Towards an upper bound theorem for the Minkowski sum of convex polytopes

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joint & on-going work with Eleni Tzanaki and Christos Konaxis

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The general problem

- We are given r convex d-polytopes $P_1, P_2, \ldots, P_r$ in $\mathbb{E}^d$ let $P = P_1 \oplus P_2 \oplus \cdots \oplus P_r$ be the Minkowski sum of these polytopes.

Question

What is the (exact) maximum number of k-faces $f_k(P)$ of P, where $0 \leq k \leq d-1$?

- In other words we seek to find a function $\Phi_k(d,r)$ such that, for all possible $P_1, P_2, \ldots, P_r$, we have

$$f_k(P) \leq \Phi_k(d,r)$$

and $\Phi_k(d,r)$ is as small as possible (ideally: *tight*).

Minkowski sums

- Given two sets P and Q, their Minkowski sum is defined as

$$P \oplus Q = \{p + q \,|\, p \in P, q \in Q\}.$$

[Image from www.cgal.org]

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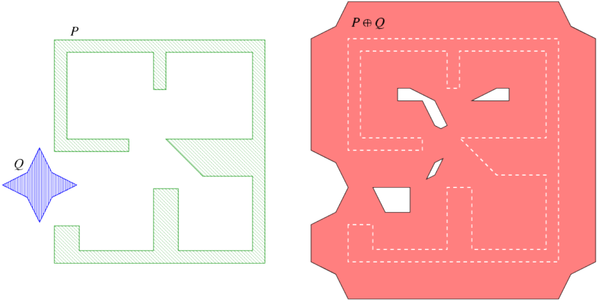

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② Important in many many applications. To name a few:

- Combinatorial Geometry, Computational Geometry, Computer Algebra
- Graphics, Robotics, Motion Planning, Assembly Planning, Computer-Aided Design
- Game Theory, Biology, Operations Research

Some facts

❶ If P_1 and P_2 are convex, then $P_1 \oplus P_2$ is also convex, i.e., $P_1 \oplus P_2 = \mathrm{conv}(\{p + q \,|\, p \in P_1, q \in P_2\})$.

- In particular, if P_1 and P_2 are convex polytopes, so is $P_1 \oplus P_2$.

❷ For the convex polytope case, $f_k(P_1 \oplus P_2)$ is maximized if P_1 and P_2 are in *general position* (cf. [Fukuda & Weibel 2007]).

Bounds in $\mathbb{E}^2$

- For 2-polytopes (polygons) the following worst-case bounds are well known (at least since the 1990's):
  - If $P_1, P_2, \ldots, P_r$ are convex, then:
$$f_k(P_1 \oplus P_2 \oplus \cdots \oplus P_r) \leq \sum_{i=1}^{r} n_i, \qquad k = 0, 1.$$
  - If P_1 is convex and P_2 is non-convex, then:
$$f_k(P_1 \oplus P_2) = \Theta(n_1 n_2), \qquad k = 0, 1.$$
  - If both P_1 and P_2 are non-convex, then:
$$f_k(P_1 \oplus P_2) = \Theta(n_1^2 n_2^2), \qquad k = 0, 1.$$

where n_i is the number of vertices (or edges) of P_i.

Asymptotic bounds in $\mathbb{E}^3$

- For 3-polytopes the following worst-case asymptotic bounds are known (see, e.g., [Fogel, Halperin & Weibel 2009]):
  - If both P_1 and P_2 are convex, the complexity of $P_1 \oplus P_2$ is in $\Theta(n_1 n_2)$.
  - If both P_1 and P_2 are non-convex, the complexity of $P_1 \oplus P_2$ is in $\Theta(n_1^3 n_2^3)$.
- For two 3-polytopes where one is convex and one is non-convex, see, e.g., [Sharir 2004].

Asymptotic bounds in $\mathbb{E}^d$

- Utilizing the *Cayley trick* it is easy to deduce that the complexity of $P_1 \oplus P_2$ is in $O((n_1 + n_2)^{\lfloor \frac{d+1}{2} \rfloor})$.
  - For $d \geq 2$ even this is tight.
  - It is also tight for $n_1 = n_2 = \Theta(n)$.

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- For $d \geq 3$ odd, the worst-case complexity of $P_1 \oplus P_2$ is in $\Theta(n_1 n_2^{\lfloor \frac{d}{2} \rfloor} + n_2 n_1^{\lfloor \frac{d}{2} \rfloor})$ (cf. [K & Tzanaki 2011a]).

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- For $r \geq d \geq 2$, [Weibel 2012] has shown that the the number of vertices of $P_1 \oplus \cdots \oplus P_r$ is at most $\binom{r}{d-1} n^{d-1}$, where n is the number of vertices of each polytope.
- For $2 \leq r \leq d-1$, we have shown that the worst-case complexity of $P_1 \oplus \cdots \oplus P_r$ is in $\Theta(n^{\lfloor\frac{d+r-1}{2}\rfloor})$, where, again, n is the number of vertices of each polytope (cf. [K & Tzanaki 2011b].

Summary of known exact tight worst-case bounds

Exact tight worst-case bounds for the number of faces of the Minkowski sum of r d-polytopes are known in the following cases:

d	r	k	in terms of:
≥ 2	≥ 2	$0, \ldots, d-1$ (all faces)	# of non-parallel edges
2	2	$0, 1$ (all faces)	# of vertices or # of facets
3	2	$0, 1, 2$ (all faces)	# of vertices or # of facets
3	≥ 2	2 (facets)	# of facets
≥ 2	≥ 2	0 (vertices)	# of vertices
≥ 4	$2, \ldots, \lfloor \frac{d}{2} \rfloor$	$0, \ldots, \lfloor \frac{d}{2} \rfloor - r$	# of vertices
≥ 3	$2, \ldots, d-1$	$0, \ldots, \lfloor \frac{d+r-1}{2} \rfloor - r$	# of vertices
≥ 2	$2, 3$	$0, \ldots, d-1$ (all faces)	# of vertices
≥ 2	≥ 4	$0, \ldots, d-1$ (all faces)	# of vertices

- *This talk: $d \geq 2$, $r = 2, 3$ and $0 \leq k \leq d-1$ (in terms of the number of vertices of the d-polytopes).*

Our results for two polytopes (in more detail)

Result

Let P_1, P_2 be d-polytopes, $d \geq 2$, with $n_j \geq d+1$ vertices, $j = 1, 2$. Then:

$$f_{k-1}(P_1 \oplus P_2) \leq f_k(C_{d+1}(n_1+n_2)) - \sum_{i=0}^{\lfloor \frac{d+1}{2} \rfloor} \binom{d+1-i}{k+1-i} \sum_{j=1}^{2} \binom{n_j-d-2+i}{i},$$

where $1 \leq k \leq d$, and $C_d(n)$ stands for the cyclic d-polytope with n vertices. These bounds are tight.

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Let P_1, P_2 *be* d*-polytopes,* $d \geq 2$*, with* $n_j \geq d+1$ *vertices,* $j = 1, 2$*. Then:*

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where $1 \leq k \leq d$*, and* $C_d(n)$ *stands for the cyclic* d*-polytope with* n *vertices. These bounds are tight.*

Theorem (Upper Bound Theorem [McMullen 1970])

Let P *be a* d*-polytope,* $d \geq 2$*, with* $n \geq d+1$ *vertices. Then:*

$$f_{k-1}(P) \leq f_{k-1}(C_d(n)) = \sum_{i=0}^{\frac{d}{2}}{}^{*} \left(\binom{d-i}{k-i} + \binom{i}{k-d+i} \right) \binom{n-d-1+i}{i}$$

where $1 \leq k \leq d$*, and* $C_d(n)$ *stands for the cyclic* d*-polytope with* n *vertices.*

Our results for three polytopes (again in detail)

Result

Let P_1, P_2, P_3 *be* d*-polytopes,* $d \geq 2$*, with* $n_j \geq d+1$ *vertices,* $j = 1, 2, 3$*. Then:*

$$f_{k-1}(P_1 \oplus P_2 \oplus P_3) \leq f_{k+1}(C_{d+2}(n_{[3]})) - \sum_{i=0}^{\lfloor \frac{d+2}{2} \rfloor} \binom{d+2-i}{k+2-i} \sum_{\emptyset \subset S \subset [3]} (-1)^{|S|} \binom{n_S - d - 3 + i}{i} - \delta \binom{\lfloor \frac{d}{2} \rfloor + 1}{k - \lfloor \frac{d}{2} \rfloor} \sum_{i=1}^{3} \binom{n_i - \lfloor \frac{d}{2} \rfloor - 2}{\lfloor \frac{d}{2} \rfloor + 1}$$

where $[3] = \{1, 2, 3\}$, $\delta = d - 2\lfloor \frac{d}{2} \rfloor$, $1 \leq k \leq d$, *and* $n_S = \sum_{i \in S} n_i$, $\emptyset \subset S \subseteq [3]$*.*
These bounds are tight.

f-vectors/h-vectors/g-vectors

- Given a d-polytope P, its f-vector $\boldsymbol{f}(P)$ is the $(d+1)$-dimensional vector
$$\boldsymbol{f}(P) = (f_{-1}(P), f_0(P), \ldots, f_{d-1}(P))$$
  - $f_k(P)$ is the number of k-faces of P; $f_{-1}(P) = 1$ (empty set).
- The h-vector $\boldsymbol{h}(P)$ of a simplicial d-polytope P is the $(d+1)$-dimensional vector
$$\boldsymbol{h}(P) = (h_0(P), h_1(P), \ldots, h_d(P))$$
where
$$h_k(P) := \textstyle\sum_{i=0}^{k}(-1)^{k-i}\binom{d-i}{d-k}f_{i-1}(P), \quad 0 \leq k \leq d.$$
- The g-vector of P is the $(\lfloor\frac{d}{2}\rfloor + 1)$-dimensional vector
$$\boldsymbol{g}(P) = (g_0(P), g_1(P), \ldots, g_{\lfloor\frac{d}{2}\rfloor}(P))$$
where $g_0(P) = 1$, and $g_k(P) = h_k(P) - h_{k-1}(P)$, $1 \leq k \leq \lfloor\frac{d}{2}\rfloor$.

Dehn-Sommerville eqs. and bounds on $\boldsymbol{f}(P)$, $\boldsymbol{h}(P)$, $\boldsymbol{g}(P)$

- For every simplicial d-polytope P, the, so called, *Dehn-Sommerville equations* hold:

$$h_{d-k}(P) = h_k(P), \qquad 0 \leq k \leq d$$

- For all $k \geq 0$ we have: $f_{k-1}(P) \leq \binom{n}{k}$, where n is the number of vertices of P.
- For all $k \geq 0$ we have: $h_k(P) \leq \binom{n-d-1+k}{k}$.
- For all $k \geq 0$ we have: $g_k(P) \leq \binom{n-d-2+k}{k}$.
- $g_{d+1-k}(P) = -g_k(P)$, $0 \leq k \leq d+1$ (we can extend the definition of $\boldsymbol{g}(P)$ using the Dehn-Sommerville equations for P).
- The maximal values for the f-, h-, and g-vector of a polytope P are all attained *"simultaneously"*, and, in particular, when P is a neighborly polytope.

Shellings

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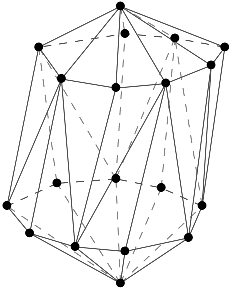

Definition

Let $\mathcal{C}$ be a pure simplicial polytopal d-complex. A shelling $\mathbb{S}(\mathcal{C})$ of $\mathcal{C}$ is a linear ordering $F_1, F_2, \ldots, F_s$ of the facets of $\mathcal{C}$ such that for every $i < j$ there exists an $\ell < j$ such that the intersection $F_i \cap F_j$ is contained in $F_\ell \cap F_j$, and such that $F_\ell \cap F_j$ is a facet of F_j.

- Every polytopal complex that has a shelling is called *shellable*.
- The boundary complex of a polytope of always shellable (cf. [Bruggesser & Mani 1971]).
- Given a shellable complex $\mathcal{C}$, the star/link of a vertex $v \in \mathrm{vert}(\mathcal{C})$ is also shellable.

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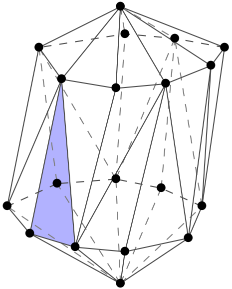

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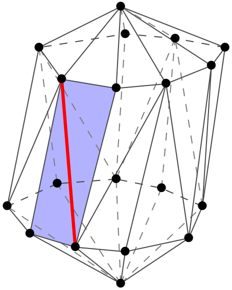

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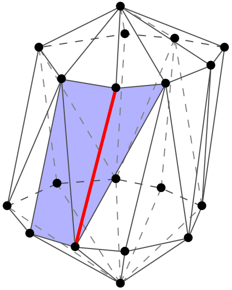

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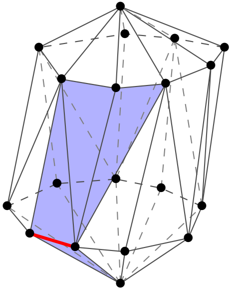

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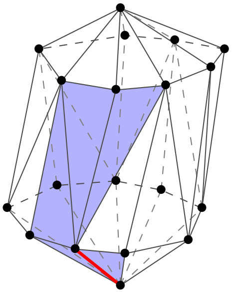

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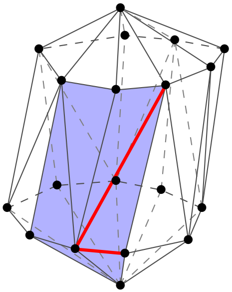

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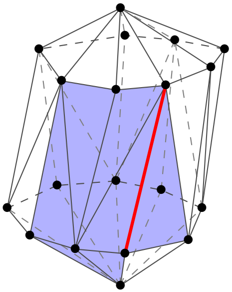

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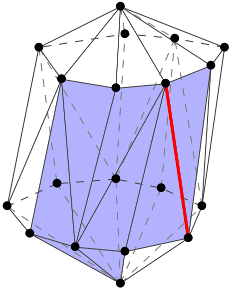

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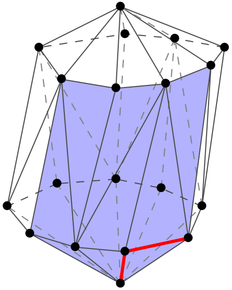

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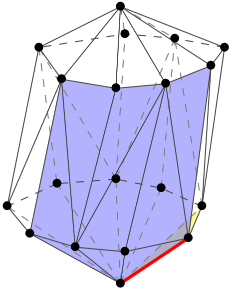

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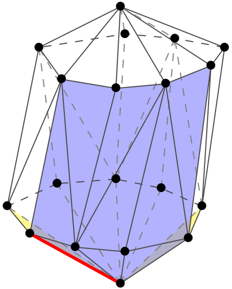

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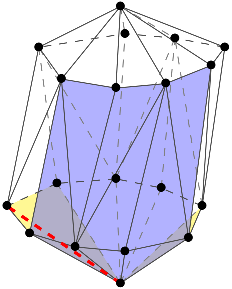

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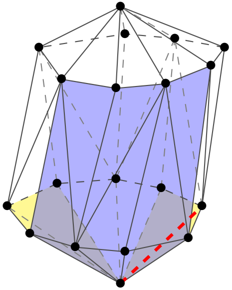

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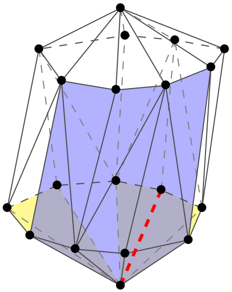

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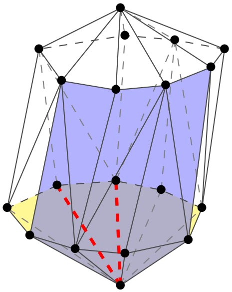

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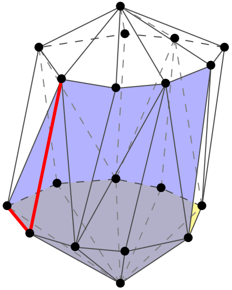

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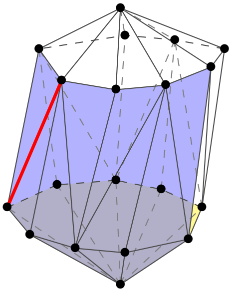

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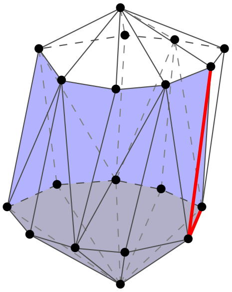

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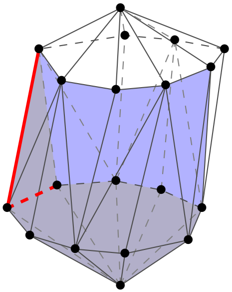

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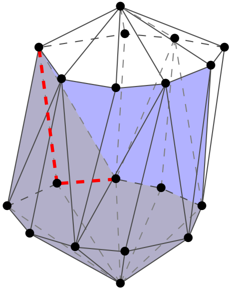

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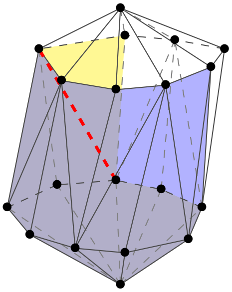

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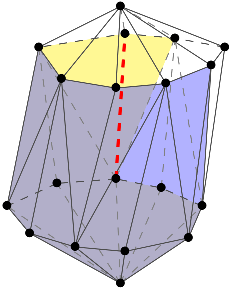

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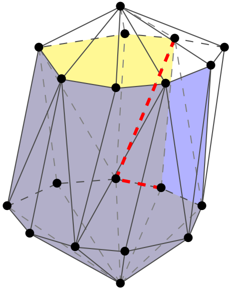

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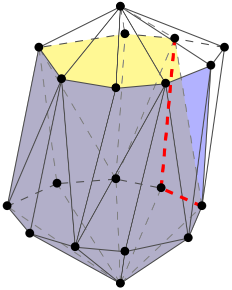

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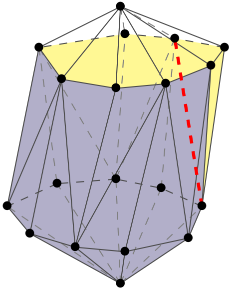

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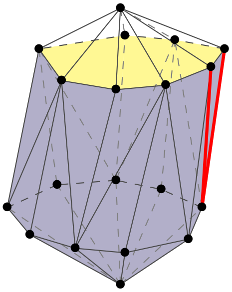

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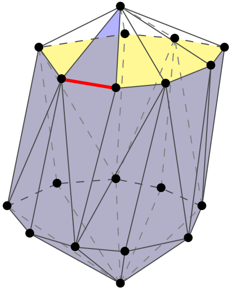

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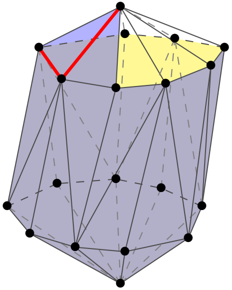

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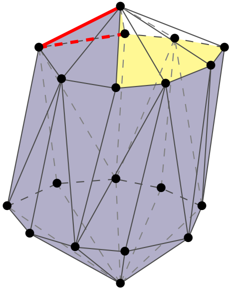

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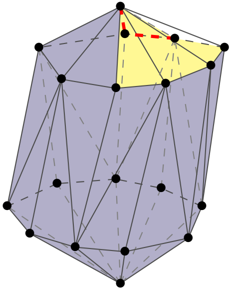

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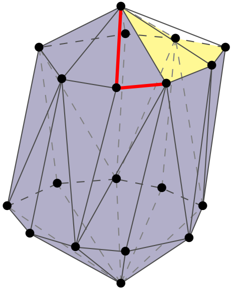

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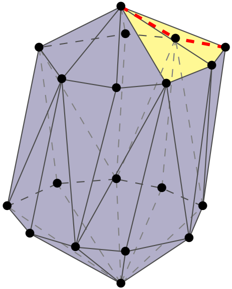

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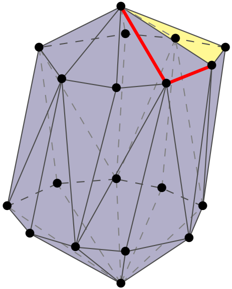

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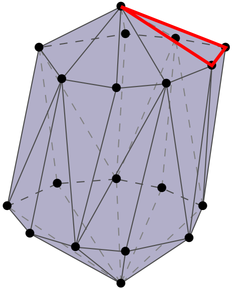

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Shellings, restrictions and h-vectors

Consider a pure shellable simplicial polytopal complex $\mathcal{C}$ and let $\mathbb{S}(\mathcal{C}) = \{F_1, \ldots, F_s\}$ be a shelling order of its facets.

- The *restriction* $R(F_j)$ of a facet F_j is the set of all vertices $v \in F_j$ such that $F_j \setminus \{v\}$ is contained in one of the earlier facets.

  Also $R(F_1) = \emptyset$ and $R(F_i) \neq R(F_j)$ for all $i \neq j$.
- The vertex set $R(F_j)$ forms a face G of F_j. G is called the *minimal new face* at the j-th shelling step.
- For a polytope P, $h_k(P)$ counts the number of facets of P whose restriction has size k (and this is independent of the chosen shelling).

f-vectors, h-vectors and shellings: an example

Skip figures

$\boldsymbol{f}(\mathcal{Q}) = (1, 19, 51, 34)$

$\boldsymbol{h}(\mathcal{Q}) = (1, 16, 16, 1)$

$\boldsymbol{f}(\mathcal{Q}/v) = (1, 6, 6)$

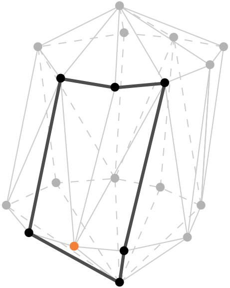
$\boldsymbol{h}(\mathcal{Q}/v) = (1, 4, 1)$

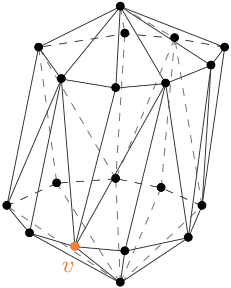
$h_0(\mathcal{Q}/v) \leq h_0(\mathcal{Q})$ $\boxed{0} \leq \boxed{0}$

$h_1(\mathcal{Q}/v) \leq h_1(\mathcal{Q})$ $\boxed{0} \leq \boxed{0}$

$h_2(\mathcal{Q}/v) \leq h_2(\mathcal{Q})$ $\boxed{0} \leq \boxed{0}$

$h_3(\mathcal{Q}/v) \leq h_3(\mathcal{Q})$ $\boxed{0} \leq \boxed{0}$





f-vectors, h-vectors and shellings: an example

Skip figures

$\boldsymbol{f}(\mathcal{Q}) = (1, 19, 51, 34)$

$\boldsymbol{h}(\mathcal{Q}) = (1, 16, 16, 1)$

$\boldsymbol{f}(\mathcal{Q}/v) = (1, 6, 6)$

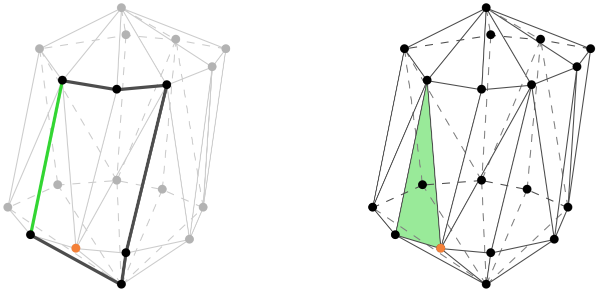
$\boldsymbol{h}(\mathcal{Q}/v) = (1, 4, 1)$

$h_0(\mathcal{Q}/v) \leq h_0(\mathcal{Q})$ $\quad \boxed{1} \leq \boxed{1}$

$h_1(\mathcal{Q}/v) \leq h_1(\mathcal{Q})$ $\quad \boxed{0} \leq \boxed{0}$

$h_2(\mathcal{Q}/v) \leq h_2(\mathcal{Q})$ $\quad \boxed{0} \leq \boxed{0}$

$h_3(\mathcal{Q}/v) \leq h_3(\mathcal{Q})$ $\quad \boxed{0} \leq \boxed{0}$



f-vectors, h-vectors and shellings: an example

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$\boldsymbol{f}(\mathcal{Q}) = (1, 19, 51, 34)$

$\boldsymbol{h}(\mathcal{Q}) = (1, 16, 16, 1)$

$\boldsymbol{f}(\mathcal{Q}/v) = (1, 6, 6)$

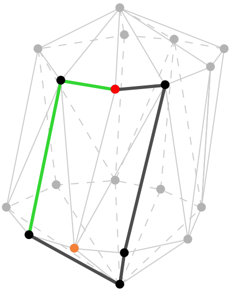
$\boldsymbol{h}(\mathcal{Q}/v) = (1, 4, 1)$

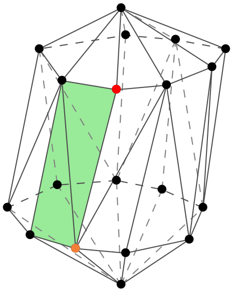
$h_0(\mathcal{Q}/v) \leq h_0(\mathcal{Q})$ $\boxed{1} \leq \boxed{1}$

$h_1(\mathcal{Q}/v) \leq h_1(\mathcal{Q})$ $\boxed{1} \leq \boxed{1}$

$h_2(\mathcal{Q}/v) \leq h_2(\mathcal{Q})$ $\boxed{0} \leq \boxed{0}$

$h_3(\mathcal{Q}/v) \leq h_3(\mathcal{Q})$ $\boxed{0} \leq \boxed{0}$





f-vectors, h-vectors and shellings: an example

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$\boldsymbol{f}(\mathcal{Q}) = (1, 19, 51, 34)$

$\boldsymbol{h}(\mathcal{Q}) = (1, 16, 16, 1)$

$\boldsymbol{f}(\mathcal{Q}/v) = (1, 6, 6)$

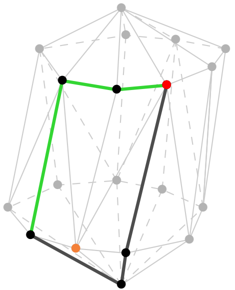
$\boldsymbol{h}(\mathcal{Q}/v) = (1, 4, 1)$

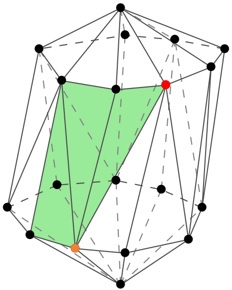
$h_0(\mathcal{Q}/v) \leq h_0(\mathcal{Q})$ $\boxed{1} \leq \boxed{1}$

$h_1(\mathcal{Q}/v) \leq h_1(\mathcal{Q})$ $\boxed{2} \leq \boxed{2}$

$h_2(\mathcal{Q}/v) \leq h_2(\mathcal{Q})$ $\boxed{0} \leq \boxed{0}$

$h_3(\mathcal{Q}/v) \leq h_3(\mathcal{Q})$ $\boxed{0} \leq \boxed{0}$





f-vectors, h-vectors and shellings: an example

Skip figures

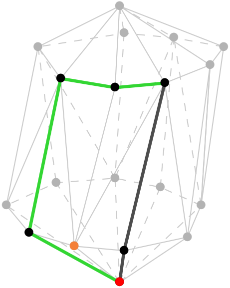
$\boldsymbol{f}(\mathcal{Q}) = (1, 19, 51, 34)$

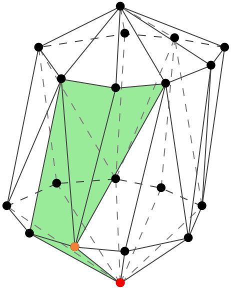
$\boldsymbol{h}(\mathcal{Q}) = (1, 16, 16, 1)$

$\boldsymbol{f}(\mathcal{Q}/v) = (1, 6, 6)$

$\boldsymbol{h}(\mathcal{Q}/v) = (1, 4, 1)$

$h_0(\mathcal{Q}/v) \leq h_0(\mathcal{Q})$	$\boxed{1} \leq \boxed{1}$
$h_1(\mathcal{Q}/v) \leq h_1(\mathcal{Q})$	$\boxed{3} \leq \boxed{3}$
$h_2(\mathcal{Q}/v) \leq h_2(\mathcal{Q})$	$\boxed{0} \leq \boxed{0}$
$h_3(\mathcal{Q}/v) \leq h_3(\mathcal{Q})$	$\boxed{0} \leq \boxed{0}$





f-vectors, h-vectors and shellings: an example

Skip figures

$\boldsymbol{f}(\mathcal{Q}) = (1, 19, 51, 34)$

$\boldsymbol{h}(\mathcal{Q}) = (1, 16, 16, 1)$

$\boldsymbol{f}(\mathcal{Q}/v) = (1, 6, 6)$

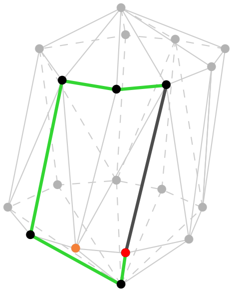
$\boldsymbol{h}(\mathcal{Q}/v) = (1, 4, 1)$

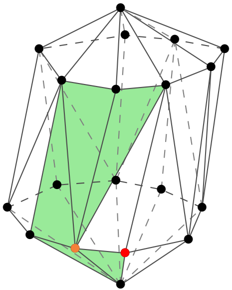
$h_0(\mathcal{Q}/v) \leq h_0(\mathcal{Q})$ $\boxed{1} \leq \boxed{1}$

$h_1(\mathcal{Q}/v) \leq h_1(\mathcal{Q})$ $\boxed{4} \leq \boxed{4}$

$h_2(\mathcal{Q}/v) \leq h_2(\mathcal{Q})$ $\boxed{0} \leq \boxed{0}$

$h_3(\mathcal{Q}/v) \leq h_3(\mathcal{Q})$ $\boxed{0} \leq \boxed{0}$





f-vectors, h-vectors and shellings: an example

Skip figures

$\boldsymbol{f}(\mathcal{Q}) = (1, 19, 51, 34)$

$\boldsymbol{h}(\mathcal{Q}) = (1, 16, 16, 1)$

$\boldsymbol{f}(\mathcal{Q}/v) = (1, 6, 6)$

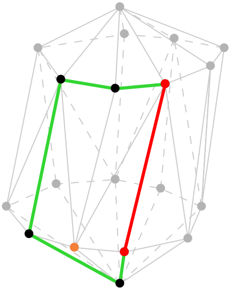
$\boldsymbol{h}(\mathcal{Q}/v) = (1, 4, 1)$

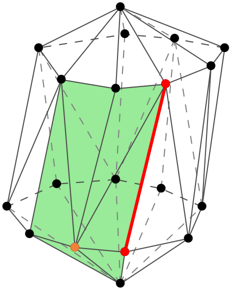
$h_0(\mathcal{Q}/v) \leq h_0(\mathcal{Q})$ $\boxed{1} \leq \boxed{1}$

$h_1(\mathcal{Q}/v) \leq h_1(\mathcal{Q})$ $\boxed{4} \leq \boxed{4}$

$h_2(\mathcal{Q}/v) \leq h_2(\mathcal{Q})$ $\boxed{1} \leq \boxed{1}$

$h_3(\mathcal{Q}/v) \leq h_3(\mathcal{Q})$ $\boxed{0} \leq \boxed{0}$





f-vectors, h-vectors and shellings: an example

Skip figures

$\boldsymbol{f}(\mathcal{Q}) = (1, 19, 51, 34)$

$\boldsymbol{h}(\mathcal{Q}) = (1, 16, 16, 1)$

$\boldsymbol{f}(\mathcal{Q}/v) = (1, 6, 6)$

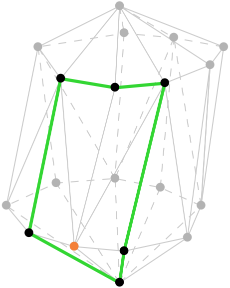
$\boldsymbol{h}(\mathcal{Q}/v) = (1, 4, 1)$

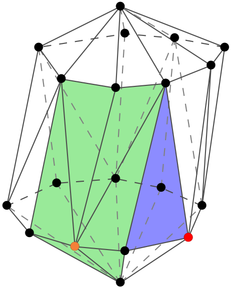
$h_0(\mathcal{Q}/v) \leq h_0(\mathcal{Q})$ $\boxed{1} \leq \boxed{1}$

$h_1(\mathcal{Q}/v) \leq h_1(\mathcal{Q})$ $\boxed{4} \leq \boxed{5}$

$h_2(\mathcal{Q}/v) \leq h_2(\mathcal{Q})$ $\boxed{1} \leq \boxed{1}$

$h_3(\mathcal{Q}/v) \leq h_3(\mathcal{Q})$ $\boxed{0} \leq \boxed{0}$





f-vectors, h-vectors and shellings: an example

Skip figures

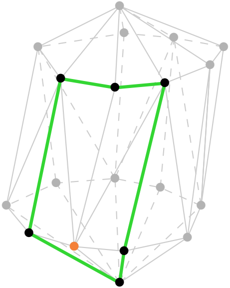
$\boldsymbol{f}(\mathcal{Q}) = (1, 19, 51, 34)$

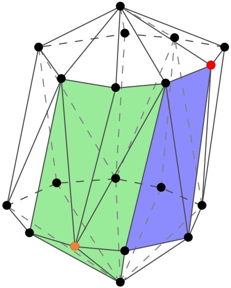
$\boldsymbol{h}(\mathcal{Q}) = (1, 16, 16, 1)$

$\boldsymbol{f}(\mathcal{Q}/v) = (1, 6, 6)$

$\boldsymbol{h}(\mathcal{Q}/v) = (1, 4, 1)$

$h_0(\mathcal{Q}/v) \leq h_0(\mathcal{Q})$	$\boxed{1} \leq \boxed{1}$
$h_1(\mathcal{Q}/v) \leq h_1(\mathcal{Q})$	$\boxed{4} \leq \boxed{6}$
$h_2(\mathcal{Q}/v) \leq h_2(\mathcal{Q})$	$\boxed{1} \leq \boxed{1}$
$h_3(\mathcal{Q}/v) \leq h_3(\mathcal{Q})$	$\boxed{0} \leq \boxed{0}$





f-vectors, h-vectors and shellings: an example

Skip figures

$\boldsymbol{f}(\mathcal{Q}) = (1, 19, 51, 34)$

$\boldsymbol{h}(\mathcal{Q}) = (1, 16, 16, 1)$

$\boldsymbol{f}(\mathcal{Q}/v) = (1, 6, 6)$

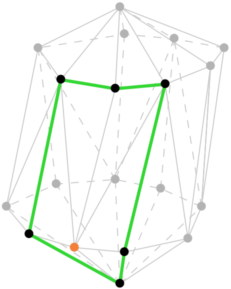
$\boldsymbol{h}(\mathcal{Q}/v) = (1, 4, 1)$

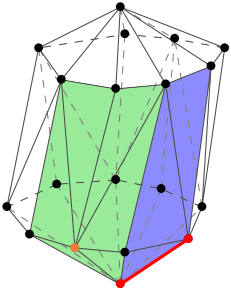
$h_0(\mathcal{Q}/v) \leq h_0(\mathcal{Q})$ $\quad \boxed{1} \leq \boxed{1}$

$h_1(\mathcal{Q}/v) \leq h_1(\mathcal{Q})$ $\quad \boxed{4} \leq \boxed{6}$

$h_2(\mathcal{Q}/v) \leq h_2(\mathcal{Q})$ $\quad \boxed{1} \leq \boxed{2}$

$h_3(\mathcal{Q}/v) \leq h_3(\mathcal{Q})$ $\quad \boxed{0} \leq \boxed{0}$





f-vectors, h-vectors and shellings: an example

Skip figures

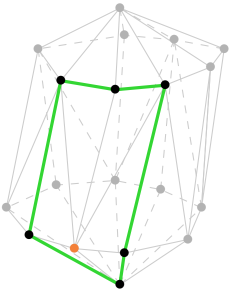
$\boldsymbol{f}(\mathcal{Q}) = (1, 19, 51, 34)$

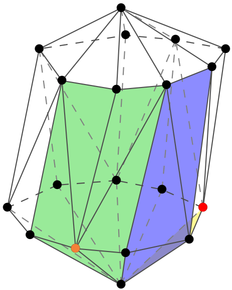
$\boldsymbol{h}(\mathcal{Q}) = (1, 16, 16, 1)$

$\boldsymbol{f}(\mathcal{Q}/v) = (1, 6, 6)$

$\boldsymbol{h}(\mathcal{Q}/v) = (1, 4, 1)$

$h_0(\mathcal{Q}/v) \le h_0(\mathcal{Q})$	$\boxed{1} \le \boxed{1}$
$h_1(\mathcal{Q}/v) \le h_1(\mathcal{Q})$	$\boxed{4} \le \boxed{7}$
$h_2(\mathcal{Q}/v) \le h_2(\mathcal{Q})$	$\boxed{1} \le \boxed{2}$
$h_3(\mathcal{Q}/v) \le h_3(\mathcal{Q})$	$\boxed{0} \le \boxed{0}$

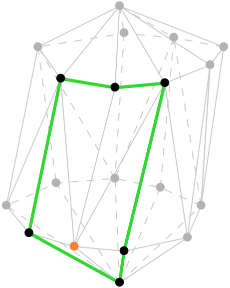


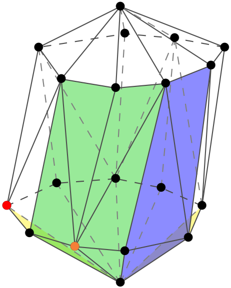


f-vectors, h-vectors and shellings: an example

Skip figures

$\boldsymbol{f}(\mathcal{Q}) = (1, 19, 51, 34)$	$h_0(\mathcal{Q}/v) \leq h_0(\mathcal{Q})$	$\boxed{1} \leq \boxed{1}$
$\boldsymbol{h}(\mathcal{Q}) = (1, 16, 16, 1)$	$h_1(\mathcal{Q}/v) \leq h_1(\mathcal{Q})$	$\boxed{4} \leq \boxed{8}$
$\boldsymbol{f}(\mathcal{Q}/v) = (1, 6, 6)$	$h_2(\mathcal{Q}/v) \leq h_2(\mathcal{Q})$	$\boxed{1} \leq \boxed{2}$
$\boldsymbol{h}(\mathcal{Q}/v) = (1, 4, 1)$	$h_3(\mathcal{Q}/v) \leq h_3(\mathcal{Q})$	$\boxed{0} \leq \boxed{0}$





f-vectors, h-vectors and shellings: an example

Skip figures

$\boldsymbol{f}(\mathcal{Q}) = (1, 19, 51, 34)$

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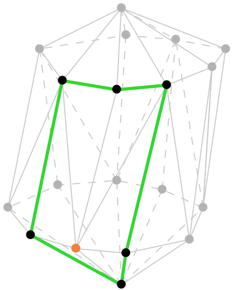
$\boldsymbol{h}(\mathcal{Q}/v) = (1, 4, 1)$

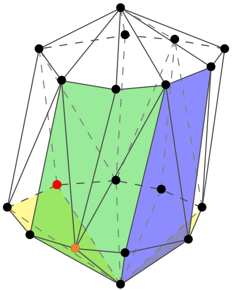
$h_0(\mathcal{Q}/v) \leq h_0(\mathcal{Q})$ $\boxed{1} \leq \boxed{1}$

$h_1(\mathcal{Q}/v) \leq h_1(\mathcal{Q})$ $\boxed{4} \leq \boxed{9}$

$h_2(\mathcal{Q}/v) \leq h_2(\mathcal{Q})$ $\boxed{1} \leq \boxed{2}$

$h_3(\mathcal{Q}/v) \leq h_3(\mathcal{Q})$ $\boxed{0} \leq \boxed{0}$





f-vectors, h-vectors and shellings: an example

Skip figures

$\boldsymbol{f}(\mathcal{Q}) = (1, 19, 51, 34)$

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$\boldsymbol{f}(\mathcal{Q}/v) = (1, 6, 6)$

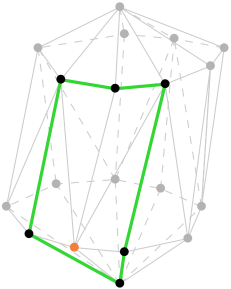
$\boldsymbol{h}(\mathcal{Q}/v) = (1, 4, 1)$

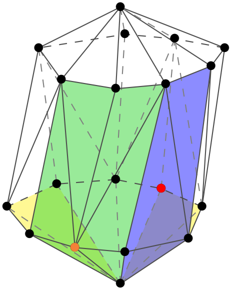
$h_0(\mathcal{Q}/v) \leq h_0(\mathcal{Q})$ $\quad \boxed{1} \leq \boxed{1}$

$h_1(\mathcal{Q}/v) \leq h_1(\mathcal{Q})$ $\quad \boxed{4} \leq \boxed{10}$

$h_2(\mathcal{Q}/v) \leq h_2(\mathcal{Q})$ $\quad \boxed{1} \leq \boxed{2}$

$h_3(\mathcal{Q}/v) \leq h_3(\mathcal{Q})$ $\quad \boxed{0} \leq \boxed{0}$





f-vectors, h-vectors and shellings: an example

▸ Skip figures

$\boldsymbol{f}(\mathcal{Q}) = (1, 19, 51, 34)$

$\boldsymbol{h}(\mathcal{Q}) = (1, 16, 16, 1)$

$\boldsymbol{f}(\mathcal{Q}/v) = (1, 6, 6)$

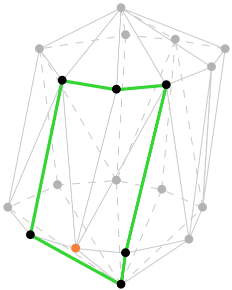
$\boldsymbol{h}(\mathcal{Q}/v) = (1, 4, 1)$

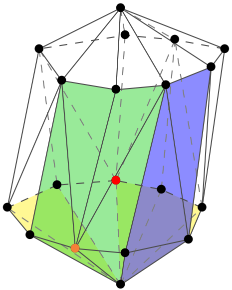
$h_0(\mathcal{Q}/v) \leq h_0(\mathcal{Q})$ $\quad \boxed{1} \leq \boxed{1}$

$h_1(\mathcal{Q}/v) \leq h_1(\mathcal{Q})$ $\quad \boxed{4} \leq \boxed{11}$

$h_2(\mathcal{Q}/v) \leq h_2(\mathcal{Q})$ $\quad \boxed{1} \leq \boxed{2}$

$h_3(\mathcal{Q}/v) \leq h_3(\mathcal{Q})$ $\quad \boxed{0} \leq \boxed{0}$





f-vectors, h-vectors and shellings: an example

Skip figures

$\boldsymbol{f}(\mathcal{Q}) = (1, 19, 51, 34)$

$\boldsymbol{h}(\mathcal{Q}) = (1, 16, 16, 1)$

$\boldsymbol{f}(\mathcal{Q}/v) = (1, 6, 6)$

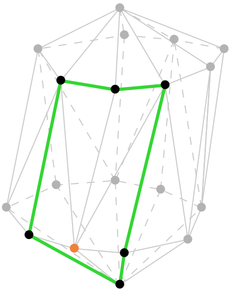
$\boldsymbol{h}(\mathcal{Q}/v) = (1, 4, 1)$

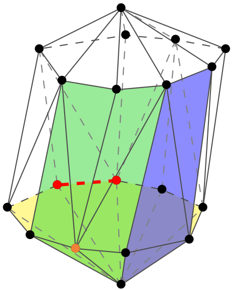
$h_0(\mathcal{Q}/v) \leq h_0(\mathcal{Q})$ $\boxed{1} \leq \boxed{1}$

$h_1(\mathcal{Q}/v) \leq h_1(\mathcal{Q})$ $\boxed{4} \leq \boxed{11}$

$h_2(\mathcal{Q}/v) \leq h_2(\mathcal{Q})$ $\boxed{1} \leq \boxed{3}$

$h_3(\mathcal{Q}/v) \leq h_3(\mathcal{Q})$ $\boxed{0} \leq \boxed{0}$





f-vectors, h-vectors and shellings: an example

Skip figures

$\boldsymbol{f}(\mathcal{Q}) = (1, 19, 51, 34)$

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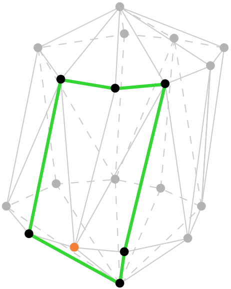
$\boldsymbol{h}(\mathcal{Q}/v) = (1, 4, 1)$

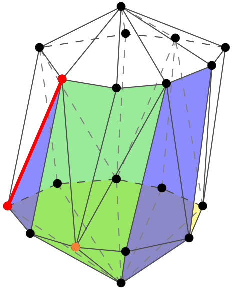
$h_0(\mathcal{Q}/v) \leq h_0(\mathcal{Q})$ $\boxed{1} \leq \boxed{1}$

$h_1(\mathcal{Q}/v) \leq h_1(\mathcal{Q})$ $\boxed{4} \leq \boxed{11}$

$h_2(\mathcal{Q}/v) \leq h_2(\mathcal{Q})$ $\boxed{1} \leq \boxed{4}$

$h_3(\mathcal{Q}/v) \leq h_3(\mathcal{Q})$ $\boxed{0} \leq \boxed{0}$





f-vectors, h-vectors and shellings: an example

Skip figures

$\boldsymbol{f}(\mathcal{Q}) = (1, 19, 51, 34)$

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$\boldsymbol{f}(\mathcal{Q}/v) = (1, 6, 6)$

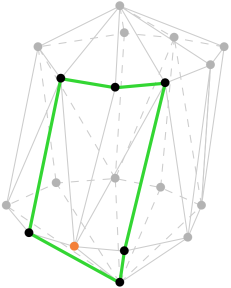
$\boldsymbol{h}(\mathcal{Q}/v) = (1, 4, 1)$

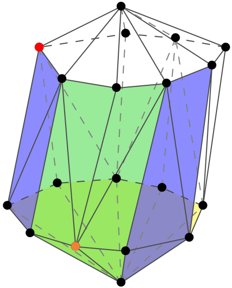
$h_0(\mathcal{Q}/v) \leq h_0(\mathcal{Q})$ $\boxed{1} \leq \boxed{1}$

$h_1(\mathcal{Q}/v) \leq h_1(\mathcal{Q})$ $\boxed{4} \leq \boxed{12}$

$h_2(\mathcal{Q}/v) \leq h_2(\mathcal{Q})$ $\boxed{1} \leq \boxed{4}$

$h_3(\mathcal{Q}/v) \leq h_3(\mathcal{Q})$ $\boxed{0} \leq \boxed{0}$





f-vectors, h-vectors and shellings: an example

Skip figures

$\boldsymbol{f}(\mathcal{Q}) = (1, 19, 51, 34)$

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$\boldsymbol{f}(\mathcal{Q}/v) = (1, 6, 6)$

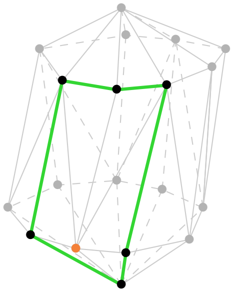
$\boldsymbol{h}(\mathcal{Q}/v) = (1, 4, 1)$

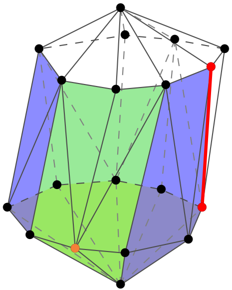
$h_0(\mathcal{Q}/v) \leq h_0(\mathcal{Q})$ $\boxed{1} \leq \boxed{1}$

$h_1(\mathcal{Q}/v) \leq h_1(\mathcal{Q})$ $\boxed{4} \leq \boxed{12}$

$h_2(\mathcal{Q}/v) \leq h_2(\mathcal{Q})$ $\boxed{1} \leq \boxed{5}$

$h_3(\mathcal{Q}/v) \leq h_3(\mathcal{Q})$ $\boxed{0} \leq \boxed{0}$





f-vectors, h-vectors and shellings: an example

Skip figures

$\boldsymbol{f}(\mathcal{Q}) = (1, 19, 51, 34)$

$\boldsymbol{h}(\mathcal{Q}) = (1, 16, 16, 1)$

$\boldsymbol{f}(\mathcal{Q}/v) = (1, 6, 6)$

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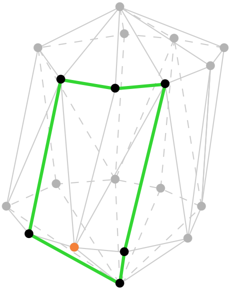

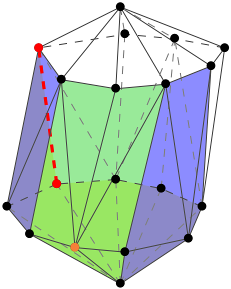

f-vectors, h-vectors and shellings: an example

▸ Skip figures

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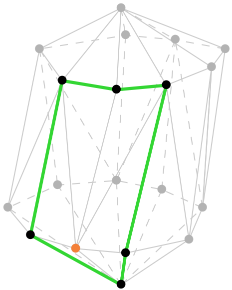
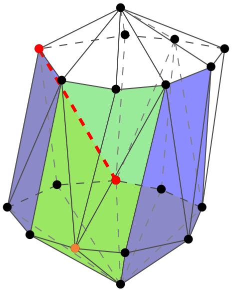
$h_2(\mathcal{Q}/v) \leq h_2(\mathcal{Q})$ $\quad \boxed{1} \leq \boxed{7}$

$h_3(\mathcal{Q}/v) \leq h_3(\mathcal{Q})$ $\quad \boxed{0} \leq \boxed{0}$





f-vectors, h-vectors and shellings: an example

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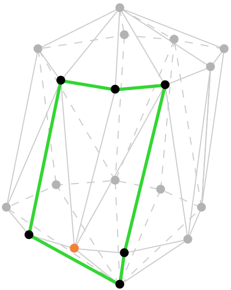
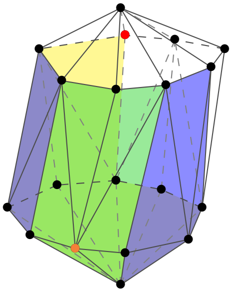
$h_2(\mathcal{Q}/v) \leq h_2(\mathcal{Q})$ $\quad \boxed{1} \leq \boxed{7}$

$h_3(\mathcal{Q}/v) \leq h_3(\mathcal{Q})$ $\quad \boxed{0} \leq \boxed{0}$





f-vectors, h-vectors and shellings: an example

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$h_2(\mathcal{Q}/v) \leq h_2(\mathcal{Q})$ $\quad \boxed{1} \leq \boxed{7}$

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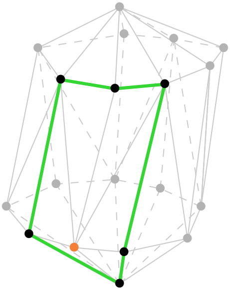

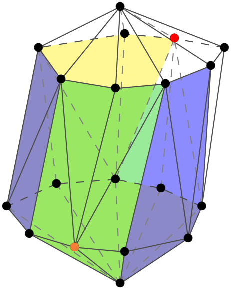

f-vectors, h-vectors and shellings: an example

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$h_2(\mathcal{Q}/v) \leq h_2(\mathcal{Q})$ $\boxed{1} \leq \boxed{8}$

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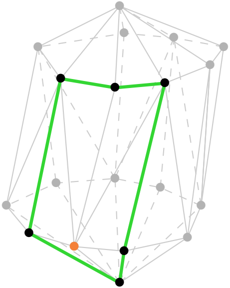

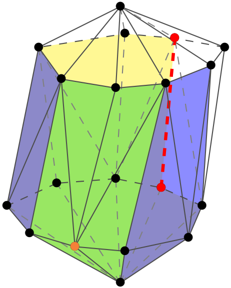

f-vectors, h-vectors and shellings: an example

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$h_2(\mathcal{Q}/v) \leq h_2(\mathcal{Q})$ $\quad \boxed{1} \leq \boxed{9}$

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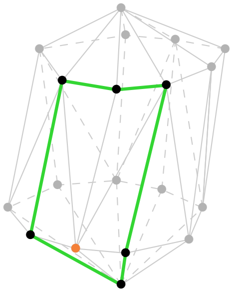

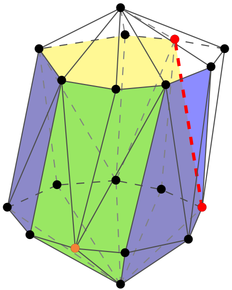

f-vectors, h-vectors and shellings: an example

Skip figures

$\boldsymbol{f}(\mathcal{Q}) = (1, 19, 51, 34)$

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$h_1(\mathcal{Q}/v) \leq h_1(\mathcal{Q})$ $\boxed{4} \leq \boxed{15}$

$h_2(\mathcal{Q}/v) \leq h_2(\mathcal{Q})$ $\boxed{1} \leq \boxed{9}$

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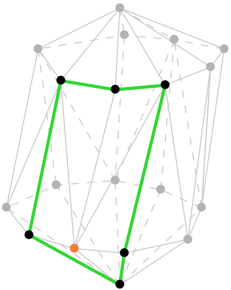

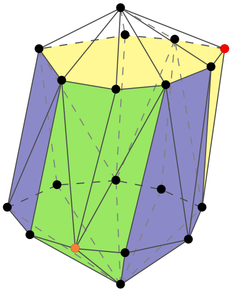

f-vectors, h-vectors and shellings: an example

Skip figures

$\boldsymbol{f}(\mathcal{Q}) = (1, 19, 51, 34)$

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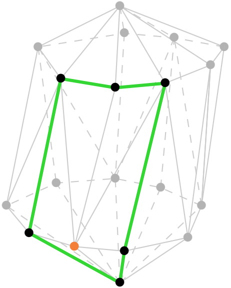
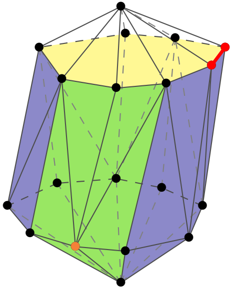
$h_2(\mathcal{Q}/v) \leq h_2(\mathcal{Q})$ $\boxed{1} \leq \boxed{10}$

$h_3(\mathcal{Q}/v) \leq h_3(\mathcal{Q})$ $\boxed{0} \leq \boxed{0}$





f-vectors, h-vectors and shellings: an example

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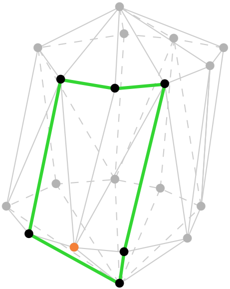
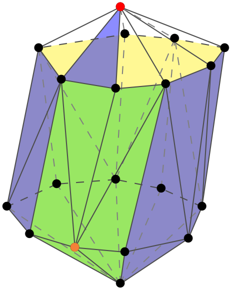
$h_2(\mathcal{Q}/v) \leq h_2(\mathcal{Q})$ $\boxed{1} \leq \boxed{10}$

$h_3(\mathcal{Q}/v) \leq h_3(\mathcal{Q})$ $\boxed{0} \leq \boxed{0}$





f-vectors, h-vectors and shellings: an example

▸ Skip figures

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$h_2(\mathcal{Q}/v) \leq h_2(\mathcal{Q})$ $\boxed{1} \leq \boxed{11}$

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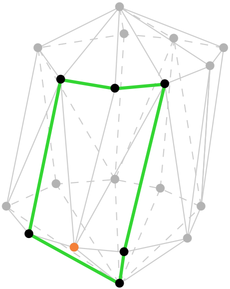

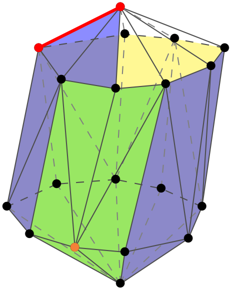

f-vectors, h-vectors and shellings: an example

Skip figures

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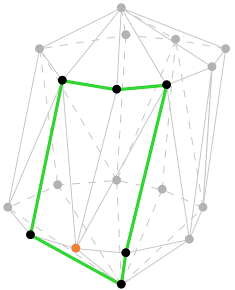

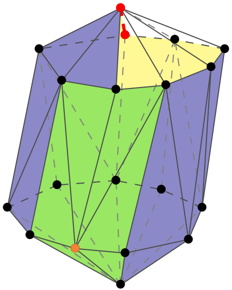

f-vectors, h-vectors and shellings: an example

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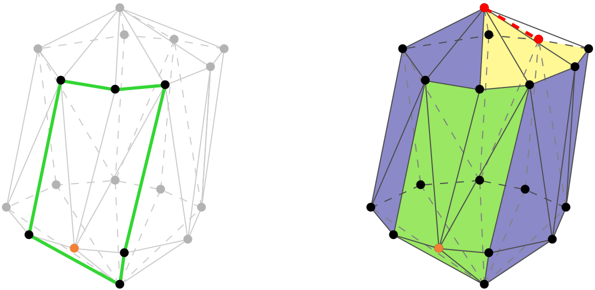

f-vectors, h-vectors and shellings: an example

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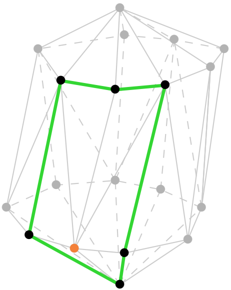

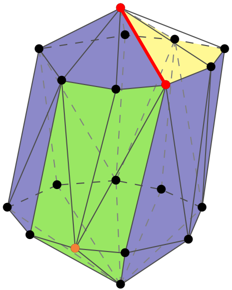

f-vectors, h-vectors and shellings: an example

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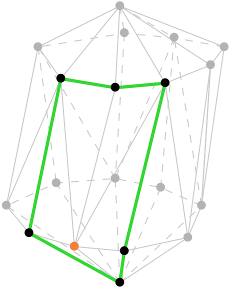

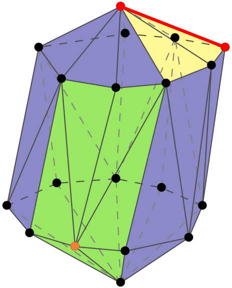

f-vectors, h-vectors and shellings: an example

Skip figures

$\boldsymbol{f}(\mathcal{Q}) = (1, 19, 51, 34)$ $\qquad h_0(\mathcal{Q}/v) \leq h_0(\mathcal{Q})$ $\qquad \boxed{1} \leq \boxed{1}$

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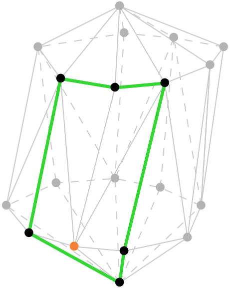

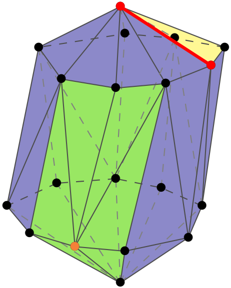

f-vectors, h-vectors and shellings: an example

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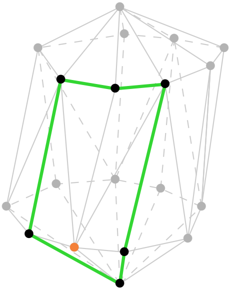
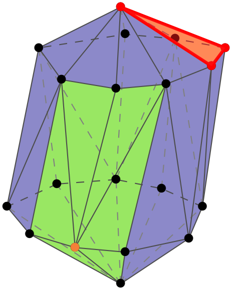


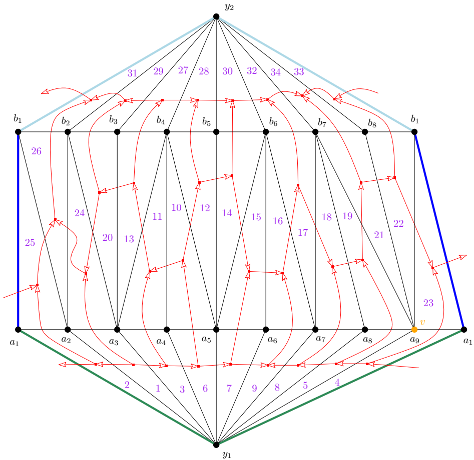


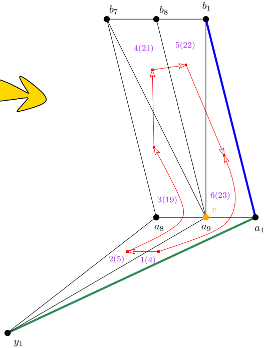
The dual graph $G^{\Delta}(\partial Q)$

$\boldsymbol{h}(\mathcal{Q}) = (1, 16, 16, 1)$

$\boldsymbol{h}(\mathcal{Q}/v) = (1, 4, 1)$

$h_k(\cdot)$ counts the number of vertices in the dual graph with in-degree k.





Using the Dehn-Sommerville equations

For a simplicial d-polytope we have:

$$f_{k-1}(P) = \sum_{i=0}^{d} \binom{d-i}{k-i} h_i(P) = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \binom{d-i}{k-i} h_i(P) + \sum_{i=\lfloor \frac{d}{2} \rfloor+1}^{d} \binom{d-i}{k-i} h_i(P)$$

$$= \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \binom{d-i}{k-i} h_i(P) + \sum_{i=0}^{\lceil \frac{d}{2} \rceil -1} \binom{i}{k-d+i} h_{d-i}(P)$$

$$= \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \binom{d-i}{k-i} h_i(P) + \sum_{i=0}^{\lfloor \frac{d-1}{2} \rfloor} \binom{i}{k-d+i} h_i(P)$$

$$= \sum_{i=0}^{\frac{d}{2}}{}^{*} \left(\binom{d-i}{k-i} + \binom{i}{k-d+i} \right) h_i(P)$$

The recurrence relation for the h-vector

For any n-vertex simplicial shellable $(d-1)$-complex we have:

$$(k+1)h_{k+1}(\mathcal{C}) + (d-k)h_k(\mathcal{C}) = \sum_{v \in \mathrm{vert}(\mathcal{C})} h_k(\mathcal{C}/v)$$

But on the other hand:

$$h_k(\mathcal{C}/v) \leq h_k(\mathcal{C}),$$

which gives:

$$(k+1)h_{k+1}(\mathcal{C}) + (d-k)h_k(\mathcal{C}) \leq \sum_{v \in \mathrm{vert}(\mathcal{C})} h_k(\mathcal{C}) = n \cdot h_k(\mathcal{C})$$

or, equivalently,

$$h_{k+1}(\mathcal{C}) \leq \frac{n-d+k}{k+1}\, h_k(\mathcal{C}).$$

Solving the recurrence relation and finishing up

Using the fact that $h_0(P) = 1$, we can solve the recurrence relation and get:

$$h_k(P) \leq \binom{n-d-1+k}{k}, \qquad k \geq 0.$$

Substituting this bound in relations above, we get:

$$\begin{aligned} f_{k-1}(P) &= \sum_{i=0}^{\frac{d}{2}}{}^{*} \left(\binom{d-i}{k-i} + \binom{i}{k-d+i} \right) h_i(P) \\ &\leq \sum_{i=0}^{\frac{d}{2}}{}^{*} \left(\binom{d-i}{k-i} + \binom{i}{k-d+i} \right) \cdot \binom{n-d-1+k}{k} \end{aligned}$$

The above bound is *tight* for n-vertex $\lfloor \frac{d}{2} \rfloor$-neighborly d-polytopes (e.g., the cyclic d-polytopes $C_d(n)$).

McMullen's methodology in a glance

① Consider the h-vector $\boldsymbol{h}(P)$ of P.

② Use the Dehn-Sommerville-like equations for P.

③ Prove a recurrence relation for the elements of $\boldsymbol{h}(P)$.

④ Prove bounds for the elements of $\boldsymbol{h}(P)$ (using ③).

⑤ Compute bounds for the elements of the f-vector $\boldsymbol{f}(P)$ of P using ② and the bounds on the elements of $\boldsymbol{h}(P)$.

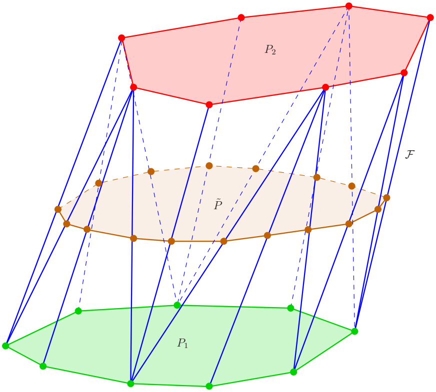
Our methodology

For the upper bound we proceed in way analogous to that of [McMullen 1970] for proving the UBT for polytopes:

- ➔ We use the *Cayley embedding* to define a set of faces $\mathcal{F}$ such that $f_k(\mathcal{F}) = f_{k-1}(P_1 \oplus P_2)$, $1 \le k \le d$.
- ① We define the h-vector $\boldsymbol{h}(\mathcal{F})$ of $\mathcal{F}$.
- ② We establish Dehn-Sommerville-like equations for $\boldsymbol{h}(\mathcal{F})$.
- ③ We prove a recurrence relation for the elements of $\boldsymbol{h}(\mathcal{F})$.
- ④ We prove bounds for the elements of $\boldsymbol{h}(\mathcal{F})$ (using ③).
- ⑤ We compute bounds for the elements of the f-vector $\boldsymbol{f}(\mathcal{F})$ of $\mathcal{F}$ using ② and the bounds on the elements of $\boldsymbol{h}(\mathcal{F})$

  $\rightsquigarrow$ bounds on the elements of $\boldsymbol{f}(P_1 \oplus P_2)$.

The Cayley embedding & the Cayley trick

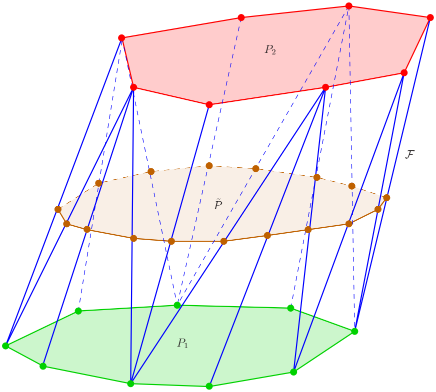


- *Cayley embedding:*
  Given two d-polytopes P_1 and P_2, embed P_1 (resp., P_2) in the hyperplane of $\mathbb{E}^{d+1}$ with equation $\{x_{d+1} = 0\}$ (resp., $\{x_{d+1} = 1\}$).
- *Cayley trick:*
  The intersection of the *Cayley polytope* $P = \text{conv}(\{P_1, P_2\})$ with the hyperplane $\{x_{d+1} = \lambda\}$, $\lambda \in (0, 1)$, is the weighted Minkowski sum $(1 - \lambda)P_1 \oplus \lambda P_2$.

Remark

For any two values of $\lambda \in (0, 1)$, the weighted Minkowski sums are combinatorially equivalent to each other and to $P_1 \oplus P_2$.

The Cayley embedding & the Cayley trick



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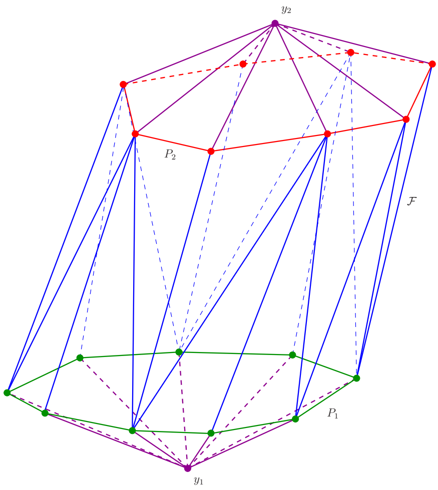
Remark

Call $\mathcal{F}$ the set of faces of P intersected by $\{x_{d+1} = \lambda\}$. Then for all $1 \leq k \leq d$:

$$f_k(\mathcal{F}) = f_{k-1}(P_1 \oplus P_2).$$

Double stellar subdivision of the Cayley embedding



- W.l.o.g., we can assume that P is *almost* simplicial (except possibly its facets P_1 and P_2).
- Add two points y_1 and y_2 so that they are beyond the facets P_1 and P_2 of P, and beneath any other facet of P.
- Let $Q = \mathrm{conv}(V_1 \cup V_2 \cup \{y_1, y_2\})$. Observe that Q *is simplicial*.

Remark

The faces of ∂Q that are not faces of $\mathcal{F}$ are exactly the faces of the star $\mathcal{S}_j$ of y_j in ∂Q, $j = 1, 2$.

Remark

The link $\partial Q / y_j$ of y_j in ∂Q is the boundary complex ∂P_j, $j = 1, 2$.

Dehn-Sommerville equations for $\mathcal{F}$

- We can show that, for all $0 \leq k \leq d+1$:

$$h_k(\partial Q) = h_k(\mathcal{F}) + h_k(\partial P_1) + h_k(\partial P_2).$$

- Using the Dehn-Sommerville equations for Q, we get:

$$h_{d+1-k}(\mathcal{F}) + h_{d+1-k}(\partial P_1) + h_{d+1-k}(\partial P_2) = h_k(\mathcal{F}) + h_k(\partial P_1) + h_k(\partial P_2).$$

- After using the Dehn-Sommerville equations for ∂P_j:

$$h_{d+1-k}(\mathcal{F}) + h_{k-1}(\partial P_1) + h_{k-1}(\partial P_2) = h_k(\mathcal{F}) + h_k(\partial P_1) + h_k(\partial P_2).$$

- Since $g_k(\partial P_j) = h_k(\partial P_j) - h_{k-1}(\partial P_j)$, we arrive at the relation:

$$h_{d+1-k}(\mathcal{F}) = h_k(\mathcal{F}) + g_k(\partial P_1) + g_k(\partial P_2) = h_k(\mathcal{K}).$$

where $\mathcal{K}$ is the *closure* of $\mathcal{F}$ under inclusion.

The recurrence relation for the elements of $h(\mathcal{F})$

Lemma

For all $0 \leq k \leq d$,

$$h_{k+1}(\mathcal{F}) \leq \frac{n_1 + n_2 - d - 1 + k}{k+1}\, h_k(\mathcal{F}) + \frac{n_1}{k+1}\, g_k(\partial P_2) + \frac{n_2}{k+1}\, g_k(\partial P_1).$$

Sketch of proof.

Since Q is a $(d+1)$-polytope, and P_1, P_2 are d-polytopes, we have: (cf. [McMullen 1970]):

$$(k+1)h_{k+1}(\partial Q) + (d+1-k)h_k(\partial Q) = \sum_{v \in V} h_k(\partial Q/v), \qquad 0 \leq k \leq d.$$

$$(k+1)h_{k+1}(\partial P_j) + (d-k)h_k(\partial P_j) = \sum_{v \in V_j} h_k(\partial P_j/v), \quad 0 \leq k \leq d-1.$$

From these relations and after some algebra, we arrive at:

$$(k+1)h_{k+1}(\mathcal{F}) + (d+1-k)h_k(\mathcal{F}) = \sum_{i=1}^{2} \left(\sum_{v \in V_i} \left[h_k(\mathcal{K}/v) - g_k(\partial P_i/v) \right] \right).$$

The recurrence relation for the elements of $h(\mathcal{F})$

Claim

The following relations hold, for $0 \leq k \leq d$:

$$h_k(\mathcal{K}/v) - g_k(\partial P_i/v) \leq h_k(\mathcal{K}) - g_k(\partial P_i), \qquad v \in V_i.$$

Sketch of proof.

If the claim is true, we get:

$$\sum_{v\in V_1} [h_k(\mathcal{K}/v) - g_k(\partial P_1/v)] \leq \sum_{v\in V_1} [h_k(\mathcal{K}) - g_k(\partial P_1)] = n_1[h_k(\mathcal{F}) + g_k(\partial P_2)],$$
$$\sum_{v\in V_2} [h_k(\mathcal{K}/v) - g_k(\partial P_2/v)] \leq \sum_{v\in V_2} [h_k(\mathcal{K}) - g_k(\partial P_2)] = n_2[h_k(\mathcal{F}) + g_k(\partial P_1)].$$

We thus conclude that, for $0 \leq k \leq d$:

$$(k+1)h_{k+1}(\mathcal{F}) + (d+1-k)h_k(\mathcal{F}) \leq (n_1+n_2)h_k(\mathcal{F}) + n_1 g_k(\partial P_2) + n_2 g_k(\partial P_1)$$

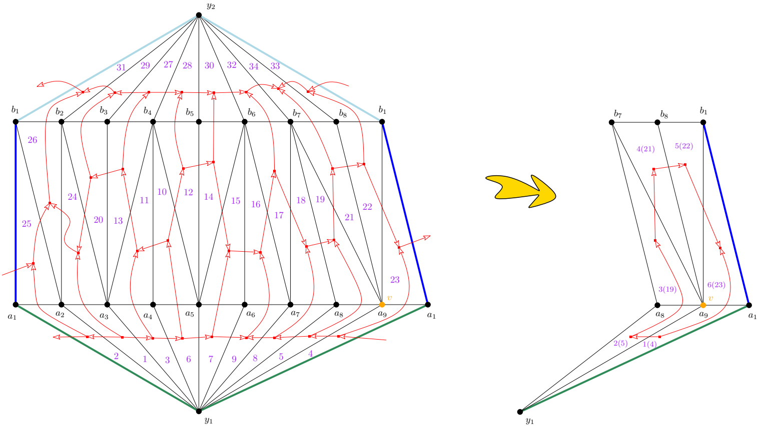
Solving for $h_{k+1}(\mathcal{F})$ gives the relation in the statement of the lemma. □

Proof of the Claim by ... example

- Let $\mathcal{X}_1 = \mathcal{K} \setminus \partial P_1 = \mathcal{K}_1 \setminus \mathcal{S}_1$, where $\mathcal{K}_1 = \mathcal{K} \cup \mathcal{S}_1$ and $\mathcal{S}_i = \mathrm{star}(y_i, \partial Q)$, $i = 1, 2$.

$\rightsquigarrow$ $h_k(\mathcal{X}_1) = h_k(\mathcal{K}_1) - h_k(\mathcal{S}_1) = h_k(\mathcal{K}) - g_k(\partial P_1)$

$h_k(\mathcal{X}_1/v) = h_k(\mathcal{K}_1/v) - h_k(\mathcal{S}_1/v) = h_k(\mathcal{K}/v) - g_k(\partial P_1/v)$, for all $v \in \mathrm{vert}(\partial P_1)$

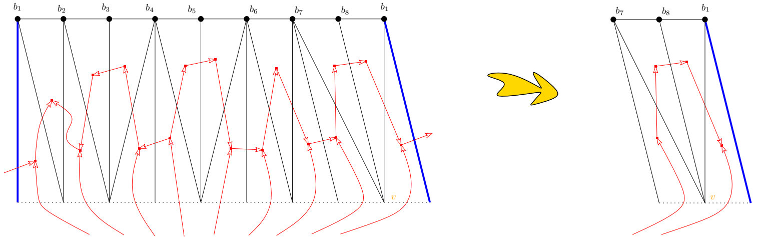


Proof of the Claim by ... example

- Let $\mathcal{X}_1 = \mathcal{K} \setminus \partial P_1 = \mathcal{K}_1 \setminus \mathcal{S}_1$, where $\mathcal{K}_1 = \mathcal{K} \cup \mathcal{S}_1$ and $\mathcal{S}_i = \mathrm{star}(y_i, \partial Q)$, $i = 1, 2$.

- In a shelling of ∂Q that shells $\mathcal{S}_1$ first and $\mathcal{S}_2$ last, $h_k(\mathcal{X}_1)$ counts the # of vertices of in-degree k in $G^{\Delta}(\partial Q)$, that are dual to facets in $\mathcal{K}$.

- Analogously: In the induced shelling of $\partial Q/v$, $h_k(\mathcal{X}_1/v)$ counts the # of vertices of in-degree k in $G^{\Delta}(\partial Q/v)$, that are dual to facets in $\mathcal{K}/v$.



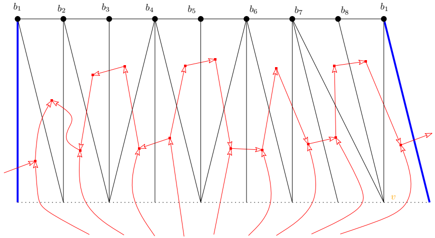
$\boldsymbol{h}(\mathcal{X}_1) = (0, 8, 9, 0)$ $\boldsymbol{h}(\mathcal{X}_1/v) = (0, 3, 1)$

Proof of the Claim by ... example

- Let $\mathcal{X}_1 = \mathcal{K} \setminus \partial P_1 = \mathcal{K}_1 \setminus \mathcal{S}_1$, where $\mathcal{K}_1 = \mathcal{K} \cup \mathcal{S}_1$ and $\mathcal{S}_i = \mathrm{star}(y_i, \partial Q)$, $i = 1, 2$.

Since $G^{\Delta}(\partial Q/v)$ is (isomorphic to) a subgraph of $G^{\Delta}(\partial Q)$, we deduce that

$$h_k(\mathcal{K}/v) - g_k(\partial P_1/v) = h_k(\mathcal{X}_1/v) \leq h_k(\mathcal{X}_1) = h_k(\mathcal{K}) - g_k(\partial P_1).$$



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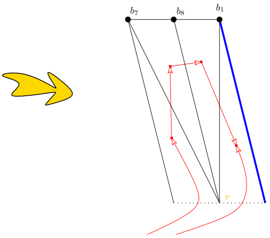

$\boldsymbol{h}(\mathcal{X}_1/v) = (0, 3, 1)$

Upper bounds for the elements of $\boldsymbol{h}(\mathcal{F})$

Lemma

For all $0 \leq k \leq d+1$,

$$h_k(\mathcal{F}) \leq \binom{n_1+n_2-d-2+k}{k} - \binom{n_1-d-2+k}{k} - \binom{n_2-d-2+k}{k}.$$

Sketch of proof.

The upper bound follows by induction on k, from the recurrence relation for $\boldsymbol{h}(\mathcal{F})$, and the upper bounds for $\boldsymbol{g}(\partial P_1)$ and $\boldsymbol{g}(\partial P_2)$. $\square$

Bounds for the elements of $f(\mathcal{F})$

$$f_{k-1}(\mathcal{F}) = \sum_{i=0}^{d+1} \binom{d+1-i}{k-i} h_i(\mathcal{F}) = \sum_{i=0}^{\lfloor \frac{d+1}{2} \rfloor} \binom{d+1-i}{k-i} h_i(\mathcal{F}) + \sum_{i=\lfloor \frac{d+1}{2} \rfloor+1}^{d+1} \binom{d+1-i}{k-i} h_i(\mathcal{F})$$

$$= \sum_{i=0}^{\lfloor \frac{d+1}{2} \rfloor} \binom{d+1-i}{k-i} h_i(\mathcal{F}) + \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \binom{i}{k-d-1+i} h_{d+1-i}(\mathcal{F})$$

$$= \sum_{i=0}^{\lfloor \frac{d+1}{2} \rfloor} \binom{d+1-i}{k-i} h_i(\mathcal{F}) + \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \binom{i}{k-d-1+i} (h_i(\mathcal{F}) + g_i(\partial P_1) + g_i(\partial P_2))$$

$$= \sum_{i=0}^{\frac{d+1}{2}}{}^{*} \left(\binom{d+1-i}{k-i} + \binom{i}{k-d-1+i} \right) h_i(\mathcal{F}) + \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \binom{i}{k-d-1+i} (g_i(\partial P_1) + g_i(\partial P_2))$$

$$\leq \cdots = \cdots = \cdots$$

$$= f_{k-1}(C_{d+1}(n_1 + n_2)) - \sum_{i=0}^{\lfloor \frac{d+1}{2} \rfloor} \binom{d+1-i}{k-i} \sum_{j=1}^{2} \binom{n_j - d - 2 + i}{i}$$

Summary & analogy with McMullen's UBT proof

① $h_k(P) = \sum_{i=1}^{k} (-1)^{k-i} \binom{d-i}{d-k} f_{i-1}(P), \qquad 0 \leq k \leq d$

➔ $f_k(\mathcal{F}) = f_{k-1}(P_1 \oplus P_2), \qquad 1 \leq k \leq d$

① $h_k(\mathcal{F}) = \sum_{i=1}^{k} (-1)^{k-i} \binom{d+1-i}{d+1-k} f_{i-1}(\mathcal{F}), \qquad 0 \leq k \leq d+1$

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➔ $f_k(\mathcal{F}) = f_{k-1}(P_1 \oplus P_2), \qquad 1 \le k \le d$

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③ $h_{k+1}(\mathcal{F}) \le \frac{n_1+n_2-d-1+k}{k+1} h_k(\mathcal{F}) + \frac{n_1}{k+1} g_k(\partial P_2) + \frac{n_2}{k+1} g_k(\partial P_1), \qquad 0 \le k \le d$

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④ $h_k(P) \le \binom{n-d-1+k}{k}, \qquad 0 \le k \le d$

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④ $h_k(\mathcal{F}) \le \binom{n_1+n_2-d-2+k}{k} - \binom{n_1-d-2+k}{k} - \binom{n_2-d-2+k}{k}, \qquad 0 \le k \le d+1$

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① $h_k(P) = \sum_{i=1}^{k} (-1)^{k-i} \binom{d-i}{d-k} f_{i-1}(P), \qquad 0 \le k \le d$

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④ $h_k(P) \le \binom{n-d-1+k}{k}, \qquad 0 \le k \le d$

⑤ $f_{k-1}(P) = \sum_{i=0}^{\frac{d}{2}}{}^{*} \left(\binom{d-i}{k-i} + \binom{i}{k-d+i} \right) h_i(P), \qquad 0 \le k \le d$

➔ $f_k(\mathcal{F}) = f_{k-1}(P_1 \oplus P_2), \qquad 1 \le k \le d$

① $h_k(\mathcal{F}) = \sum_{i=1}^{k} (-1)^{k-i} \binom{d+1-i}{d+1-k} f_{i-1}(\mathcal{F}), \qquad 0 \le k \le d+1$

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③ $h_{k+1}(\mathcal{F}) \le \frac{n_1+n_2-d-1+k}{k+1} h_k(\mathcal{F}) + \frac{n_1}{k+1} g_k(\partial P_2) + \frac{n_2}{k+1} g_k(\partial P_1), \qquad 0 \le k \le d$

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⑤ $f_{k-1}(\mathcal{F}) = \sum_{i=0}^{\frac{d+1}{2}}{}^{*} \left(\binom{d+1-i}{k-i} + \binom{i}{k-d-1+i} \right) h_i(\mathcal{F}) + \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \binom{i}{k-d-1+i} \sum_{j=1}^{2} g_i(\partial P_j),$
$0 \le k \le d+1$

⇝ $f_{k-1}(P_1 \oplus P_2) = \sum_{i=0}^{\frac{d+1}{2}}{}^{*} \left(\binom{d+1-i}{k+1-i} + \binom{i}{k-d+i} \right) h_i(\mathcal{F}) + \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \binom{i}{k-d+i} \sum_{j=1}^{2} g_i(\partial P_j),$
$1 \le k \le d$

The upper bound theorem

Lemma

For all $0 \leq k \leq d+1$*:*

$$f_{k-1}(\mathcal{F}) \leq f_{k-1}(C_{d+1}(n_1+n_2)) - \sum_{i=0}^{\lfloor \frac{d+1}{2} \rfloor} \binom{d+1-i}{k-i} \left(\binom{n_1-d-2+i}{i} + \binom{n_2-d-2+i}{i} \right),$$

where $C_d(n)$ *stands for the cyclic* d*-polytope with* n *vertices.*

Using the fact that, for all $0 \leq k \leq d$, $f_{k-1}(P_1 \oplus P_2) = f_k(\mathcal{F})$, we have:

Theorem (UBT4MS)

Let P_1 *and* P_2 *be two* d*-polytopes in* $\mathbb{E}^d$, $d \geq 2$, *with* $n_1 \geq d+1$ *and* $n_2 \geq d+1$ *vertices, respectively. Let also* P *be the Cayley polytope of* P_1 *and* P_2*. Then, for* $1 \leq k \leq d$, *we have:*

$$f_{k-1}(P_1 \oplus P_2) \leq f_k(C_{d+1}(n_1+n_2)) - \sum_{i=0}^{\lfloor \frac{d+1}{2} \rfloor} \binom{d+1-i}{k+1-i} \left(\binom{n_1-d-2+i}{i} + \binom{n_2-d-2+i}{i} \right),$$

where $C_d(n)$ *stands for the cyclic* d*-polytope with* n *vertices.*

The lower bounds

To prove that the upper bounds are tight, we need to construct two d-polytopes P_1 and P_2 that satisfy two conditions:

① Both P_1 and P_2 are $\lfloor \frac{d}{2} \rfloor$-neighborly

$\iff \quad g_k(P_j) = \binom{n_j-d-2+k}{k}$, for all $0 \leq k \leq \lfloor \frac{d}{2} \rfloor$, $j = 1, 2$.

② For all $0 \leq k \leq \lfloor \frac{d+1}{2} \rfloor$,

$$h_k(\mathcal{F}) = \binom{n_1+n_2-d-2+k}{k} - \binom{n_1-d-2+k}{k} - \binom{n_2-d-2+k}{k}.$$

The lower bounds

To prove that the upper bounds are tight, we need to construct two d-polytopes P_1 and P_2 that satisfy two conditions:

① Both P_1 and P_2 are $\lfloor \frac{d}{2} \rfloor$-neighborly

$\Longleftrightarrow \quad g_k(P_j) = \binom{n_j - d - 2 + k}{k}$, for all $0 \leq k \leq \lfloor \frac{d}{2} \rfloor$, $j = 1, 2$.

② For all $0 \leq k \leq \lfloor \frac{d+1}{2} \rfloor$,

$$h_k(\mathcal{F}) = \binom{n_1 + n_2 - d - 2 + k}{k} - \binom{n_1 - d - 2 + k}{k} - \binom{n_2 - d - 2 + k}{k}.$$

The second condition is equilavent to the condition:

② For all $0 \leq k \leq \lfloor \frac{d+1}{2} \rfloor$,

$$f_{k-1}(\mathcal{F}) = \binom{n_1 + n_2}{k} - \binom{n_1}{k} - \binom{n_2}{k}.$$

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for all $2 \leq k \leq d+1$. According to [Fukuda & Weibel 2007], this bound is tight for $d \geq 4$ and for all k with $2 \leq k \leq \lfloor \frac{d}{2} \rfloor$, and is attained when P_1 and P_2 are *cyclic* d-polytopes with disjoint vertex sets.

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✔ Hence, condition ① is satisfied (cyclic polytopes are neighborly).

✔ For condition ② we observe that we get:

$$f_{k-1}(\mathcal{F}) = f_{k-2}(P_1 \oplus P_2) = \binom{n_1+n_2}{k} - \binom{n_1}{k} - \binom{n_2}{k},$$

for all $2 \leq k \leq \lfloor \frac{d}{2} \rfloor = \lfloor \frac{d+1}{2} \rfloor$, while for $k = 0$ and $k = 1$ it is easy to see that equality is also satisfied.

Lower bounds for $d \geq 3$ odd – First step

- Define the following two moment-like curves in $\mathbb{E}^d$.

$$\begin{aligned}\boldsymbol{\gamma}_1(t;\zeta) &= (t, \zeta t^d, t^2, t^3, \ldots, t^{d-1}),\\ \boldsymbol{\gamma}_2(t;\zeta) &= (\zeta t^d, t, t^2, t^3, \ldots, t^{d-1}),\end{aligned} \qquad t > 0, \qquad \zeta \geq 0.$$

  Specifically, denote $\boldsymbol{\gamma}_j(\cdot;0)$ by $\boldsymbol{\gamma}_j(\cdot)$.
- Embed $\boldsymbol{\gamma}_1(t;\zeta)$ in $\{x_{d+1} = 0\}$ and $\boldsymbol{\gamma}_2(t;\zeta)$ in $\{x_{d+1} = 1\}$.
- Choose n_1 points on $\boldsymbol{\gamma}_1(t)$ of the form $t_i = \alpha_i \tau$.
- Choose n_2 points on $\boldsymbol{\gamma}_2(t)$ of the form $t_i = \beta_i$.
- Let U_1 and U_2 be the two point sets, and let $Q_j = \operatorname{conv}(U_j)$, $Q = \operatorname{conv}(\{U_1, U_2\})$.
- There exists a sufficiently small positive value $\tau^\star$ for τ, such that for all $0 \leq k \leq \lfloor \frac{d+1}{2} \rfloor$:

$$f_{k-1}(\mathcal{F}_{Q^\star}) = \binom{n_1+n_2}{k} - \binom{n_1}{k} - \binom{n_2}{k}.$$

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✔ $Q^\star$ satisfies condition ②.

✗ The $Q_j^\star$'s are $(d-1)$-dimensional.

Lower bounds for $d \geq 3$ odd – Second step

- Fix a small enough value $\tau^\star$ for τ; let $U_j^\star$ be the corresponding vertex sets.
- Choose some positive ζ, call V_j the d-dimensional vertex sets we get from $U_j^\star$, and let $P_j = \mathrm{conv}(V_j)$, $P = \mathrm{conv}(\{V_1, V_2\})$.
- For any $\zeta > 0$, P_1 and P_2 are $\lfloor \frac{d}{2} \rfloor$-neighborly.
- For $\zeta > 0$ small enough, P satisfies

$$f_{k-1}(\mathcal{F}_P) = \binom{n_1+n_2}{k} - \binom{n_1}{k} - \binom{n_2}{k}, \qquad 0 \leq k \leq \lfloor \tfrac{d+1}{2} \rfloor.$$

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✔ By choosing a small enough positive ζ, conditions ① and ② are satisfied for P_1 and P_2.

Upper bounds for three polytopes

- Same methodology as for two polytopes:
  - Consider the Cayley polytope of the polytopes
  - Make it simplicial (by adding vertices)
  - Derive Dehn-Sommerville-like equations
  - Prove bounds on certain h-vectors
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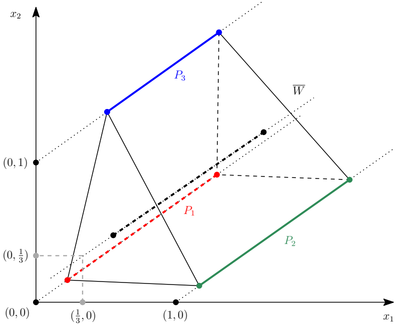
- Analysis is much harder
- Idea seems to generalize for more summands (later on in the talk)

The Cayley trick (for three polytopes)



- *Cayley embedding:*
  Given three d-polytopes P_1, P_2, P_3, and the (standard) affine basis $\boldsymbol{e}_0$, $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ of $\mathbb{E}^2$ we embed each P_i in $\mathbb{E}^{d+2}$ using the inclusion $\mu_i(\boldsymbol{x}) = (\boldsymbol{e_{i-1}}, \boldsymbol{x})$.
- *Cayley trick:*
  The intersection of $\mathcal{C}_{[3]} = \mathrm{conv}(\{P_1, P_2, P_3\})$ with the d-flat of $\mathbb{E}^{d+2}$

  $$\overline{W} = \{\tfrac{1}{3}\boldsymbol{e}_0 + \tfrac{1}{3}\boldsymbol{e}_1 + \tfrac{1}{3}\boldsymbol{e}_2\} \times \mathbb{E}^d,$$

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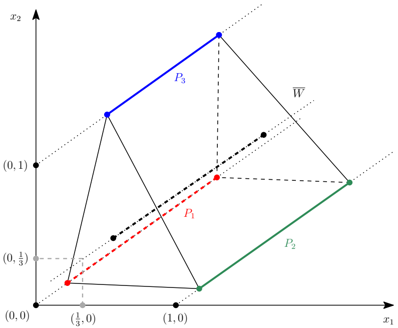

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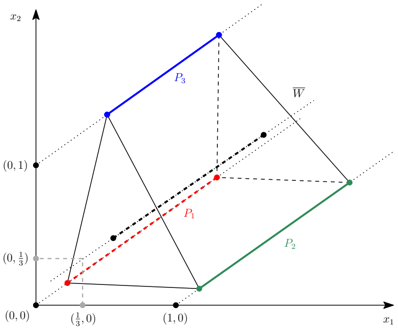

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- *Minkowski Sum:* $\mathcal{F}_{[3]}$ is the set of faces in $\mathcal{C}_{[3]}$ having at least one vertex from each $P_i, \quad 1 \leq i \leq 3$
- $f_k(P_1 \oplus P_2 \oplus P_3) = f_{k+2}(\mathcal{F}_{[3]})$
  for all $0 \leq k \leq d$

Dehn-Sommerville-like equations

- Call $\mathcal{K}_R$ the closure of $\mathcal{F}_R$ under sub-face inclusion, $\emptyset \subset R \subseteq [3]$.
- $\mathcal{K}_R$ is a pure simplicial complex $(d+|R|-2)$-complex
- We have:

$$f_k(\mathcal{K}_R) = \sum_{\emptyset \subset S \subseteq R} f_k(\mathcal{F}_S), \qquad f_k(\mathcal{F}_R) = \sum_{\emptyset \subset S \subseteq R} (-1)^{|R|-|S|} f_k(\mathcal{K}_S)$$

We can prove the following analogue of the Dehn-Sommerville equations:

Lemma

For any $\emptyset \subset R \subseteq [3]$ and for all $0 \leq k \leq d+|R|-1$, we have:

$$h_{d+|R|-1-k}(\mathcal{F}_R) = h_k(\mathcal{K}_R).$$

Towards the recurrence relation for $\boldsymbol{h}(\mathcal{F}_{[3]})$

Define the m-order g-vector $\boldsymbol{g}^{(m)}(\mathcal{Y})$ of $\mathcal{Y}$ as follows:

$$g_k^{(m)}(\mathcal{Y}) = \begin{cases} h_k(\mathcal{Y}), & m = 0, \\ g_k^{(m-1)}(\mathcal{Y}) - g_{k-1}^{(m-1)}(\mathcal{Y}), & m > 0. \end{cases}$$

Lemma

For all $k \geq 0$, we have:

$$(k+1)h_{k+1}(\mathcal{F}_{[3]}) + (d+2-k)h_k(\mathcal{F}_{[3]}) = \sum_{\emptyset \subset S \subseteq [3]} (-1)^{3-|S|} \sum_{v \in V_S} g_k^{(3-|S|)}(\mathcal{K}_S/v)$$

where $V_S = \cup_{i \in S} V_i$, and $\mathcal{K}_S/v$ denotes the empty set if $v \notin V_S$.

The *Link/Non-link* inequality

Lemma

For all $k \geq 0$, and for all $v \in V_R$, we have:

$$\sum_{\emptyset \subset S \subseteq [3]} (-1)^{3-|S|} \sum_{v \in V_S} g_k^{(3-|S|)}(\mathcal{K}_S/v) \leq \sum_{\emptyset \subset S \subseteq [3]} (-1)^{3-|S|} \sum_{v \in V_S} g_k^{(3-|S|)}(\mathcal{K}_S),$$

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where $\mathcal{K}_S/v$ denotes the empty set if $v \notin V_S$.

- Similar in spirit (though more involved) argument as in the case of two polytopes

The recurrence relation for $\boldsymbol{h}(\mathcal{F}_{[3]})$

Lemma

For all $0 \leq k \leq d+1$, we have:

$$h_{k+1}(\mathcal{F}_{[3]}) \leq \tfrac{n_{[3]}-d-2+k}{k+1}\, h_k(\mathcal{F}_{[3]}) + \sum_{i=1}^{3} \tfrac{n_i}{k+1}\, g_k(\mathcal{F}_{[3]\setminus\{i\}}).$$

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- Using induction we can fairly easily prove the following bounds:

$$h_k(\mathcal{F}_{[3]}) \leq \sum_{\emptyset \subset S \subseteq [3]} (-1)^{3-|S|} \binom{n_S - d - 3 + k}{k}, \quad n_S = \sum_{i \in S} n_i.$$

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- More involved can also be proved for $h_k(\mathcal{K}_{[3]})$

Putting everything together

$$
\begin{aligned}
f_{k-1}(\mathcal{F}_{[3]}) &= \sum_{i=0}^{d+2} \binom{d+2-i}{k-i} h_i(\mathcal{F}_{[3]}) \\
&= \sum_{i=0}^{\lfloor \frac{d+2}{2} \rfloor} \binom{d+2-i}{k-i} h_i(\mathcal{F}_{[3]}) + \sum_{i=\lfloor \frac{d+2}{2} \rfloor +1}^{d+2} \binom{d+2-i}{k-i} h_i(\mathcal{F}_{[3]}) \\
&= \sum_{i=0}^{\lfloor \frac{d+2}{2} \rfloor} \binom{d+2-i}{k-i} h_i(\mathcal{F}_{[3]}) + \sum_{j=0}^{\lfloor \frac{d+1}{2} \rfloor} \binom{j}{k-d-2+j} h_{d+2-j}(\mathcal{F}_{[3]}) \\
&= \sum_{i=0}^{\lfloor \frac{d+2}{2} \rfloor} \binom{d+2-i}{k-i} h_i(\mathcal{F}_{[3]}) + \sum_{j=0}^{\lfloor \frac{d+1}{2} \rfloor} \binom{j}{k-d-2+j} h_j(\mathcal{K}_{[3]}) \\
&\leq \sum_{i=0}^{\lfloor \frac{d+2}{2} \rfloor} \binom{d+2-i}{k-i} \boxed{} + \sum_{j=0}^{\lfloor \frac{d+1}{2} \rfloor} \binom{j}{k-d-2+j} \boxed{} \\
&= \cdots \\
&= <\text{final result}>
\end{aligned}
$$

The tightness construction

- Define the following two moment-like curves in $\mathbb{E}^d$.

$$\begin{aligned}
\boldsymbol{\gamma}_1(t;\zeta) &= (\ t, \zeta t^2, \zeta t^3, t^4, \ldots, t^{d-1}), \\
\boldsymbol{\gamma}_2(t;\zeta) &= (\zeta t,\ \ t^2, \zeta t^3, t^4, \ldots, t^{d-1}), \qquad t > 0, \qquad \zeta \geq 0. \\
\boldsymbol{\gamma}_3(t;\zeta) &= (\zeta t, \zeta t^2,\ \ t^3, t^4, \ldots, t^{d-1}),
\end{aligned}$$

  Specifically, denote $\boldsymbol{\gamma}_j(\cdot;0)$ by $\boldsymbol{\gamma}_j(\cdot)$.
- Embed $\boldsymbol{\gamma}_i(t;\zeta)$ in $\mathbb{E}^{d+2}$ using the lifting map $\mu_i(\cdot)$.
- Choose n_i points on $\boldsymbol{\gamma}_i(t)$ of the form $t_{i,j} = x_{i,j}\tau^{\nu_i}$, $1 \leq j \leq n_i$, where $\nu_1 > \nu_2 > \nu_3 \geq 0$ (τ is a non-negative parameter).
- Set $\zeta = \tau^M$, for M sufficiently large.
- Let U_i be the three point sets, $U_R = \cup_{i\in R} U_i$, and let $\mathcal{C}_R = \mathrm{conv}(U_R)$, $\emptyset \subset R \subseteq [3]$.
- $\mathcal{C}_R$, $\emptyset \subset R \subseteq [3]$, is obviously the Cayley polytope of the polytopes P_i with $i \in R$.
- $\mathcal{F}_R$ is (as before) the set of faces of $\mathcal{C}_R$ that contain vertices from the P_i's, $i \in R$ and those only.

The tightness construction (contd.)

- There exists a sufficiently small positive value $\hat{\tau}_R$, such that for all $\tau \in (0, \hat{\tau}_R)$ and $2 \leq k \leq \lfloor \frac{d+1}{2} \rfloor$:

$$f_{k-1}(\mathcal{F}_R) = \sum_{\emptyset \subset S \subseteq R} (-1)^{2-|S|} \binom{n_S}{k}. \tag{1}$$

  where $|R| = 2$.

- There exists a sufficiently small positive value $\hat{\tau}_{[3]}$, such that for all $\tau \in (0, \hat{\tau}_{[3]})$ and $3 \leq k \leq \lfloor \frac{d+2}{2} \rfloor$:

$$f_{k-1}(\mathcal{F}_{[3]}) = \sum_{\emptyset \subset S \subseteq [3]} (-1)^{3-|S|} \binom{n_S}{k}. \tag{2}$$

- Choose positive $\tau^\star$ to be smaller than: (1) all $\hat{\tau}_R$, $|R| = 2$, and (2) $\hat{\tau}_{[3]}$. Then conditions (1) and (2) are satisfied for $\tau \leftarrow \tau^\star$.

These two conditions are necessary and sufficient for the h-vectors of $\mathcal{F}_{[3]}$ and $\mathcal{K}_{[3]}$ to take their element-wise maximal values.

Bounds for four or more d-polytopes

- Consider $r \geq 4$ d-polytopes $P_1, \ldots, P_r$.
  Let $\mathcal{C}$ be their Cayley polytope in $\mathbb{R}^{d+r-1}$.
- Call $\mathcal{F}_R$, where $\emptyset \subset R \subseteq [r]$ the set of faces of $\mathcal{C}$ that have at least one vertex from each P_i, $i \in R$.
- Call $\mathcal{K}_R$ the *closure* of $\mathcal{F}_R$ (under inclusion).

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$$f_{k-1}(\mathcal{F}_{[r]}) = f_{k-r}(P_1 \oplus P_2 \oplus \cdots \oplus P_r), \qquad r \leq k \leq d+r-1.$$

- Our goal is to compute tight bounds for $f_{k-1}(\mathcal{F}_{[r]})$, $r \leq k \leq d+r-1$.

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- Our goal is to compute tight bounds for $f_{k-1}(\mathcal{F}_{[r]})$, $r \leq k \leq d+r-1$.
- Exploit the fact that:

$$f_{k-1}(\mathcal{F}_{[r]}) = \sum_{i=0}^{\lfloor \frac{d+r-1}{2} \rfloor} \binom{d+r-1-i}{k-i} h_i(\mathcal{F}_{[r]}) + \sum_{i=0}^{\lfloor \frac{d+r-2}{2} \rfloor} \binom{i}{k-d-r+1+i} h_{d+r-1-k}(\mathcal{F}_{[r]})$$

Some facts

- R below denotes any non-empty subset of $[r]$.
- The Dehn-Sommerville equations for $\mathcal{F}_R$ are:

$$h_{d+|R|-1-k}(\mathcal{F}_R) = h_k(\mathcal{K}_R), \qquad 0 \leq k \leq d+|R|-1.$$

- The following relation holds:

$$(k+1)h_{k+1}(\mathcal{F}_R)+(d+|R|-1-k)h_k(\mathcal{F}_R) = \sum_{\emptyset\subset S\subseteq R}(-1)^{|R|-|S|}\sum_{v\in V_S} g_k^{(|R|-|S|)}(\mathcal{K}_S/v)$$

- The recurrence relation for $\boldsymbol{h}(\mathcal{F}_R)$ is* :

$$h_{k+1}(\mathcal{F}_R) \leq \tfrac{n_R-d-|R|+1+k}{k+1}h_k(\mathcal{F}_R) + \sum_{i=1}^{|R|}\tfrac{n_i}{k+1}g_k(\mathcal{F}_{R\setminus\{i\}}), \qquad (3)$$

where $n_R = \sum_{i\in R} n_i$.

- Using (3), we quite easily get:

$$h_k(\mathcal{F}_R) \leq \sum_{\emptyset\subset S\subseteq R}(-1)^{|R|-|S|}\binom{n_S-d-|R|+k}{k}, \qquad 0 \leq k \leq d+|R|-1.$$

Some non-facts

- Not at all straightforward to derive *good* bounds for $h_k(\mathcal{K}_R)$.
- Tightness construction for three polytopes seems to generalize to r polytopes, for $2 \leq r < d$.
- For $r \geq d$, we need to further generalize the construction in conjunction with some results by [Weibel 2012].

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What to take home with you?

- The Cayley polytope includes a lot of information that we need/can take advantage of it.
- It seems that we can generalize McMullen's proof for the UBT to get analogous bounds for the Minkowski sum of convex polytopes (already completely done for two polytopes).
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THANK YOU FOR YOUR ATTENTION