Three Ways to Cover a Graph

Kolja Knauer  
Université Montpellier 2

Torsten Ueckerdt  
Karlsruhe Institute of Technology

GT Combi du LIX, June 3, 2013
Interval graphs
Intersection graphs of intervals

every $v$ represented by an interval
graph edges $\iff$ interval intersections

• classical graph class
• efficient recognition
• chordal & perfect
• many applications
Intersection graphs of systems of intervals

every $\nu$ represented by $\leq k$ intervals

graph edges $\iff$ interval intersections
Intersection graphs of systems of intervals

every \( v \) represented by \( \leq k \) intervals

graph edges \( \iff \) interval intersections

on one line
Intersection graphs of systems of intervals

every \( v \) represented by \( \leq k \) intervals

graph edges \( \iff \) interval intersections

at most one on each of \( k \) lines

on one line

Track number
Gyárfás, West '95

Interval number
Harary, Trotter '79
Intersection graphs of systems of intervals

every $v$ represented by $\leq k$ intervals

graph edges $\Leftrightarrow$ interval intersections

at most one on each of $k$ lines

on one line

at most one on each line

Track number
Gyárfás, West '95

Local track number

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Harary, Trotter '79
### Some Results

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- Kostochka, West ’99
- Scheinermann, West ’83
- Gonçalves, Ochem ’09
- KU ’12
Intersection graphs of systems of intervals

- at most one on each of \( k \) lines
- at most one on each line
- on one line

- **Track number**
  Gyárfás, West '95

- **Local track number**

- **Interval number**
  Harary, Trotter '79
Intersection graphs of systems of intervals

at most one on each of $k$ lines

edges covered by $\leq k$ interval graphs

Track number
Gyárfás, West '95

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on one line

at most one on each line
Intersection graphs of systems of intervals

at most one on each of \( k \) lines

at most one on each line

edges covered by interval graphs, \( \leq k \) at each vertex

\( \leq k \) interval graphs

Track number
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Intersection graphs of systems of intervals

edges covered by \( \leq k \) interval graphs, \( \leq k \) at each vertex

at most one on each of \( k \) lines

homomorphism from an interval graph, each vertex hit \( \leq k \) times

at most one on each line

Track number
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Harary, Trotter ’79
Global, Local, and Folded Covers
  ◦ Templates = Interval Graphs

Formal Definitions

Local and Folded Linear Arboricity
  ◦ Templates = Collections of Paths

Interrelations
  ◦ Templates = Forests, Pseudo-Forests, Star Forests

What is known and what is open
More Formally

\[ \varphi : T_1 \sqcup \cdots \sqcup T_k \rightarrow G \]
edge-surjective homomorphism

\[ \varphi \text{ injective} \iff \varphi \text{ restricted to each } T_i \text{ injective} \]

\[ \text{size of } \varphi \iff \# \text{ template graphs in preimage} \]
More Formally

ϕ cover \iff \varphi : T_1 \sqcup \cdots \sqcup T_k \to G

edge-surjective homomorphism

ϕ injective \iff \varphi \text{ restricted to each } T_i \text{ injective}

size of ϕ \iff \# \text{ template graphs in preimage}

c_T^g(G) = \min \{ \text{size of } \varphi : \varphi \text{ injective cover of } G \}

global

c_T^\ell(G) = \min \{ \max_{v \in V(G)} |\varphi^{-1}(v)| : \varphi \text{ injective cover of } G \}

local

c_T^f(G) = \min \{ \max_{v \in V(G)} |\varphi^{-1}(v)| : \varphi \text{ cover of } G \text{ of size 1} \}

folded
Basic Properties

We consider template classes that are closed under disjoint union.

Lemma:

1) \( c_T^g (G) \geq c_T^\ell (G) \geq c_T^f (G) \) for every \( G \)

   define \( c_T^i (G) := \sup \{ c_T^i (G) : G \in \mathcal{G} \} \) (\( \mathcal{G} \) graph class)

2) \( c_T^i (\mathcal{G}) \leq c_T^i (\mathcal{G}') \) \( \mathcal{G} \subseteq \mathcal{G}' \)

3) \( c_T^i (\mathcal{G}) \geq c_T' (\mathcal{G}) \) \( \mathcal{T} \subseteq \mathcal{T}' \)
Global Covering Number
- star arboricity
- caterpillar arboricity
- clique covering number
- track number
- linear arboricity
- arboricity
- outer-thickness
- edge-chromatic number
- thickness
- bipartite dimension

Local Covering Number
- bipartite degree

Folded Covering Number
- bar visibility number
- interval number
- splitting number

Unifying Concept
Global, Local, and Folded Covers
  - Templates = Interval Graphs

Formal Definitions

Local and Folded Linear Arboricity
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What is known and what is open
Global and Local Linear Arboricity

host graph

$G = \text{Petersen Graph}$

template class

$\mathcal{T} = \{\text{linear forests}\}$
Global and Local Linear Arboricity

template class

$T = \{\text{linear forests}\}$

host graph

$G = \text{Petersen Graph}$
Global and Local Linear Arboricity

linear arboricity

\[ c^T_g(G) = \text{la}(G) = 2 \]

host graph

\[ G = \text{Petersen Graph} \]

template class

\[ \mathcal{T} = \{\text{linear forests}\} \]
Global and Local Linear Arboricity

linear arboricity
\( c_g^T (G) = \text{la}(G) = 2 \)

\( \text{host graph} \quad G = \text{Petersen Graph} \)

\( \mathcal{T} = \{ \text{linear forests} \} \)

Akiyama et. al. ’80

Linear Arboricity Conjecture

\( \text{la}(G) \leq \left\lceil \frac{\Delta + 1}{2} \right\rceil \)
Global and Local Linear Arboricity

local linear arboricity

\[ c_\ell^\mathcal{T}(G) = l\alpha_\ell(G) = 2 \]

host graph

\[ G = \text{Petersen Graph} \]

template class

\[ \mathcal{T} = \{ \text{linear forests} \} \]
Global and Local Linear Arboricity

Local linear arboricity
\[ c^T_\ell(G) = \text{la}_\ell(G) = 2 \]

Petersen Graph

Local Linear Arboricity Conjecture
\[ \text{la}_\ell(G) \leq \left\lceil \frac{\Delta+1}{2} \right\rceil \]
Folded Linear Arboricity

folded linear arboricity

$$c^T_f(G) = \text{la}_f(G) = 2$$

host graph

\(G = \text{Petersen Graph}\)

template class

\(\mathcal{T} = \{\text{linear forests}\}\)
Folded Linear Arboricity

folded linear arboricity
\[ c_f(T)(G) = \text{la}_f(G) = 2 \]

host graph
\[ G = \text{Petersen Graph} \]

\[ \mathcal{T} = \{ \text{linear forests} \} \]

Folded Linear Arboricity Theorem [KU]
\[ \text{la}_f(G) \leq \left\lceil \frac{\Delta + 1}{2} \right\rceil \]
Folded Linear Arboricity Theorem [KU]

\[ l_{af}(G') \leq \left\lceil \frac{\Delta + 1}{2} \right\rceil \]
Folded Linear Arboricity Theorem [KU]

\[ \text{la}_f(G') \leq \left\lceil \frac{\Delta + 1}{2} \right\rceil \]

Proof: (easy)

\( \Delta \text{ even:} \)
- add vertices and edges to obtain Eulerian
- take Eulertour
- all visited \( \leq \frac{\Delta}{2} \) times
- start-vertex once more
- \( 1 + \frac{\Delta}{2} = \left\lceil \frac{\Delta + 1}{2} \right\rceil \)
Folded Linear Arboricity Theorem [KU]

\[ \lambda_f(G) \leq \left\lceil \frac{\Delta + 1}{2} \right \rceil \]

Proof: (easy)

\(\Delta\) even:
- add vertices and edges to obtain Eulerian
- take Euler tour
- all visited \(\leq \frac{\Delta}{2}\) times
- start-vertex once more
- \(1 + \frac{\Delta}{2} = \left\lceil \frac{\Delta + 1}{2} \right \rceil\)

\(\Delta\) odd:
- add vertices and edges to obtain Eulerian
- take Euler tour
- all visited \(\leq \frac{\Delta + 1}{2}\) times
- start-vertex once more
- start on added vertex
- \(\left\lceil \frac{\Delta + 1}{2} \right \rceil\)
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What is known and what is open
\[ \mathcal{F} = \{ \text{forests} \} \]

\[ \mathcal{P} = \{ \text{pseudo-forests} \} \]

\[ \mathcal{S} = \{ \text{star forests} \} \]
\[ F = \{ \text{forests} \} \]
\[ P = \{ \text{pseudo-forests} \} \]
\[ S = \{ \text{star forests} \} \]

**Arboricity**

\[ c_g^F (G) = a(G) \]

[Nash-Williams ’64]

\[ a(G) = \max_{S \subseteq V(G)} \left\lfloor \frac{|E[S]|}{|S| - 1} \right\rfloor \]
\[ F = \{ \text{forests} \} \]

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\[ S = \{ \text{star forests} \} \]

**Arboricity**

\[ c^F_g (G') = a(G') \]

**Pseudo-Arboricity**

\[ c^P_g (G') = p(G') \]

[Nash-Williams '64]

\[
a(G) = \max_{S \subseteq V(G)} \left\lfloor \frac{|E[S]|}{|S| - 1} \right\rfloor
\]

[Picard et al. '82]

\[
p(G) = \max_{S \subseteq V(G)} \left\lfloor \frac{|E[S]|}{|S|} \right\rfloor
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**Arboricity**
\[ c_{\mathcal{F}}^g (G') = a(G') \]

**Pseudo-Arboricity**
\[ c_{\mathcal{P}}^g (G') = p(G) \]

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\[ a(G) = \max_{S \subseteq V(G)} \left\lceil \frac{|E[S]|}{|S| - 1} \right\rceil \]

[Picard et al. ’82]  
\[ p(G) = \max_{S \subseteq V(G)} \left\lceil \frac{|E[S]|}{|S|} \right\rceil \]

\[ p(G) \leq a(G) \leq p(G') + 1 \]
\[ F = \{ \text{forests} \} \]

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**Arboricity**

\[ c_g^F(G) = a(G) \]

**Pseudo-Arboricity**

\[ c_g^P(G) = p(G) \]

**Star Arboricity**

\[ c_g^S(G) = sa(G) \]

[Nash-Williams '64]  
[Picard et al. '82]

\[
 a(G) = \max_{S \subseteq V(G)} \left[ \frac{|E[S]|}{|S| - 1} \right] \quad p(G) = \max_{S \subseteq V(G)} \left[ \frac{|E[S]|}{|S|} \right]
\]

\[
p(G) \leq a(G) \leq p(G) + 1
\]
\[ c_{g}^{\mathcal{F}}(G) = a(G) \]

Arboricity

\[ c_{g}^{\mathcal{P}}(G) = p(G) \]

Pseudo-Arboricity

\[ c_{g}^{\mathcal{S}}(G) = sa(G) \]

Star Arboricity

\[ a(G) = \max_{S \subseteq V(G)} \left\lceil \frac{|E[S]|}{|S| - 1} \right\rceil \]

[Nash-Williams ’64]

\[ p(G) = \max_{S \subseteq V(G)} \left\lceil \frac{|E[S]|}{|S|} \right\rceil \]

[Picard et al. ’82]

Local Star Arboricity

\[ p(G) \leq a(G) \leq sa_{\ell}(G) \leq p(G) + 1 \]
\[ \mathcal{F} = \{ \text{forests} \} \]
\[ \mathcal{P} = \{ \text{pseudo-forests} \} \]
\[ \mathcal{S} = \{ \text{star forests} \} \]

**Arboricity**
\[ c_{g}^{\mathcal{F}}(G) = a(G) \]

[Nash-Williams '64]
\[ a(G) = \max_{S \subseteq \mathcal{V}(G)} \left[ \frac{|E[S]|}{|S| - 1} \right] \]

**Pseudo-Arboricity**
\[ c_{g}^{\mathcal{P}}(G) = p(G) \]

[Picard et al. '82]
\[ p(G) = \max_{S \subseteq \mathcal{V}(G)} \left[ \frac{|E[S]|}{|S|} \right] \]

**Star Arboricity**
\[ c_{g}^{\mathcal{S}}(G) = sa(G) \]

Local Star Arboricity
\[ c_{\ell}^{\mathcal{S}}(G) = sa_{\ell}(G) \]

\[ p(G) \leq a(G) \leq sa_{\ell}(G) \leq p(G) + 1 \]
**Thm.** We have \( p(G) \leq a(G) \leq sa_\ell(G) \leq p(G) + 1. \)

(where any of these inequalities can be strict)

Moreover, \( p(G) = sa_\ell(G) \) iff \( G \) has an orientation with:

- \( \text{outdeg}(v) \leq p(G) \) for every \( v \in V(G) \)
- \( \text{outdeg}(v) = p(G) \) only if \( \text{deg}(v) = p(G) \)
**Thm.:** We have \[ p(G) \leq a(G) \leq sa_{\ell}(G) \leq p(G) + 1. \]

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**Proofsketch:**

[Diagram of graphs showing different orientations and degrees]
**Thm.:** We have\[ p(G) \leq a(G) \leq sa_\ell(G) \leq p(G) + 1. \]
(where any of these inequalities can be strict)

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**Proofsketch:**

orient edges towards center

\[ p(G) \leq sa_\ell(G) \]

outdeg\((v) \leq sa_\ell(G)\)
**Thm.:** We have  
\[ p(G) \leq a(G) \leq sa_\ell(G) \leq p(G) + 1. \]  
(where any of these inequalities can be strict)  

Moreover,  
\[ p(G) = sa_\ell(G) \]  
iff \( G \) has an orientation with:  
- \( \text{outdeg}(v) \leq p(G) \) for every \( v \in V(G) \)  
- \( \text{outdeg}(v) = p(G) \) only if \( \text{deg}(v) = p(G) \)

**Proofsketch:**

- Orient edges towards the center.
- Orient the stars of incoming edges.

\[ p(G) \leq sa_\ell(G) \leq p(G) + 1 \]
Thm.: We have $p(G) \leq a(G) \leq sa_\ell(G) \leq p(G) + 1$.

(where any of these inequalites can be strict)

Moreover, $p(G) = sa_\ell(G)$ iff $G$ has an orientation with:

- outdeg($v$) $\leq p(G)$ for every $v \in V(G)$
- outdeg($v$) $= p(G)$ only if $\deg(v) = p(G)$

Remains to show $a(G) \leq sa_\ell(G)$:

- W.l.o.g. $p(G) = sa_\ell(G)$
- Orientation with max outdeg $p(G)$
  attained only at degree-$p(G)$ vertices
- Remove degree-$p(G)$ vertices
- $p(G') \leq p(G) - 1$, thus $a(G') \leq p(G')$
- Reinsert degree-$p(G)$ vertices
- $a(G) \leq p(G) = sa_\ell(G)$
Thm.: We have $p(G) \leq a(G) \leq sa_\ell(G) \leq p(G) + 1$.

(where any of these inequalities can be strict)

Moreover, $p(G) = sa_\ell(G)$ iff $G$ has an orientation with:

- $\text{outdeg}(v) \leq p(G)$ for every $v \in V(G)$
- $\text{outdeg}(v) = p(G)$ only if $\text{deg}(v) = p(G)$

Remains to show $a(G) \leq sa_\ell(G)$:

- W.l.o.g. $p(G) = sa_\ell(G)$
- Orientation with max outdeg $p(G)$ attained only at degree-$p(G)$ vertices
- Remove degree-$p(G)$ vertices
- $p(G') \leq p(G) - 1$, thus $a(G') \leq p(G')$
- Reinsert degree-$p(G)$ vertices
- $a(G) \leq p(G) = sa_\ell(G)$

every edge into a different forest
Theorem
We have $p(G) \leq a(G) \leq sa_\ell(G) \leq p(G) + 1$.

Corollary
Local star arboricity can be computed in polynomial time.

[Hakimi, Mitchem, Schmeichel ’96]
Deciding $sa(G) \leq 2$ is NP-complete.

[Alon, McDiarmid, Reed ’92]
$sa(G) \leq 2a(G)$ and this is best possible.
Global, Local, and Folded Covers
  Templates = Interval Graphs

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What is known and what is open
What else is known

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KU '12
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- Kostochka, West '99
- Algor, Alon '89
- Alon et. al. '92
- Gonçalves '07
- Scheinermann, West '83
- Ding et. al. '98
- KU '12
- Hakimi et. al. '96
What is open

Local

Linear Arboricity Conjecture

$$\text{lala}(G) \leq \left\lceil \frac{\Delta+1}{2} \right\rceil$$

Local track number of planars

$$3 \leq t_\ell \leq 4$$

How much can $c_\ell^T(G)$ and $c_f^T(G)$ differ?

Are there $T$ and $k$, where $c_g^T(G) \leq k$ is poly, but $c_\ell^T(G) \leq k$ or $c_f^T(G) \leq k$ NP-hard?
What is open

Local Linear Arboricity Conjecture

\[ l\alpha_l(G) \leq \left\lceil \frac{\Delta + 1}{2} \right\rceil \]

Local track number of planars

\[ 3 \leq t_l \leq 4 \]

How much can \( c_T^\ell(G) \) and \( c_f^\ell(G) \) differ?

Are there \( T \) and \( k \), where \( c_g^T(G) \leq k \) is poly, but \( c_T^\ell(G) \leq k \) or \( c_f^T(G) \leq k \) NP-hard?

...three ways to pack a graph