Factorisations d’un élément de Coxeter dans les groupes de réflexions (complexes)

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travail commun avec
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Part 1: the objects
Minimal factorizations of a full cycle – Cayley’s formula

• In the symmetric group $S_n$ we consider factorizations of the full cycle $(1, 2, \ldots, n)$ into a product of $(n - 1)$ transpositions.

• **Theorem** [Cayley’s formula] The number of such factorizations is

$$\# \{\tau_1 \tau_2 \cdots \tau_{n-1} = (1, 2, \ldots, n)\} = n^{n-2}$$
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[a Cayley tree with labelled edges: there are $(n - 1)! n^{n-2}$ of them]
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- A factorization of an arbitrary full cycle:

  $$(1\ 2)\ (1\ 6)\ (3\ 5)(1\ 7)\ (1\ 3)\ (3\ 4)\ (1\ 8)$$

  $$=(1\ 8\ 5\ 3\ 4\ 7\ 6\ 2)$$

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Diagram: a Cayley tree with labelled edges.
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  - planar (∼ factorizations of minimal length)
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- Let us keep the one-face condition but consider an arbitrary genus \( g \geq 0 \)

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h_{n,g} = \# \{ \tau_1 \tau_2 \cdots \tau_{n-1+2g} = (1, 2, \ldots, n) \} =?
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- **Theorem** [Shapiro-Shapiro-Vainshtein 1997] The generating function of one-face Hurwitz numbers is

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F(t) = \sum_{g \geq 0} \frac{t^{n-1+2g}}{(n-1+2g)!} h_{n,g} = \frac{1}{n!} \left( e^{\frac{nt}{2}} - e^{-\frac{nt}{2}} \right)^{n-1}.
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\[ \sim \frac{1}{n!} (tn)^{n-1} = \frac{t^{n-1}}{(n-1)!} n^{n-2} \]

\[ \rightarrow \text{at order 1, this is Cayley's formula.} \]
Reflection groups (I)

• Let $V$ be a complex vector space, $n = \dim_{\mathbb{C}} V$.

A **reflection** is an element $\tau \in \text{GL}(V)$ such that $\ker(\text{id} - \tau)$ is a hyperplane and $\tau$ has finite order. In other words $\tau \approx \text{Diag}(1, 1, \ldots, 1, \zeta)$ for $\zeta$ a root of unity.

• A **complex reflection group** is a finite subgroup of $\text{GL}(V)$ generated by reflections. We can always assume $W \subset \text{U}(V)$ for some inner product.
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- permutation matrices: $S_n \subset \text{GL}(\mathbb{C}^n)$ generated by transpositions.
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take an $n \times n$ permutation matrix

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  take an $n \times n$ permutation matrix
  replace entries by $r$-th roots of unity
  product of all entries is an $r/p$-th root of unity.
Reflection groups (II)

- If $W \subset \GL(V)$ is irreducible (=no stable subspace) then $\dim V$ is called its rank. If $W$ is irreducible and is generated by $\dim V$ reflections then it is well-generated.

- $S_n \subset \GL(\mathbb{C}^n)$ is not irreducible since $V_0 = \{\sum_i x_i = 0\}$ is stable.

- $S_n \subset \GL(V_0)$ is irreducible. It has rank $(n - 1)$. It is well-generated, take $s_i = (i \; i + 1)$ for $1 \leq i < n$. 
If $\mathcal{W} \subset \text{GL}(V)$ is irreducible (=no stable subspace) then $\dim V$ is called its rank. If $\mathcal{W}$ is irreducible and is generated by $\dim V$ reflections then it is well-generated.

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If $\mathcal{W}$ is irreducible and well-generated there is a notion of Coxeter element that plays the same role as the full cycle for the symmetric group. In general: it is an element having an eigenvalue $\zeta$ a primitive $d$-th root of unity with $d$ as large as possible.

For real groups, it is the product (in any order) of the $(n - 1)$ generators. The Coxeter number, $h$, is the order of the Coxeter element.
Deligne’s formula

**Theorem** [Deligne-Tits-Zagier 74, Bessis 07] Let $W$ be an irreducible well-generated complex reflection group of rank $n$. Then the number of factorizations of a Coxeter element into a product of $n$ reflections is

$$\#\{\tau_1 \tau_2 \ldots \tau_n = \text{cox. element}\} = \frac{n!}{|W|} h^n.$$
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- **Translation** for the symmetric group $S_m$.
  - **cox. element** = full cycle; its order $h = m$
  - **reflection** = transposition
  - **rank** $n = m - 1$
  $$ \rightarrow \frac{(m-1)!}{m!} m^{m-1} = m^{m-2} \quad \text{Cayley's formula!} $$
Our result – "higher genus" factorizations in w.g.c.r.g.

- **Theorem** [C.-Stump] Let $W$ be an irreducible well-generated complex reflection group of rank $n$. Consider factorizations of a Coxeter element $c$ into reflections and let

$$h_{\ell} = \# \{ \tau_1 \tau_2 \ldots \tau_\ell = c \text{ where } \tau_i \text{ are reflections} \}$$

Then the generating function is nice:

$$F'(t) = \sum_{\ell \geq 0} \frac{t^\ell}{\ell!} h_{\ell} = \frac{1}{|W|} \left( e^{\frac{h'}{2} t} - e^{-\frac{h''}{2} t} \right)^n.$$

- Parameters: $\frac{h'}{2} = \frac{\# \text{ reflections}}{n}$ and $\frac{h''}{2} = \frac{\# \text{ reflection hyperplanes}}{n}$
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- For real groups $h' = h'' = h$ (e.g. Shapiro-Shapiro-Vainshtein for $S_m$).
Part 2: group characters
Let $\mathcal{R} = \{\text{reflections}\}$ and $c = \text{Coxeter element}$. Let $h_\ell = \# \{\tau_1 \tau_2 \ldots \tau_\ell = c \text{ where } \tau_i \in \mathcal{R} \}$.

**Lemma** [the Frobenius formula] Let $\chi_\lambda, \lambda \in \Lambda$ be the list of all irreducible characters of $W$. Then one has:

$$h_\ell = \frac{1}{|W|} \sum_{\lambda \in \Lambda} (\dim \lambda) \left( \frac{\chi_\lambda(R)}{\dim \lambda} \right)^\ell \chi_\lambda(c^{-1}).$$

where $\chi_\lambda(R) := \sum_{\tau \in \mathcal{R}} \chi_\lambda(\tau)$. 


Counting factorizations in groups (I)

- Let $\mathcal{R} = \{\text{reflections}\}$ and $c = \text{Coxeter element}$.
  Let $h_\ell = \#\{\tau_1 \tau_2 \ldots \tau_\ell = c \mid \tau_i \in \mathcal{R}\}$

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- **Sketch of a proof:** Consider the group algebra $\mathbb{C}[W]$.

  Then $h_\ell = \text{coeff. of } 1 \text{ in } \left(R^\ell c^{-1}\right)$ where $R = \sum_{\tau \in \mathcal{R}} \tau$

  $$= \frac{1}{|W|} \text{Tr} \left(R^\ell c^{-1}\right) \quad \text{since if } \sigma \in W, \text{ then } \text{Tr}_{\mathbb{C}[W]} \sigma = \begin{cases} |W| & \text{if } \sigma = 1 \\ 0 & \text{if } \sigma \neq 1 \end{cases}$$

  Now use: - the (classical) decomposition of $\mathbb{C}[W]$ as $C[W] = \bigoplus_{\lambda \in \Lambda} (\dim V^\lambda) V^\lambda$

  - the fact that $R$ is central and therefore acts as a scalar on each $V^\lambda$. 
Immediate consequence of the Frobenius formula:

- **Proposition** For a given group $W$, our generating function is a finite sum:

$$F_W(t) := \sum_{\ell \geq 0} \frac{h_{\ell}}{\ell!} = \frac{1}{|W|} \sum_{\lambda \in \Lambda} (\dim \lambda) \chi_{\lambda}(c^{-1}) \exp \left( \frac{\chi_{\lambda}(R)}{\dim \lambda} \cdot t \right)$$
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$$F_{E_8}(t) = \frac{1}{|E_8|} \left( e^{102t} + 28 e^{-1680t} + \ldots \right)$$
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$$F_{E_8}(t) = \frac{1}{|E_8|} \left( e^{102t} + 28 e^{-1680t} + \ldots \right)$$

  - ask your computer to factor it... it works!

$$F_{E_8}(t) = \frac{1}{|E_8|} \left( e^{15t} - e^{-15t} \right)^8.$$
Part 3: Classification
...and case-by-case proof
Classification and proof strategy

• **Theorem**[Sheppard, Todd, 54] Let $W$ be an irreducible complex reflection group. Then $W$ is (isomorphic to) either:
  - the symmetric group $S_n \subset GL(V_0)$
  - $G(r, p, n)$ for some integer $r \geq 2$, $p, n \geq 1$ with $p|r$.
  - one of 34 exceptional groups

• Well-generated: $S_n$, $G(r, 1, n)$ and $G(r, r, n)$ + 26 exceptional groups.
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\[ \text{finitely many groups} \]

**COMPUTER !**
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INfinitely many groups

MATHS !

finitely many groups

COMPUTER !
Example of $S_n$ (what is so special about the Coxeter element ?)

- We start from $F(t) = \frac{1}{|W|} \sum_{\lambda \in \Lambda} (\dim \lambda) \chi_\lambda(c^{-1}) \exp \left( \frac{\chi_\lambda(R)}{\dim \lambda} \cdot t \right)$

Here $\Lambda = \{\text{partitions of } n\}$ and $c^{-1} = \text{full cycle.}$
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- Crucial fact: There are very few partitions $\lambda$ such that $\chi_\lambda(c^{-1}) \neq 0$. 
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### Murnaghan-Nakayama rule

$\lambda = [3, 3, 2, 1]$ and $\sigma = (1, 3, 4)(2, 8, 9)(5, 7)(6)$

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[Diagram of ribbon strip]
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Example of $S_n$ (what is so special about the Coxeter element?) – 2

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Murnaghan Nakayama.
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- # S.Y.T. Murnaghan Nakayama.

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Use combinatorial rules (e.g. Jucys Murphy or Murnaghan-Nakayama)

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\[ n - k \]

BIG sum

SMALL sum
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Other infinite families – $G(r, 1, n)$ and $G(r, r, n)$

- We need some **combinatorial representation theory** for these groups

- $G(r, 1, n) \rightarrow$ standard [MacDonald, Serre...]

$r$-tuples of partitions of total size $n$
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- In both cases: - there are **only $O(r^2n)$** characters to consider
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- **Conclusion:** The formulas are nice but we don’t UNDERSTAND them!
Extended conclusion

- We end up with a nice formula but a classification dependent proof… (which is not so nice)
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• Hope: the rep-theoretic approach could lead to classification-free proofs

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Extended conclusion

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Thank you !