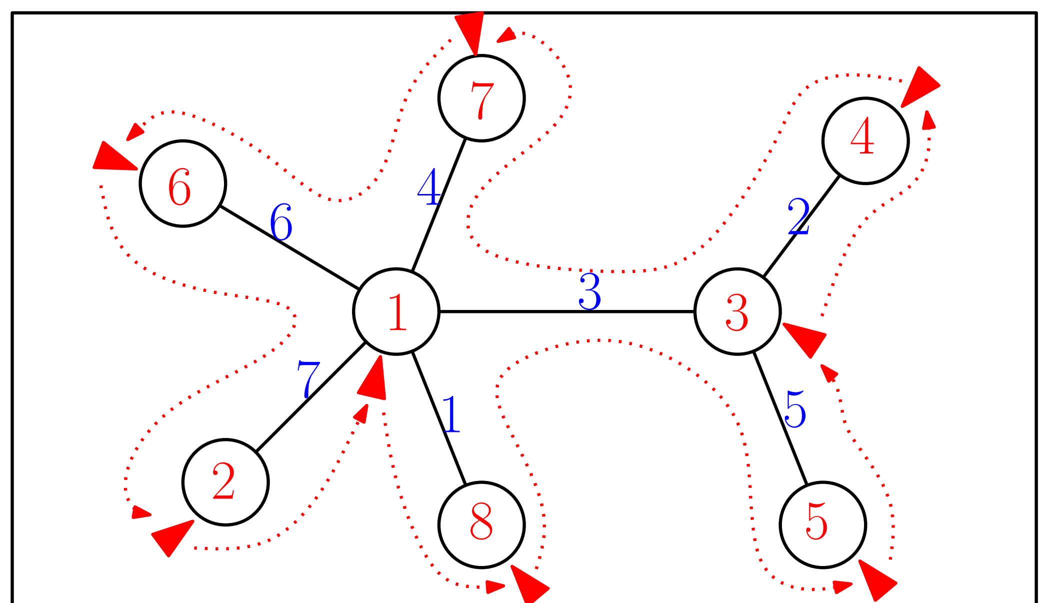
- **Theorem** [Cayley's formula] The number of such factorizations is

$$\boxed{\#\{\tau_1\tau_2\ldots\tau_{n-1} = (1, 2, \ldots, n)\} = n^{n-2}}$$

$$\underset{(1\ 2)}{\tau_7}\ \underset{(1\ 6)}{\tau_6}\ \underset{(3\ 5)}{\tau_5}\ \underset{(1\ 7)}{\tau_4}\ \underset{(1\ 3)}{\tau_3}\ \underset{(3\ 4)}{\tau_2}\ \underset{(1\ 8)}{\tau_1}$$

$$= (1\ 8\ 5\ 3\ 4\ 7\ 6\ 2)$$

a factorization of an arbitrary full cycle



a Cayley tree with labelled edges : there are $(n-1)!n^{n-2}$ of them

Hurwitz numbers, (Shapiro-Shapiro-Vainshtein) Jackson

- From a topological viewpoint, we are considering two restrictions:
  - planar ($\sim$ factorizations of minimal length)
  - one-face ($\sim$ factorizations of a full cycle)

Hurwitz numbers, (Shapiro-Shapiro-Vainshtein) Jackson

- From a topological viewpoint, we are considering two restrictions:
  - planar ($\sim$ factorizations of minimal length)
  - one-face ($\sim$ factorizations of a full cycle)

- Let us keep the one-face condition but consider an arbitrary genus $g \geq 0$

$$h_{n,g} = \#\{\tau_1 \tau_2 \dots \tau_{n-1+2g} = (1, 2, \dots, n)\} = \text{?}$$

- **Theorem** [Shapiro-Shapiro-Vainshtein 1997] Jackson 88 The generating function of one-face Hurwitz numbers is

$$\boxed{F(t) = \sum_{g \geq 0} \frac{t^{n-1+2g}}{(n-1+2g)!} h_{n,g} = \frac{1}{n!} \left(e^{\frac{nt}{2}} - e^{-\frac{nt}{2}} \right)^{n-1}.}$$

Hurwitz numbers, (Shapiro-Shapiro-Vainshtein) Jackson

- From a topological viewpoint, we are considering two restrictions:
  - planar ($\sim$ factorizations of minimal length)
  - one-face ($\sim$ factorizations of a full cycle)

- Let us keep the one-face condition but consider an arbitrary genus $g \geq 0$

$$h_{n,g} = \#\{\tau_1\tau_2\ldots\tau_{n-1+2g} = (1,2,\ldots,n)\} = \textbf{?}$$

- **Theorem** [Shapiro-Shapiro-Vainshtein 1997] Jackson 88 The generating function of one-face Hurwitz numbers is

$$\boxed{F(t) = \sum_{g\geq 0} \frac{t^{n-1+2g}}{(n-1+2g)!} h_{n,g} = \underbrace{\frac{1}{n!}\left(e^{\frac{nt}{2}} - e^{-\frac{nt}{2}}\right)^{n-1}}_{\sim \frac{1}{n!}(tn)^{n-1} = \frac{t^{n-1}}{(n-1)!}n^{n-2}}.}$$

$\rightarrow$ at order 1, this is Cayley's formula.

Reflection groups (I)

- Let V be a complex vector space, $n = \dim_{\mathbb{C}} V$.

A reflection is an element $\tau \in \mathrm{GL}(V)$ such that $\ker(\mathrm{id} - \tau)$ is a hyperplane and τ has finite order. In other words $\tau \approx \mathrm{Diag}(1, 1, \ldots, 1, \zeta)$ for ζ a root of unity.

- A complex reflection group is a finite subgroup of $\mathrm{GL}(V)$ generated by reflections. We can always assume $W \subset U(V)$ for some inner product.

Reflection groups (I)

• Let V be a complex vector space, $n = \dim_{\mathbb{C}} V$.

A reflection is an element $\tau \in \mathrm{GL}(V)$ such that $\ker(\mathrm{id} - \tau)$ is a hyperplane and τ has finite order. In other words $\tau \approx \mathrm{Diag}(1, 1, \ldots, 1, \zeta)$ for ζ a root of unity.

• A complex reflection group is a finite subgroup of $\mathrm{GL}(V)$ generated by reflections. We can always assume $W \subset U(V)$ for some inner product.

Examples

- permutation matrices: $\mathbb{S}_n \subset \mathrm{GL}(\mathbb{C}^n)$ generated by transpositions.

Reflection groups (I)

- Let V be a complex vector space, $n = \dim_{\mathbb{C}} V$.

A reflection is an element $\tau \in \mathrm{GL}(V)$ such that $\ker(\mathrm{id} - \tau)$ is a hyperplane and τ has finite order. In other words $\tau \approx \mathrm{Diag}(1, 1, \ldots, 1, \zeta)$ for ζ a root of unity.

- A complex reflection group is a finite subgroup of $\mathrm{GL}(V)$ generated by reflections. We can always assume $W \subset U(V)$ for some inner product.

Examples

- permutation matrices: $\mathbb{S}_n \subset \mathrm{GL}(\mathbb{C}^n)$ generated by transpositions.

- finite Coxeter groups (same definition, but over $\mathbb{R}$)

Reflection groups (I)

• Let V be a complex vector space, $n = \dim_{\mathbb{C}} V$.

A reflection is an element $\tau \in \mathrm{GL}(V)$ such that $\ker(\mathrm{id} - \tau)$ is a hyperplane and τ has finite order. In other words $\tau \approx \mathrm{Diag}(1, 1, \ldots, 1, \zeta)$ for ζ a root of unity.

• A complex reflection group is a finite subgroup of $\mathrm{GL}(V)$ generated by reflections. We can always assume $W \subset U(V)$ for some inner product.

Examples

- permutation matrices: $\mathbb{S}_n \subset \mathrm{GL}(\mathbb{C}^n)$ generated by transpositions.
- finite Coxeter groups (same definition, but over $\mathbb{R}$)
- complex reflection group $G(r, 1, n) \subset \mathrm{GL}(\mathbb{C}^n)$ with $r, n \geq 1$

$$\begin{pmatrix} 0 & \zeta & 0 \\ \zeta^2 & 0 & 0 \\ 0 & 0 & \zeta^5 \end{pmatrix}$$

take an $n \times n$ permutation matrix
replace entries by r-th roots of unity

Reflection groups (I)

- Let V be a complex vector space, $n = \dim_{\mathbb{C}} V$.

A reflection is an element $\tau \in \mathrm{GL}(V)$ such that $\ker(\mathrm{id} - \tau)$ is a hyperplane and τ has finite order. In other words $\tau \approx \mathrm{Diag}(1, 1, \ldots, 1, \zeta)$ for ζ a root of unity.

- A complex reflection group is a finite subgroup of $\mathrm{GL}(V)$ generated by reflections. We can always assume $W \subset U(V)$ for some inner product.

Examples

- permutation matrices: $\mathbb{S}_n \subset \mathrm{GL}(\mathbb{C}^n)$ generated by transpositions.
- finite Coxeter groups (same definition, but over $\mathbb{R}$)
- complex reflection group $G(r, p, n) \subset \mathrm{GL}(\mathbb{C}^n)$ with $r, p, n \geq 1$ and $p|r$

$$\begin{pmatrix} 0 & \zeta & 0 \\ \zeta^2 & 0 & 0 \\ 0 & 0 & \zeta^5 \end{pmatrix}$$

take an $n \times n$ permutation matrix
replace entries by r-th roots of unity
product of all entries is an r/p-th root of unity.

Reflection groups (II)

• If $W \subset \mathrm{GL}(V)$ is irreducible (=no stable subspace) then $\dim V$ is called its rank. If W is irreducible and is generated by $\dim V$ reflections then it is well-generated.

- $\mathbb{S}_n \subset \mathrm{GL}(\mathbb{C}^n)$ is not irreducible since $V_0 = \{\sum_i x_i = 0\}$ is stable.

- $\mathbb{S}_n \subset \mathrm{GL}(V_0)$ is irreducible. It has rank $(n-1)$. It is well-generated, take $s_i = (i\ i+1)$ for $1 \leq i < n$.

Reflection groups (II)

- If $W \subset \mathrm{GL}(V)$ is irreducible (=no stable subspace) then $\dim V$ is called its rank. If W is irreducible and is generated by $\dim V$ reflections then it is well-generated.

- $\mathbb{S}_n \subset \mathrm{GL}(\mathbb{C}^n)$ is not irreducible since $V_0 = \{\sum_i x_i = 0\}$ is stable.

- $\mathbb{S}_n \subset \mathrm{GL}(V_0)$ is irreducible. It has rank $(n-1)$. It is well-generated, take $s_i = (i\ i+1)$ for $1 \leq i < n$.

- If W is irreducible and well-generated there is a notion of Coxeter element that plays the same role as the full cycle for the symmetric group.

In general: it is an element having an eigenvalue ζ a primitive d-th root of unity with d as large as possible.

For real groups, it is the product (in any order) of the $(n-1)$ generators.

The Coxeter number, h, is the order of the Coxeter element.

Deligne's formula

- **Theorem** [Deligne-Tits-Zagier 74, Bessis 07] Let W be an irreducible well-generated complex reflection group of rank n. Then the number of factorizations of a Coxeter element into a product of n reflections is

$$\#\{\tau_1\tau_2\ldots\tau_n = \text{cox. element}\} = \frac{n!}{|W|}h^n.$$

Deligne's formula

- **Theorem** [Deligne-Tits-Zagier 74, Bessis 07] Let W be an irreducible well-generated complex reflection group of rank n. Then the number of factorizations of a Coxeter element into a product of n reflections is

$$\boxed{\#\{\tau_1\tau_2\ldots\tau_n = \text{cox. element}\} = \frac{n!}{|W|}h^n.}$$

Deligne's formula

- **Theorem** [Deligne-Tits-Zagier 74, Bessis 07] Let W be an irreducible well-generated complex reflection group of rank n. Then the number of factorizations of a Coxeter element into a product of n reflections is

$$\boxed{\#\{\tau_1\tau_2\ldots\tau_n = \text{cox. element}\} = \frac{n!}{|W|}h^n.}$$

- Translation for the symmetric group $\mathbb{S}_m$.

  - cox. element $=$ full cycle; its order $h = m$
  - reflection $=$ transposition
  - rank $n = m-1$

$\rightarrow \frac{(m-1)!}{m!}m^{m-1} = m^{m-2}$ Cayley's formula!

Our result – "higher genus" factorizations in w.g.c.r.g.

- **Theorem** [C.-Stump] Let W be an irreducible well-generated complex reflection group of rank n. Consider factorizations of a Coxeter element c into reflections and let

$$h_\ell = \#\big\{\tau_1\tau_2\ldots\tau_\ell = c \text{ where } \tau_i \text{ are reflections}\big\}$$

Then the generating function is nice:

$$F(t) = \sum_{\ell\geq 0} \frac{t^\ell}{\ell!} h_\ell = \frac{1}{|W|}\left(e^{\frac{h'}{2}t} - e^{-\frac{h''}{2}t}\right)^n.$$

- Parameters: $\frac{h'}{2} = \frac{\#\text{reflections}}{n}$ and $\frac{h''}{2} = \frac{\#\text{reflection hyperplanes}}{n}$

Our result – “higher genus” factorizations in w.g.c.r.g.

- **Theorem** [C.-Stump] Let W be an irreducible well-generated complex reflection group of rank n. Consider factorizations of a Coxeter element c into reflections and let

$$h_\ell = \#\{\tau_1\tau_2\ldots\tau_\ell = c \text{ where } \tau_i \text{ are reflections}\}$$

Then the generating function is nice:

$$\boxed{F(t) = \sum_{\ell\geq 0}\frac{t^\ell}{\ell!}h_\ell = \frac{1}{|W|}\left(e^{\frac{h'}{2}t} - e^{-\frac{h''}{2}t}\right)^n}$$

- Parameters: $\frac{h'}{2} = \frac{\#\text{reflections}}{n}$ and $\frac{h''}{2} = \frac{\#\text{reflection hyperplanes}}{n}$

Our result – “higher genus” factorizations in w.g.c.r.g.

- **Theorem** [C.-Stump] Let W be an irreducible well-generated complex reflection group of rank n. Consider factorizations of a Coxeter element c into reflections and let

$$h_\ell = \#\{\tau_1\tau_2\ldots\tau_\ell = c \text{ where } \tau_i \text{ are reflections}\}$$

Then the generating function is nice:

$$\boxed{F(t) = \sum_{\ell\geq 0}\frac{t^\ell}{\ell!}h_\ell = \frac{1}{|W|}\left(e^{\frac{h'}{2}t} - e^{-\frac{h''}{2}t}\right)^n}$$

- Parameters: $\frac{h'}{2} = \frac{\#\text{reflections}}{n}$ and $\frac{h''}{2} = \frac{\#\text{reflection hyperplanes}}{n}$
- Known that $\frac{h'}{2} + \frac{h''}{2} = h$ Coxeter number $\rightarrow$ Deligne's formula at $t \sim 0$

Our result – “higher genus” factorizations in w.g.c.r.g.

- **Theorem** [C.-Stump] Let W be an irreducible well-generated complex reflection group of rank n. Consider factorizations of a Coxeter element c into reflections and let

$$h_\ell = \#\{\tau_1\tau_2\ldots\tau_\ell = c \text{ where } \tau_i \text{ are reflections}\}$$

Then the generating function is nice:

$$\boxed{F(t) = \sum_{\ell\geq 0}\frac{t^\ell}{\ell!}h_\ell = \frac{1}{|W|}\left(e^{\frac{h'}{2}t} - e^{-\frac{h''}{2}t}\right)^n}$$

- Parameters: $\frac{h'}{2} = \frac{\#\text{reflections}}{n}$ and $\frac{h''}{2} = \frac{\#\text{reflection hyperplanes}}{n}$
- Known that $\frac{h'}{2} + \frac{h''}{2} = h$ Coxeter number $\rightarrow$ Deligne's formula at $t \sim 0$
- For real groups $h' = h'' = h$ (e.g. Shapiro-Shapiro-Vainshtein for $\mathbb{S}_m$).

Part 2: group characters

Counting factorizations in groups (I)

- Let $\mathcal{R} = \{\text{reflections}\}$ and $c =$ Coxeter element.

  Let $h_\ell = \#\{\tau_1\tau_2\ldots\tau_\ell = c$ where $\tau_i \in \mathcal{R}\}$

- **Lemma** [the Frobenius formula] Let $\chi_\lambda, \lambda \in \Lambda$ be the list of all irreducible characters of W. Then one has:

$$h_\ell = \frac{1}{|W|}\sum_{\lambda\in\Lambda}(\dim\lambda)\left(\frac{\chi_\lambda(R)}{\dim\lambda}\right)^\ell \chi_\lambda(c^{-1}).$$

where $\chi_\lambda(R) := \sum_{\tau\in\mathcal{R}}\chi_\lambda(\tau)$.

Counting factorizations in groups (I)

- Let $\mathcal{R} = \{\text{reflections}\}$ and $c =$ Coxeter element.
  Let $h_\ell = \#\{\tau_1\tau_2\ldots\tau_\ell = c \text{ where } \tau_i \in \mathcal{R}\}$

- **Lemma** [the Frobenius formula] Let $\chi_\lambda, \lambda \in \Lambda$ be the list of all irreducible characters of W. Then one has:

$$h_\ell = \frac{1}{|W|}\sum_{\lambda\in\Lambda}(\dim\lambda)\left(\frac{\chi_\lambda(R)}{\dim\lambda}\right)^\ell \chi_\lambda(c^{-1}).$$

where $\chi_\lambda(R) := \sum_{\tau\in\mathcal{R}}\chi_\lambda(\tau)$.

- **Sketch of a proof:** Consider the group algebra $\mathbb{C}[W]$.

Then $h_\ell =$ coeff. of 1 in $\left(R^\ell c^{-1}\right)$ where $R = \sum_{\tau\in\mathcal{R}}\tau$

$$= \frac{1}{|W|}\mathrm{Tr}\left(R^\ell c^{-1}\right)$$

since if $\sigma \in W$, then $\mathrm{Tr}_{\mathbb{C}[W]}\sigma = \begin{cases} |W| \text{ if } \sigma = 1 \\ 0 \text{ if } \sigma \neq 1 \end{cases}$

Now use: - the (classical) decomposition of $\mathbb{C}[W]$ as $C[W] = \bigoplus_{\lambda\in\Lambda}(\dim V^\lambda)V^\lambda$

- the fact that R is central and therefore acts as a scalar on each V^λ.

Counting factorizations in groups (II)

Immediate consequence of the Frobenius formula:

- **Proposition** For a given group W, our generating function is a **finite sum**:

$$F_W(t) := \sum_{\ell \geq 0} \frac{h_\ell}{\ell!} = \frac{1}{|W|} \sum_{\lambda \in \Lambda} (\dim \lambda) \chi_\lambda(c^{-1}) \exp\left(\frac{\chi_\lambda(R)}{\dim \lambda} \cdot t \right)$$

Counting factorizations in groups (II)

Immediate consequence of the Frobenius formula:

- **Proposition** For a given group W, our generating function is a **finite sum**:

$$\boxed{F_W(t) := \sum_{\ell \geq 0} \frac{h_\ell}{\ell!} = \frac{1}{|W|} \sum_{\lambda \in \Lambda} (\dim \lambda) \chi_\lambda(c^{-1}) \exp\left(\frac{\chi_\lambda(R)}{\dim \lambda} \cdot t \right)}$$

Counting factorizations in groups (II)

Immediate consequence of the Frobenius formula:

- **Proposition** For a given group W, our generating function is a **finite sum**:

$$\boxed{F_W(t) := \sum_{\ell \geq 0} \frac{h_\ell}{\ell!} = \frac{1}{|W|} \sum_{\lambda \in \Lambda} (\dim \lambda) \chi_\lambda(c^{-1}) \exp\left(\frac{\chi_\lambda(R)}{\dim \lambda} \cdot t \right)}$$

- Now you can prove the main theorem for your favorite fixed group, e.g. the group $W = W(\mathcal{E}_8)$.

Counting factorizations in groups (II)

Immediate consequence of the Frobenius formula:

- **Proposition** For a given group W, our generating function is a **finite sum**:

$$\boxed{F_W(t) := \sum_{\ell \geq 0} \frac{h_\ell}{\ell!} = \frac{1}{|W|} \sum_{\lambda \in \Lambda} (\dim \lambda) \chi_\lambda(c^{-1}) \exp\left(\frac{\chi_\lambda(R)}{\dim \lambda} \cdot t \right)}$$

- Now you can prove the main theorem for your favorite fixed group, e.g. the group $W = W(\mathcal{E}_8)$.

  - plug your computer in

Counting factorizations in groups (II)

Immediate consequence of the Frobenius formula:

- **Proposition** For a given group W, our generating function is a **finite sum**:

$$\boxed{F_W(t) := \sum_{\ell \geq 0} \frac{h_\ell}{\ell!} = \frac{1}{|W|} \sum_{\lambda \in \Lambda} (\dim \lambda) \chi_\lambda(c^{-1}) \exp\left(\frac{\chi_\lambda(R)}{\dim \lambda} \cdot t \right)}$$

- Now you can prove the main theorem for your favorite fixed group, e.g. the group $W = W(\mathcal{E}_8)$.

  - plug your computer in
  - ask for the character table of E_8

Counting factorizations in groups (II)

Immediate consequence of the Frobenius formula:

- **Proposition** For a given group W, our generating function is a **finite sum**:

$$\boxed{F_W(t) := \sum_{\ell \geq 0} \frac{h_\ell}{\ell!} = \frac{1}{|W|} \sum_{\lambda \in \Lambda} (\dim \lambda) \chi_\lambda(c^{-1}) \exp\left(\frac{\chi_\lambda(R)}{\dim \lambda} \cdot t \right)}$$

- Now you can prove the main theorem for your favorite fixed group, e.g. the group $W = W(\mathcal{E}_8)$.

- plug your computer in
- ask for the character table of E_8
- compute the sum (many terms...)

$$F_{E_8}(t) = \tfrac{1}{|E_8|} \left(e^{102t} + 28\, e^{-1680t} + \ldots\ldots \right)$$

Counting factorizations in groups (II)

Immediate consequence of the Frobenius formula:

- **Proposition** For a given group W, our generating function is a **finite sum**:

$$\boxed{F_W(t) := \sum_{\ell \geq 0} \frac{h_\ell}{\ell!} = \frac{1}{|W|} \sum_{\lambda \in \Lambda} (\dim \lambda) \chi_\lambda(c^{-1}) \exp\left(\frac{\chi_\lambda(R)}{\dim \lambda} \cdot t \right)}$$

- Now you can prove the main theorem for your favorite fixed group, e.g. the group $W = W(\mathcal{E}_8)$.

- plug your computer in
- ask for the character table of E_8
- compute the sum (many terms...)

$$F_{E_8}(t) = \frac{1}{|E_8|} \left(e^{102t} + 28\, e^{-1680t} + \ldots\ldots \right)$$

- ask your computer to factor it... it works!

$$F_{E_8}(t) = \frac{1}{|E_8|} \left(e^{15t} - e^{-15t} \right)^8.$$

Part 3: Classification
...and case-by-case proof

Classification and proof strategy

- **Theorem**[Sheppard, Todd, 54] Let W be an irreducible complex reflection group. Then W is (isomorphic to) either:

    - the symmetric group $\mathbb{S}_n \subset \mathrm{GL}(V_0)$

    - $G(r,p,n)$ for some integer $r \geq 2$, $p, n \geq 1$ with $p|r$.

    - one of 34 exceptional groups

- Well-generated: $\mathbb{S}_n$, $G(r,1,n)$ and $G(r,r,n) + 26$ exceptional groups.

Classification and proof strategy

- **Theorem**[Sheppard, Todd, 54] Let W be an irreducible complex reflection group. Then W is (isomorphic to) either:

  - the symmetric group $\mathbb{S}_n \subset \mathrm{GL}(V_0)$
  - $G(r, p, n)$ for some integer $r \geq 2$, $p, n \geq 1$ with $p|r$.
  - one of 34 exceptional groups

- Well-generated: $\mathbb{S}_n$, $G(r, 1, n)$ and $G(r, r, n)$ + 26 exceptional groups.

finitely many groups

COMPUTER !

Classification and proof strategy

- **Theorem**[Sheppard, Todd, 54] Let W be an irreducible complex reflection group. Then W is (isomorphic to) either:

  - the symmetric group $\mathbb{S}_n \subset \mathrm{GL}(V_0)$
  - $G(r,p,n)$ for some integer $r \geq 2$, $p,n \geq 1$ with $p|r$.
  - one of 34 exceptional groups

- Well-generated: $\mathbb{S}_n$, $G(r,1,n)$ and $G(r,r,n)$ + 26 exceptional groups.

INfinitely many groups

MATHS !

finitely many groups

COMPUTER !

Example of $\mathbb{S}_n$ (what is so special about the Coxeter element ?)

- We start from $F(t) = \dfrac{1}{|W|} \sum_{\lambda \in \Lambda} (\dim \lambda) \chi_\lambda(c^{-1}) \exp\left(\dfrac{\chi_\lambda(R)}{\dim \lambda} \cdot t \right)$

Here $\Lambda = \{\text{partitions of n}\}$ and $c^{-1} = \text{full cycle}$.

Example of $\mathbb{S}_n$ (what is so special about the Coxeter element ?)

- We start from $F(t) = \frac{1}{|W|} \sum_{\lambda \in \Lambda} (\dim \lambda) \chi_\lambda(c^{-1}) \exp\left(\frac{\chi_\lambda(R)}{\dim \lambda} \cdot t \right)$

Here $\Lambda = \{\text{partitions of n}\}$ and $c^{-1} = \text{full cycle.}$

- **Crucial fact:** There are **very few** partitions λ such that $\chi_\lambda(c^{-1}) \neq 0$

Example of $\mathbb{S}_n$ (what is so special about the Coxeter element ?)

- We start from $F(t) = \dfrac{1}{|W|} \displaystyle\sum_{\lambda \in \Lambda} (\dim \lambda)\, \chi_\lambda(c^{-1}) \exp\left(\dfrac{\chi_\lambda(R)}{\dim \lambda} \cdot t \right)$

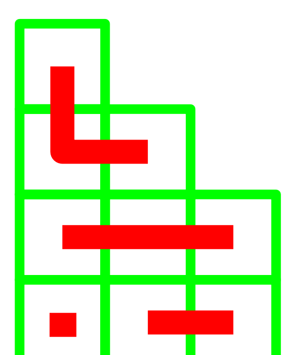
Here $\Lambda = \{$partitions of n$\}$ and $c^{-1} =$ full cycle.

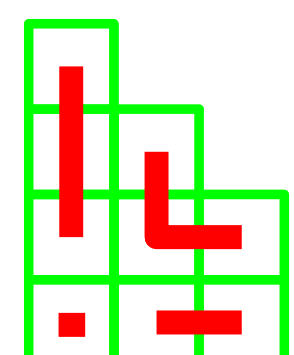
- **Crucial fact:** There are **very few** partitions λ such that $\chi_\lambda(c^{-1}) \neq 0$

Murnaghan-Nakayama rule

$\lambda = [3, 3, 2, 1]$ and $\sigma = (1, 3, 4)(2, 8, 9)(5, 7)(6)$

$$\chi_\lambda(\sigma) = \quad (-1) \quad + \quad (-1) \quad = (-2)$$





ribbon strip

Example of $\mathbb{S}_n$ (what is so special about the Coxeter element ?)

- We start from $F(t) = \dfrac{1}{|W|} \sum_{\lambda \in \Lambda} (\dim \lambda) \chi_\lambda(c^{-1}) \exp\left(\dfrac{\chi_\lambda(R)}{\dim \lambda} \cdot t \right)$

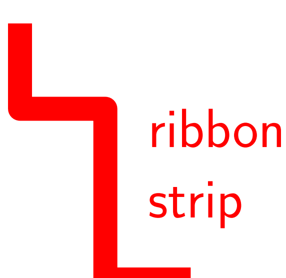
Here $\Lambda = \{$partitions of n$\}$ and $c^{-1} =$ full cycle.

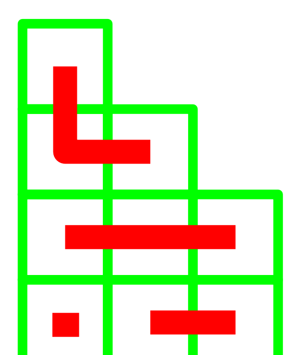
- **Crucial fact:** There are **very few** partitions λ such that $\chi_\lambda(c^{-1}) \neq 0$

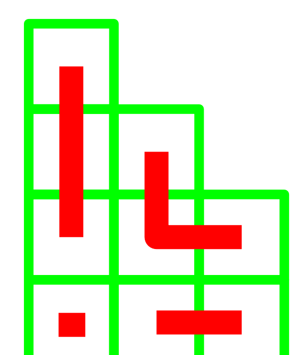
Murnaghan-Nakayama rule

$\lambda = [3, 3, 2, 1]$ and $\sigma = (1, 3, 4)(2, 8, 9)(5, 7)(6)$

$$\chi_\lambda(\sigma) = (-1) + (-1) = (-2)$$







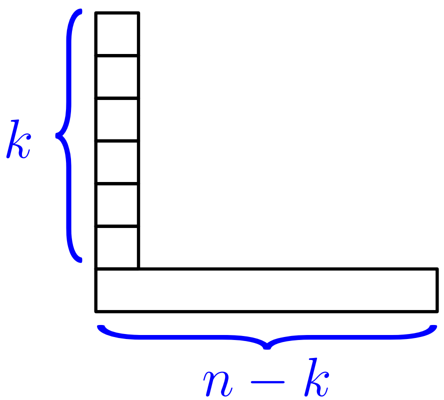
Example of $\mathbb{S}_n$ (what is so special about the Coxeter element ?) – 2

- We have $F(t) = \dfrac{1}{|W|} \displaystyle\sum_{\lambda \in \Lambda} (\dim \lambda) \chi_\lambda(c^{-1}) \exp\left(\frac{\chi_\lambda(R)}{\dim \lambda} \cdot t \right)$

BIG sum

SMALL sum

$$= \frac{1}{|W|} \sum_{k=0}^{n-1} (\dim \mathfrak{h}_k) \chi_{\mathfrak{h}_k}(c^{-1}) \exp\left(\frac{\chi_{\mathfrak{h}_k}(R)}{\dim \mathfrak{h}_k} \cdot t \right)$$



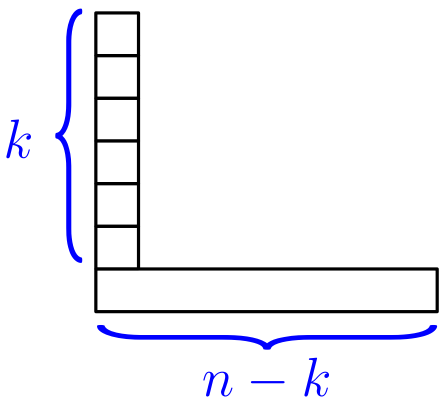
$\mathfrak{h}_k$ = hook

Example of $\mathbb{S}_n$ (what is so special about the Coxeter element ?) – 2

- We have $F(t) = \dfrac{1}{|W|} \displaystyle\sum_{\lambda\in\Lambda} (\dim\lambda)\chi_\lambda(c^{-1}) \exp\left(\frac{\chi_\lambda(R)}{\dim\lambda}\cdot t\right)$

BIG sum

SMALL sum



$\mathfrak{h}_k$ = hook

$$= \frac{1}{|W|} \sum_{k=0}^{n-1} \dim\mathfrak{h}_k \, \chi_{\mathfrak{h}_k}(c^{-1}) \exp\left(\frac{\chi_{\mathfrak{h}_k}(R)}{\dim\mathfrak{h}_k}\cdot t\right)$$

\# S.Y.T.

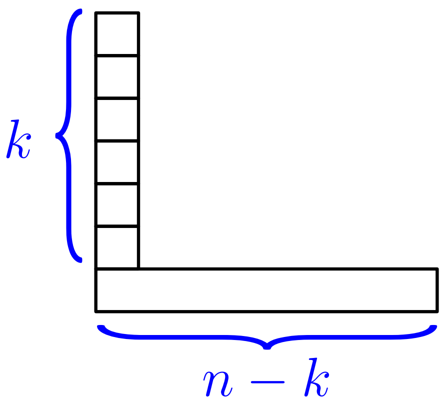
$$= \frac{1}{|W|} \sum_{k=0}^{n-1} \binom{n-1}{k}$$

Example of $\mathbb{S}_n$ (what is so special about the Coxeter element ?) – 2

- We have $F(t) = \dfrac{1}{|W|} \displaystyle\sum_{\lambda \in \Lambda} (\dim \lambda) \chi_\lambda(c^{-1}) \exp\left(\frac{\chi_\lambda(R)}{\dim \lambda} \cdot t \right)$

BIG sum

SMALL sum



$\mathfrak{h}_k$ = hook

$$= \frac{1}{|W|} \sum_{k=0}^{n-1} (\dim \mathfrak{h}_k) \chi_{\mathfrak{h}_k}(c^{-1}) \exp\left(\frac{\chi_{\mathfrak{h}_k}(R)}{\dim \mathfrak{h}_k} \cdot t \right)$$

S.Y.T.

Murnaghan Nakayama.

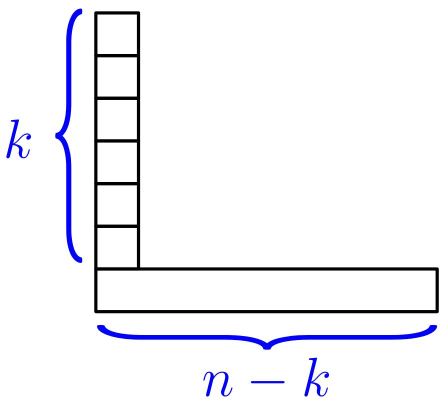
$$= \frac{1}{|W|} \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k$$

Example of $\mathbb{S}_n$ (what is so special about the Coxeter element ?) – 2

- We have $F(t) = \dfrac{1}{|W|} \displaystyle\sum_{\lambda \in \Lambda} (\dim \lambda) \chi_\lambda(c^{-1}) \exp\left(\frac{\chi_\lambda(R)}{\dim \lambda} \cdot t \right)$

BIG sum

SMALL sum



$\mathfrak{h}_k =$ hook

$$= \frac{1}{|W|} \sum_{k=0}^{n-1} (\dim \mathfrak{h}_k) \chi_{\mathfrak{h}_k}(c^{-1}) \exp\left(\frac{\chi_{\mathfrak{h}_k}(R)}{\dim \mathfrak{h}_k} \cdot t \right)$$

S.Y.T.

Murnaghan Nakayama.

Use combinatorial rules (e.g. Jucys Murphy or Murnaghan-Nakayama)

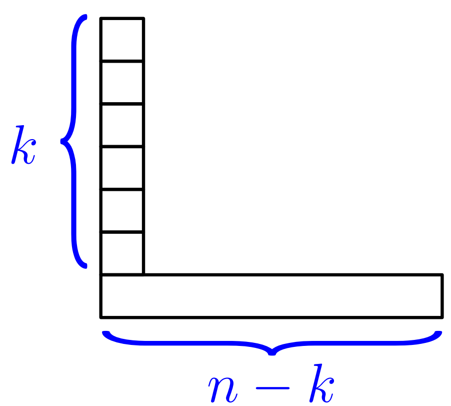
$$= \frac{1}{|W|} \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \exp\left(\frac{n(n-2k-2)}{2} \cdot t \right)$$

Example of $\mathbb{S}_n$ (what is so special about the Coxeter element ?) – 2

- We have $F(t) = \dfrac{1}{|W|} \displaystyle\sum_{\lambda \in \Lambda} (\dim \lambda) \chi_\lambda(c^{-1}) \exp\left(\frac{\chi_\lambda(R)}{\dim \lambda} \cdot t \right)$

BIG sum

SMALL sum



$\mathfrak{h}_k$ = hook

$$= \frac{1}{|W|} \sum_{k=0}^{n-1} (\dim \mathfrak{h}_k) \chi_{\mathfrak{h}_k}(c^{-1}) \exp\left(\frac{\chi_{\mathfrak{h}_k}(R)}{\dim \mathfrak{h}_k} \cdot t \right)$$

\# S.Y.T.

Murnaghan Nakayama.

Use combinatorial rules (e.g. Jucys Murphy or Murnaghan-Nakayama)

$$= \frac{1}{|W|} \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \exp\left(\frac{n(n-2k-2)}{2} \cdot t \right)$$

...IT FACTORS... PURE LUCK ! (Newton's binom formula)

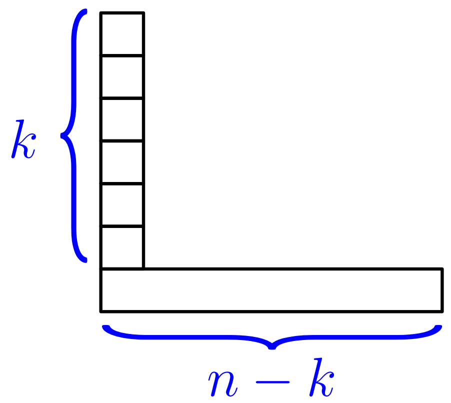
$$= \frac{1}{|W|} \left(e^{\frac{n}{2}t} - e^{-\frac{n}{2}t} \right)^{n-1}.$$

Example of $\mathbb{S}_n$ (what is so special about the Coxeter element ?) – 2

- We have $F(t) = \frac{1}{|W|}\sum_{\lambda\in\Lambda}(\dim\lambda)\chi_\lambda(c^{-1})\exp\left(\frac{\chi_\lambda(R)}{\dim\lambda}\cdot t\right)$

BIG sum

SMALL sum



$\mathfrak{h}_k$ = hook

$$= \frac{1}{|W|}\sum_{k=0}^{n-1}(\dim\mathfrak{h}_k)\chi_{\mathfrak{h}_k}(c^{-1})\exp\left(\frac{\chi_{\mathfrak{h}_k}(R)}{\dim\mathfrak{h}_k}\cdot t\right)$$

\# S.Y.T.

Murnaghan Nakayama.

Use combinatorial rules (e.g. Jucys Murphy or Murnaghan-Nakayama)

$$= \frac{1}{|W|}\sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^k \exp\left(\frac{n(n-2k-2)}{2}\cdot t\right)$$

...IT FACTORS...
PURE LUCK !
(Newton's binom formula)

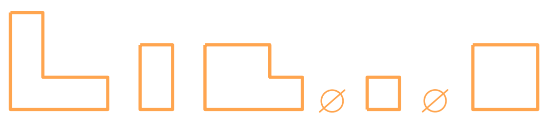
$$= \frac{1}{|W|}\left(e^{\frac{n}{2}t} - e^{-\frac{n}{2}t}\right)^{n-1}.$$

DONE!

Other infinite families – $G(r,1,n)$ and $G(r,r,n)$

- We need some combinatorial representation theory for these groups

- $G(r,1,n) \to$ standard [MacDonald, Serre...]

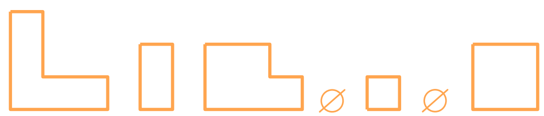


r-tuples of partitions of total size n

Other infinite families – $G(r,1,n)$ and $G(r,r,n)$

- We need some combinatorial representation theory for these groups

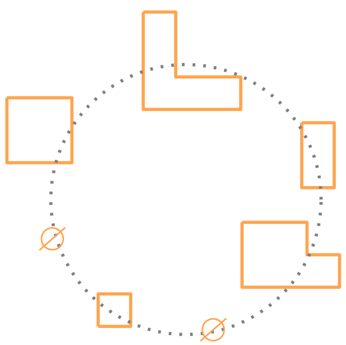
- $G(r,1,n) \to$ standard [MacDonald, Serre...]



r-tuples of partitions of total size n

- $G(r,r,n) \to$ algebraically: “easy” exercise in representation theory

  combinatorially: a bit messy so not really done anywhere...

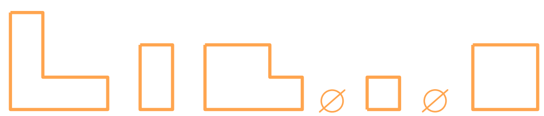


r-cycles of partitions of total size n

Other infinite families – $G(r,1,n)$ and $G(r,r,n)$

- We need some combinatorial representation theory for these groups

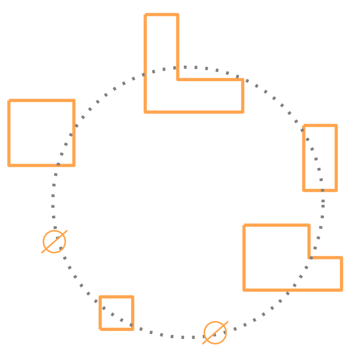
- $G(r,1,n) \to$ standard [MacDonald, Serre...]



r-tuples of partitions of total size n

- $G(r,r,n) \to$ algebraically: “easy” exercise in representation theory

  combinatorially: a bit messy so not really done anywhere...



r-cycles of partitions of total size n

- In both cases: - there are only $O(r^2 n)$ characters to consider

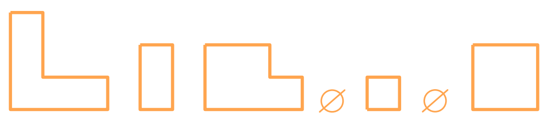
  - we can (meticulously...) compute all the pieces

  - at the end, Newton's formula collects the pieces!

Other infinite families – $G(r,1,n)$ and $G(r,r,n)$

- We need some combinatorial representation theory for these groups

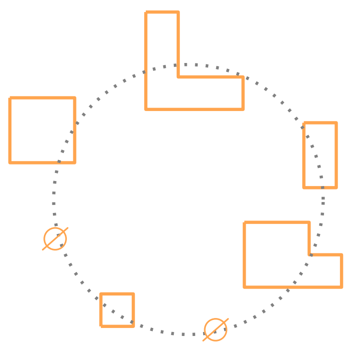
- $G(r,1,n) \to$ standard [MacDonald, Serre...]



r-tuples of partitions of total size n

- $G(r,r,n) \to$ algebraically: "easy" exercise in representation theory

  combinatorially: a bit messy so not really done anywhere...



r-cycles of partitions of total size n

- In both cases: - there are only $O(r^2 n)$ characters to consider

  - we can (meticulously...) compute all the pieces

  - at the end, Newton's formula collects the pieces!

- Conclusion: The formulas are nice but we don't UNDERSTAND them!

Extended conclusion

- We end up with a nice formula but a classification dependent proof... (which is not so nice)

Extended conclusion

- We end up with a nice formula but a classification dependent proof... (which is not so nice)

  This is a general phenomenon in this context!

  - Deligne's formula still has no classification-free proof

Extended conclusion

• We end up with a nice formula but a classification dependent proof... (which is not so nice)

This is a general phenomenon in this context!

- Deligne's formula still has no classification-free proof

- vast litterature in algebraic combinatorics on non-crossing partitions [Armstong, Bessis-Reiner, Krattenthaler-Muller...]

These results deal with refinements of the planar case (=trees for $\mathbb{S}_n$)

None of them has a classification-free proof

Extended conclusion

- We end up with a nice formula but a classification dependent proof... (which is not so nice)

  This is a general phenomenon in this context!

  - Deligne's formula still has no classification-free proof

  - vast litterature in algebraic combinatorics on non-crossing partitions [Armstong, Bessis-Reiner, Krattenthaler-Muller...]

    These results deal with refinements of the planar case (=trees for $\mathbb{S}_n$)

    None of them has a classification-free proof

- Suprisingly we are the first ones to use representation theory in this context

- Hope: the rep-theoretic approach could lead to classification-free proofs

- Why? because I hope that the non-vanishing characters have a nice geometric description... we just have to find it!

Extended conclusion

- We end up with a nice formula but a classification dependent proof... (which is not so nice)

  This is a general phenomenon in this context!

  - Deligne's formula still has no classification-free proof

  - vast litterature in algebraic combinatorics on non-crossing partitions [Armstong, Bessis-Reiner, Krattenthaler-Muller...]

    These results deal with refinements of the planar case (=trees for $\mathbb{S}_n$)

    None of them has a classification-free proof

- Suprisingly we are the first ones to use representation theory in this context

- Hope: the rep-theoretic approach could lead to classification-free proofs

- Why? because I hope that the non-vanishing characters have a nice geometric description... we just have to find it!

Thank you !